Notes on constraint damping for Maxwell's equations

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1 Basic Equations

We will consider two simple cases for Maxwell's equations: a) Maxwell's equations in vacuum, and b) Maxwell's equations in the ideal magnetohydrodynamics (MHD) limit – the induction equations.

1.1 Vacuum

We know that Maxwell's equations in vacuum can be written as wave equations for the E and B fields. In particular we will focus on the B field equations, which we can write as

$$\partial_t^2 \mathbf{B} = \nabla^2 \mathbf{B}, \tag{1}$$

where $\partial_t = \partial/\partial_t$, and the Laplacian operator $\nabla = \delta^{ij}\partial_i\partial_j$.

Of course the solutions of this equation must satisfy the no-monopole constraint of Maxwell's theory, i.e.,

$$C = \nabla \mathbf{B} = \partial_i B^i = 0, \tag{2}$$

where as usual the Einstein summation convention is employed, i.e., repeated upper and lower indices are summed over.

First lets check if eqn. (1) preserves the constraint (2). Taking two time derivatives of the constraint C we (symbolically) find

$$\partial_t^2 C = \partial_i \partial_t^2 B^i = \partial_i \nabla^2 B^i = \nabla^2(\partial_i B^i) = \nabla^2 C \tag{3}$$

In other words, the constraint C evolves like a wave equation, too. Moreover, if the constraint is 0 initially, then the B-field wave equation guarantees that the constraint will remain zero for all times.

Now we want to find a modification of Eq. (1) such that the evolution equation for the constraint is a damped wave equation. Due to the mismatch in the order of the derivatives that appear in the constraint C (first order) and those that appear in (1) (second order), it does not appear possible to add some multiple of the constraint to the right-hand-side (RHS) of (1) such that we obtain a wave equation for C with a friction term. However, it is possible if we recast

(1) to first order form. To do this we introduce the following new evolution variables

$$\partial_t B^k = A^k \tag{4}$$

and

$$\partial_j B^k = \Gamma_j{}^k \tag{5}$$

To find an evolution equation for $\Gamma_j{}^k$ we can commute a spatial with a time derivative and obtain from (5) and (4)

$$\partial_t \Gamma_j{}^k = \partial_j \partial_t B^k = \partial_j A^k \tag{6}$$

and now the first-order form of (1) becomes

$$\begin{aligned} \partial_t B^k &= A^k && (7) \\ \partial_t A^k &= \delta^{ij} \partial_i \Gamma_j{}^k && (8) \\ \partial_t \Gamma_j{}^k &= \partial_j A^k && (9) \end{aligned}$$

and the constraint C is now written as

$$C = \Gamma_i{}^i \tag{10}$$

However, the first-order system (7) is also subject to the following constraints

$$C_j{}^k \equiv \partial_j B^k - \Gamma_j{}^k = 0 \tag{11}$$

and (the ordering constraint)

$$C_{ij}{}^k \equiv \partial_i \Gamma_j{}^k - \partial_j \Gamma_i{}^k = 0, \tag{12}$$

which can be easily derived from (5) and the fact that $\partial_i \partial_j B^k = \partial_j \partial_i B^k$. The following equation will also be useful which can be derived from the previous one

$$C_{kj}{}^k = \partial_k \Gamma_j{}^k - \partial_j \Gamma_k{}^k = \partial_k \Gamma_j{}^k - \partial_j C, \tag{13}$$

We can now ask how does the constraint (10) evolve under the first-order system (7). Again we can compute the time derivative of C and use Eq. (7) to replace the time derivatives of the variables

$$\partial_t C = \partial_t \Gamma_k{}^k = \partial_k A^k. \tag{14}$$

We now take a second time derivative of the previous equation to find

$$\partial_t^2 C = \partial_k \partial_t A^k = \delta^{ij} \partial_k \partial_i \Gamma_j{}^k = \delta^{ij} \partial_i \partial_k \Gamma_j{}^k = \delta^{ij} \partial_i (\partial_j C + C_{kj}{}^k) = \nabla^2 C + \delta^{ij} \partial_i C_{kj}{}^k, \tag{15}$$

where in the last step we used Eq. (13). Thus, the constraint C again evolves like a wave equation, as it should, but slightly modified by the ordering constraint.

