TOPOLOGICAL STRINGS + DUALITIES (ICTP-SAIFR, SAO PAULO, MAY '16)

INTRO: As you have seen in Narain's lectures, topological string theory is a simplified ("bare-bones") string theory - in which the worldsheet dynamics is that of a topological field theory ("twisted 2d topological sigma model" coupled to 2d topological gravity). This enables the theory to be often exactly solvable ("in principle"). Yet it captures many of the features of the full fledged string theory:

(a) Topological strings on $CY_3 \Leftrightarrow$ II A/B string theory on $CY_3 \times \mathbb{R}^4$
- Computes exactly certain F-terms "$R^2 F^{2g-2}$".
(which are significant for BPS counting of BH microstates)

(b) Exhibits many of the non-perturbative features of string theory. ~~Ana~~ M-theory duality of physical string $\leftrightarrow$ new way of organising pert. string amplitudes.

(c) There is a version of open-closed string duality involving topological open strings (topological D-branes) which gives an exact large N duality of Chern-Simons QFT $\Leftrightarrow$ Top. closed strings on conifold.

Narain has focussed on (a). I will try to give a flavour of (b) + (c) in these lectures. Together will hopefully give you the foundation to read the literature of modern applications of topological string theory to (i) BH microstate counting (ii) refined top. strings + SUSY partition fns, integrable systems (iii) Mathematical implications (knot invariants + enumerative geom.)

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(1) REVIEW OF TOPOLOGICAL STRINGS - WORLDSHEET DEFN.

We start with a $\mathcal{N}=(2,2)$ SCFT on the worldsheet - e.g. sigma model for strings on CY_3. This has worldsheet currents $\{\underbrace{T(z)}_{\text{Stress tensor, Spin 2}}, \underbrace{G^{\pm}(z)}_{\text{Supercurr., Spin 3/2}}, \underbrace{J(z)}_{\text{R-curr., spin 1}}\}$ + antihol.

We define a modified ("twisted") stress tensor $T'(z) = T(z) \pm \frac{1}{2}\partial J(z)$.
$\hookrightarrow$ ($-/+ \Leftrightarrow$ chiral/antichiral twist)

This is equivalent to adding an addl. world sheet coupling to the usual sigma model. $= \pm\frac{i}{2}\int(\bar{\omega} J + \omega \bar{J})$

Spin conn. $\sim \partial \ln \det g \Rightarrow \delta\omega \propto \bar{\partial}\delta g$.

$\Rightarrow T'(z) = T(z) \pm \frac{1}{2}\partial J(z)$.

$\Rightarrow h' = h \mp \frac{1}{2} q$ (a.c. / c.) $\Rightarrow G^+$ has conf. wt. 1. $\Rightarrow Q^+ = \oint G^+ dz$ is a wt. zero ("scalar") object w/ $(Q^+)^2 = 0$.

$\Rightarrow G^-$ has conf. wt. 2.

The $N=2$ S.C.A. now implies $T' = \{Q^+, G^-\}$. If we define physical states in this twisted theory as those in the cohomology of Q^+ we get a simple closed truncation of the original SCFT. In fact, from the SCFT viewpoint, $Q^+ \leftrightarrow$ zero mode of $G^+(z)$ and states annihilated by this mode are nothing but <u>chiral primaries</u> of the SCFT ($h = \frac{1}{2}q$ or $h' = 0$) – Ramond sector ground states.

There is a useful, formal similarity to the bosonic string where $T(z) = \{Q_{BRST}, \underset{\text{antighost}}{b(z)}\}$ and there is a $J_{gh}(z)$.

For a general (2,2) SCFT we can have two inequivalent twists $\underbrace{(+,+) \sim (-,-)}_{\text{A-twist.}}$ or $\underbrace{(+,-) \sim (-,+)}_{\text{B-twist.}}$. Correspond to IIA/IIB theories.

For a (2,2) sigma model, like on a CY_3, the twisted action is. (A-model)

$$S' = \frac{1}{2\pi\alpha'} \int d^2z \left(\frac{1}{2} g_{IJ} \partial_z X^I \partial_{\bar z} X^J + i\psi_z^{\bar i} \tilde{D}_{\bar z} \chi^i g_{i\bar i} + i\psi_{\bar z}^{i} \tilde{D}_z \chi^{\bar i} g_{i\bar i} - R_{i\bar i j\bar j} \psi_{\bar z}^i \psi_z^{\bar i} \chi^j \chi^{\bar j} \right).$$

Here half the fermions $\psi_+^{\bar i} \to \psi_z^{\bar i}$ ($U(1)_R = -1$), Φ – 1-form on W.S. $\psi_+^i \to \chi^i$ ($U(1)_R = +1$) : (worldsheet scalar) (and opp. for $\psi_-^{\bar i}$, ψ_-^i). The $\tilde{D}_z$ now contain no w.s. spin conn. – the precise effect of adding the additional coupling.