The ordering constraint also has an evolution equation, which from Eqs. (8) and (12) becomes

$$\partial_t C_{ij}{}^k = \partial_i \partial_t \Gamma_J{}^k - \partial_j \partial_t \Gamma_i{}^k = \partial_i \partial_j A^k - \partial_j \partial_i A^k = 0. \tag{16}$$

Now, lets consider adding a multiple of the constraint (10) to the RHS of the evolution equations for $\Gamma_x{}^x$, $\Gamma_y{}^y$ and $\Gamma_z{}^z$, as follows

$$\partial_t \Gamma_x{}^x = \partial_x A^x - \frac{1}{3}\lambda C \tag{17}$$

$$\partial_t \Gamma_y{}^y = \partial_y A^y - \frac{1}{3}\lambda C \tag{18}$$

$$\partial_t \Gamma_z{}^z = \partial_z A^z - \frac{1}{3}\lambda C \tag{19}$$

where we will choose $\lambda > 0$. We are free to add the constraint as at the analytic level it should be exactly zero, so by adding zero to the RHS nothing should change. However, we can now ask again how the constraint (10) evolves under the first order system (7) but including the new modifications in Eq. (17). We can again take a time derivative of the constraint

$$\partial_t C = \partial_t \Gamma_k{}^k = \partial_k A^k - \lambda C \tag{20}$$

This equation already shows that we have now added a term (the last one) which tries to damp the constraint C.

We now take a second time derivative of the previous equation to find

$$\partial_t^2 C = \partial_k \partial_t A^k - \lambda \partial_t C = \delta^{ij} \partial_k \partial_i \Gamma_j{}^k - \lambda \partial_t C = \delta^{ij} \partial_i \partial_k \Gamma_j{}^k - \lambda \partial_t C \tag{21}$$

or

$$\partial_t^2 C = \nabla^2 C + \delta^{ij} \partial_i C_{kj}{}^k - \lambda \partial_t C \tag{22}$$

This is the equation we were after. Notice that the last term is precisely the friction term in the wave equation we were looking for. For the remaining discussion lets now drop the term corresponding the ordering constrain in (22), although it will not change things if we keep it in. We can now use the standard energy norm for a second order partial differential equation (like the wave equation)

$$E = \frac{1}{2} \int [(\partial_t C)^2 + \sum_{i=1}^{3} (\partial_i C)^2] dV \tag{23}$$

where we integrate over the volume in which we are trying to find the solution. And lets now take the time-derivative of the energy to obtain

$$
\begin{aligned}
\partial_t E &= \int [(\partial_t C)\partial_t^2 C + (\partial_i C)(\partial^i \partial_t C)] dV \\
&= \int [(\partial_t C)(\partial_i \partial^i C - \lambda \partial_t C) + (\partial_i C)(\partial^i \partial_t C)] dV \\
&= \int [(\partial_t C)(\cancel{\partial_i \partial^i C} - \lambda \partial_t C)] dV + \int \partial_i [(\partial_i C)(\partial_t C)] dV - \int \cancel{[(\partial_i \partial^i C)(\partial_t C)]} dV \\
&= -\int [\lambda (\partial_t C)^2] dV + \int_{\partial\Omega} n_i [(\partial^i C)(\partial_t C)] dA \qquad (24)
\end{aligned}
$$

where in the second line we used (22), in the third line we integrated by parts and in the last line we used Gauss' theorem to replace the volume integral by a surface integral over the boundary $\partial\Omega$ of the volume in which we are trying to find the solution. Obviously, we impose boundary conditions that respect the constraints – constraint satisfying boundary conditions – then the last term in (24) vanishes, and we simply have

$$
\partial_t E = -\int [\lambda (\partial_t C)^2] dV \leq 0. \qquad (25)
$$

In other words when $\lambda > 0$ and $\partial_t C > 0$, the energy will be a decreasing function of time, which demonstrates the friction nature of the newly added terms to Eqs. (17). Note that if we hadn't dropped the term corresponding the ordering constrain in (22), it would simply become a boundary term, hence it would vanish if constraint preserving boundary conditions are adopted, thus the discussion would not change.

The downside of the modification presented here is that we have achieved our goal at the expense of introducing new variables associated with a number of constraints, namely ${C_j}^k, {C_{ij}}^k$ which have a trivial evolution, but we did not discuss constraint damping for these constraint. It is possible to add multiples of these constraints to the RHS of the evolution equations and then achieve constraint damping even for these new constraints. However, we will leave this as an exercise and discuss a different approach in the next section.

1.2 Alternative formulation for Maxwell's equations

An alternative formulation of Maxwell's equations originally introduced by Komissarov in flat spacetime and non-covariant form in MNRAS, Volume 382, Issue 3, pp. 995-1004, and generalized for arbitrary spacetimes by Palenzuela, Lehner, Yoshida in Phys. Rev. D**81**, 084007,2010, is to embed Maxwell's equations in a larger system of evolution equations as follows. Consider the covariant form of Maxwell's equations

$$
\nabla_b F^{ab} = I^a, \;\; \nabla^*_b F^{ab} = 0 \qquad (26)
$$

where ∇_a is a covariant derivative associated with the spacetime metric, ${}^*F^{ab} = \epsilon^{abcd} F_{cd}/2$ is the dual of F_{ab} and ϵ^{abcd} is the Levi-Civita tensor. We know that given the unit vector normal to the timeslices of our spacetime we can define the electric (E) and magnetic (B) fields measured by a normal observer

$$E^a = F^{ab} n_b, \;\; B^a =^* F^{ab} n_b. \tag{27}$$

With equations (27) one can show that the field tensor F can be decomposed in 3+1 form as

$$\begin{aligned} F^{ab} &= n^a E^b - n^b E^a + \epsilon^{abcd} B_c n_d & (28) \\ {}^* F^{ab} &= n^a B^b - n^b B^a - \epsilon^{abcd} E_c n_d & (29) \end{aligned}$$

Now, we can expand the system of Maxwell's equations as follows

$$\nabla_a (F^{ab} + g^{ab}\psi) = I^b - \sigma n^b \psi, \;\; \nabla_a (*F^{ab} + g^{ab}\phi) = -\sigma n^b \psi, \tag{30}$$

which will reduce to the original Maxwell's equations when $\psi = \phi = 0$. We can now take the divergence of (30) to find

$$\nabla_a \nabla^a \psi = -\nabla_a (\sigma n^a \psi), \;\; \nabla_a \nabla^a \phi = -\nabla_a (\sigma n^a \phi), \tag{31}$$

which are wave equations with friction terms, i.e. psi and phi are being damped. For example in flat spacetime, these equations would be

$$(\partial_t^2 - \nabla^2)\psi = -\sigma \partial_t \psi, \;\; (\partial_t^2 - \nabla^2)\phi = -\sigma \partial_t \phi, \tag{32}$$

where we assumed σ is constant. This last equation demonstrates the friction terms in a more standard setting. Now, it can be shown that in terms of the E and B fields Eqs. (30) can be written as follows

$$\begin{aligned} (\partial_t - \pounds_\beta) E^i &= \epsilon^{ijk} D_j (\alpha B_k) - \alpha \gamma^{ij} D_j \psi + \alpha K E^i - \alpha J^i & (33) \\ (\partial_t - \pounds_\beta) B^i &= -\epsilon^{ijk} D_j (\alpha E_k) - \alpha \gamma^{ij} D_j \phi + \alpha K B^i & (34) \\ (\partial_t - \pounds_\beta) \psi &= -\alpha D_i E^i + \alpha q - \alpha \sigma \psi & (35) \\ (\partial_t - \pounds_\beta) \phi &= -\alpha D_i B^i - \alpha \sigma \phi & (36) \end{aligned}$$