The nett effect is that.

$$S' = \frac{i}{2\pi\alpha'} \int_\Sigma d^2z \{Q, V\} + \frac{1}{2\pi\alpha'} \int_\Sigma \Phi^*(K) \qquad \left(Q = Q^+ + \bar{Q}^+ = \oint G^+ dz + \oint \bar{G}^+ d\bar z \right)$$

$$V = g_{\bar i j} \left(\psi_z^{\bar i} \partial_{\bar z} X^j + \partial_z \chi^{\bar i} \psi_{\bar z}^j \right)$$

and $\int_\Sigma \Phi^*(K) = \int_\Sigma d^2z \left(\partial_z X^i \partial_{\bar z} X^{\bar j} g_{i\bar j} - \partial_{\bar z} X^i \partial_z X^{\bar j} g_{i\bar j} \right)$

– integral of pullback of Kahler form K on M. $= 2\pi \left(\sum n_j t_j \right) \alpha'$ (n_j: $\underbrace{\text{integer.}}_{\text{Purely topological qty.}}$)

$t_j - \int_{\Sigma_i \subset CY} (\omega + iB)$ are Kahler parameters.

Thus the action is the sum of a topological and a BRST (Q) exact piece. Therefore in a topological sector (w/ fixed $\{n_i\}$) we have the path integral (for BRST inv. operators $\{Q, \mathcal{O}_a\} = 0$)

$$\left\langle \prod_{a=1}^{n} \mathcal{O}_a \right\rangle_{\{n_i\}} = e^{-2\pi \vec{n}\cdot\vec{t}} \int [\mathcal{D}X]_n [\mathcal{D}\psi]_n [\mathcal{D}\chi]_n \, e^{-\frac{i}{2\pi\alpha'}\int d^2z \{Q, V\}} \prod_a \mathcal{O}_a .$$

We see that the second term is ind. of α' - in a Q-exact piece. Hence can evaluate in a saddle pt. approx. $\alpha' \to 0$. In this limit only $V = 0$ contribution i.e. $\partial_{\bar{z}} X^i = 0 = \partial_z X^{\bar{i}}$ i.e. holomorphic maps from W.S. to target CY. The PI thus gets contributions only from ~~between~~ these maps and we have an integral over ~~to~~ the moduli space of hol. maps $X^i : \Sigma \to M$.

The dependence of the correlator on the Kahler parameters $\{t_i\}$ is holomorphic (Though when coupled to 2d gravity, one will get a mild $\bar{t}_i$ dep.) [The correlators are also ind. of choice of cplx. structure of Σ as well as M.]

The observables which obey $\{Q, \mathcal{O}_\omega\} = 0$ are of the form.
$\mathcal{O}_\omega = \omega_{I_1 \ldots I_n}(x) \chi^{I_1} \ldots \chi^{I_n}$ $(I_1 \ldots I_n \in (i, \bar{i}))$ i.e. n-forms on M which obey $\{Q, \mathcal{O}_\omega\} = \mathcal{O}_{d\omega}$ i.e. when $d\omega = 0$ we have. ~~So~~ BRST inv. observables. Q-cohomology = de Rham cohomology.
- (p,q) forms. $\Leftrightarrow$ $U(1)_R, U(1)_L$ qtm. numbers.

[In the B-model, we can make similar arguments and write the action ind. of α'. But the saddle points now are $\partial_z X^I = \partial_{\bar{z}} X^I = 0$ i.e. point-like (constant) maps. This gets contributions only from supergravity ~~contribution~~ limit of string theory. The observables are $(0,p)$ forms U taking values in the q^{th} antisymm. power of Hol. tgt. bundle, which obey $\bar{\partial} U = 0$. $\Leftrightarrow$ $(d-q, p)$ forms in Dolbeault cohomology. (Using the Hol. d-form for a CY_d).

Finally, to couple these to 2-d "Topological gravity" corresponds simply to an integral (e.g. for the partition fn.) over moduli space.

$$F_g(\{t_j\}) = \int_{\mathcal{M}_g} \langle |G^- \mu_1 \ldots G^- \mu_{3g-3}|^2 \rangle \qquad \mu_i \sim \frac{\partial g}{\partial m_i} \text{ - Beltrami diff.}$$

- These are non-zero for CY_3 and appear as coeff. of "$R^2 F^{2g-2}$" terms
[The theory is critical when $\hat{c}=3$ when the χ zero modes $= \hat{c}(g-1)$ are absorbed by $3(g-1)$ g.m. in the G^- above. The part. fn. is non-zero for all genus g]

One can also consider open Topological string theory (again focus on A-model on CY_3). The ~~SUSY~~ bdy. condns. on the worldsheet w/ holes need to be specified. Dirichlet bdy. condns. compatible w/ SUSY (preserving 1/2 SUSY) are those where $\partial\Sigma \to M' \subset M$ w/ M' being a "Lagrangian submfld." (Kahler form ω vanishes on the submfld) of CY_3. In the case of interest where $M = T^*M'$ (phase space (q,p)), the Lagrangian submfld. is simply M' (q's).