Here, D_i is the covariant derivative associated with the 3-metric γ_{ij}, K is the trace of the extrinsic curvature $\pounds_\beta$ is the Lie derivative along the shift vector β^i and we have decomposed the 4-current in 3+1 form $I^a = q n^a + J^a$ with $J^a n_a = 0$ and $q = -n_a I^a$ the charge measured by normal observers. In flat spacetime and vacuum ($J^\alpha = 0$) the above system of equations can be written as

$$\begin{aligned} \partial_t E^i &= \epsilon^{ijk} \partial_j B_k - \delta^{ij} \partial_j \psi & (37) \\ \partial_t B^i &= -\epsilon^{ijk} \partial_j E_k - \delta^{ij} \partial_j \phi & (38) \\ \partial_t \psi &= -\partial_i E^i - \sigma \psi & (39) \\ \partial_t \phi &= -\partial_i B^i - \sigma \phi & (40) \end{aligned}$$

subject to the constraints $C_B \equiv \partial_i B^i = 0$ and $C_E \equiv \partial_i E^i = 0$. Note that σ will be the control parameter for the damping of the constraints.

1.3 Ideal MHD

In the limit of ideal MHD to solve Maxwell's equations we only need to solve an evolution equation for the magnetic field, which is known as the magnetic induction equation

$$\partial_t \mathbf{B} = -\nabla \times (\mathbf{v} \times \mathbf{B}) \tag{41}$$

where v is the 3-velocity of the MHD plasma which we will consider as constant for our purposes here. In component form this equation becomes

$$\partial_t B^i = \partial_j (v^i B^j - v^j B^i) \tag{42}$$

It is now trivial to see that Eq. (42) preserves the constraint (2)

$$\partial_t C = \partial_t(\partial_i B^i) = \partial_i(\partial_t B^i) = \partial_i \partial_j (v^i B^j - v^j B^i) = 0, \tag{43}$$

where the last equality arises because the partial derivatives commute and the quantity in the parentheses is anti-symmetric in (i,j). We will now present a modification of this equation (similar to that of the previous section) which also leeds to constraint damping and was originally introduced by Dedner et al., J. Comput. Phys. 175 , 645 (2002). Consider the larger PDE system

$$\begin{aligned}
\partial_t B^i &= \partial_j (v^i B^j - v^j B^i) - \partial^i \psi & (44)\\
\partial_t \psi &= -C - \lambda \psi = -\partial_i B^i - \lambda \psi. & (45)
\end{aligned}$$

Such a modification is perfectly legitimate as long as $\psi = 0$, which from eqn. (45) it implies that $\partial_i B^i = 0$. Notice however, that eqn. (45) is a damped equation – the last term at the RHS tries to damp ψ exponentially to zero. In fact, for very large values of λ this can practically be the case.

Lets now consider for simplicity that $v^x = const.$ and $v^y = v^z = 0$. Then Eqs. (44) and (45) become

$$\begin{aligned}
\partial_t B^x &= v^x(\partial_y B^y + \partial_z B^z) - \partial_x \psi & (46)\\
\partial_t B^y &= -v^x \partial_x B^y - \partial_y \psi & (47)\\
\partial_t B^z &= -v^x \partial_x B^x - \partial_z \psi & (48)\\
\partial_t \psi &= -(\partial_x B^x + \partial_y B^y + \partial_z B^z) - \lambda \psi. & (49)
\end{aligned}$$

It is straightforward to compute the symbol $P(n_i)$ of the above equations and convince yourselves that the above system is strongly hyperbolic, hence it admits a well-posed initial value problem. Thus, starting with $\psi = 0$ and $\partial_i B^i = 0$, the above system will "drive" the constraint $C = \partial_i B^i$ to zero. For our purposes we can simplify the above even further by restricting to 2 spatial dimensions, so that we only have to solve

$$\partial_t B^x = v^x \partial_y B^y - A\partial_x \psi \tag{50}$$
$$\partial_t B^y = -v^x \partial_x B^y - A\partial_y \psi \tag{51}$$
$$\partial_t \psi = -A((\partial_x B^x + \partial_y B^y) - \lambda\psi). \tag{52}$$

Compute the principal symbol of the above PDEs and show that its eigenvalues are ± 1 and $-v^x n^x$, i.e., two light speed modes and one advection mode, and that the principal symbol has a complete set of eigenvectors. Implement the system (50) numerically in 2 spatial dimensions, for A=0 and A=1. Set $v^x = 1$, and experiment with different value of $\lambda \sim O(1)$. Check that when $A = 0$, that some norm of the divergence constraint increases in time, while it is controlled when $A \neq 0$.