The world volume theory of the D-branes wrapped on this real 3dim. submfld. $M' \subset M$ is that of a gauge field. But the action is

$$S = \frac{k}{4\pi} \int_{M'} d^3x \, \mathrm{Tr}\left[A dA + \frac{2}{3} A^3\right]$$

- Chern-Simons action
- reduction of bosonic open string field theory w/ $Q \to d$, $\Psi \to A$

and the $* \to \wedge$ over the $U(N)$ gauge fields A. This action is purely topological - no dependence on background metric. In fact, closed string sector decoupled from the open string one.

2 M-THEORY AND TOPOLOGICAL STRINGS

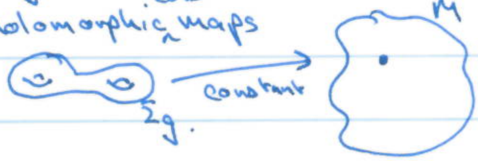
As you have seen in Narain's lectures the relation between the twisted NLσM and the untwisted NLσM (physical string theory on $CY_3 \times \mathbb{R}^4$) is quite simple. The $F_g(t,\bar{t})$ do not compute the vacuum amplitude in the physical string but rather the zero mom. limit of a scattering amplitude of two (self dual) gravitons + $(2g-2)$ graviphoton fields i.e. $R_+^2 F_+^{2g-2}$ (w/ some index contractions). These are also superpot. like terms $\quad S_{eff} = \ldots + \int d^4x \int d^4\theta \, W^{2g} \quad$ where $W_{\mu\nu} = F^+_{\mu\nu} + \theta\theta R^+_{\mu\nu} + \ldots$

For $g=0$, the corresponding $F_0(t)$ measures the prepotential that governs the space of couplings of the vector mult. These terms arise only at genus g and no other genus (since dilaton is in a h-plet)

The general form of $F_g(t)$ is:

$$F_g(t) = \underbrace{\frac{\chi(M)}{2} \int_{\mathcal{M}_g} c_{g-1}^3}_{\text{From holomorphic const maps}} + \sum_{\{n_j\} \neq 0} \alpha_{\{n_j\}} e^{-2\pi n_j t_j}$$

$\alpha_{\{n_j\}}$ → GW invariants



To obtain these contributions $\alpha_{\{n_j\}}$ is in general quite difficult. (given by an integral over the mod. sp. of hol. maps). We will see that the relation to physical string theory and the strong coupling limit of IIA theory being M-theory gives us a easier handle on $F_g(t)$. We will exploit the BPS nature of the corresponding superpot. terms. which means that we can calculate them at strong coupling of IIA on M

$= $ M-theory on $CY_3 \times S^1$. → large S^1 w/ $(R_{11} \sim g_s)$.

The lightest charged states under the graviphoton (a linear comb. of A_μ - KK field on S^1 and $A_{\mu(ij)}$ - wrapped on 2-cycles in M) are

(a) D0 branes $\sim 1/g_s$.

(b) D2 " $\sim A_c/g_s$ (wrapped on 2-cycles of M).

We can then ask what the contribution to the R_+^2 term in the 4-d eff. action is in the presence of a background $\langle F_+ \rangle = F$.

i.e. $S_{eff} = \int d^4x R_+^2 \left(\sum_g F_g(t_{i}) \times \underbrace{F^{2g-2}}_{\text{becomes a genus exp. parameter.}} \right)$

Ans: (AGNT? - using Heterotic – IIA duality)

The contribution to R_+^2 from a massive (half) hypermult. $= \underbrace{(1/2, 0) \oplus 2(0,0)}_{\text{in terms of } SU(2)_L \times SU(2)_R}$ – SUSY mult. of wrapped branes – preserving half of $N=2$ SUSY, is the same as of a single scalar particle, to the vacuum amplitude. The latter is given by a simple 1-loop calculation (Schwinger).

Schwinger's Answer (Scalar):

Basically in a background field we need to evaluate $\ln \det (D_\mu D^\mu)$. The result is

$$F(Z) = \int_\epsilon^\infty \frac{ds}{s^3} \times \left(\frac{s/2}{\sinh s/2} \right)^2 e^{-sZ/F}$$

Z → central charge (note Z and not $|Z|$) – see Witten)

F → selfdual field strength.

$$= \sum_g F_g(Z) F^{2g-2} + O(e^{-Z/F})$$

w/ $F_g = -\chi_g Z^{2-2g}$. $(g \geq 2)$ $\left(\chi_g = \frac{(-1)^{g-1} B_g}{2g(2g-2)} = \text{Euler Char. of } M_g. \right)$

$F_0 = \frac{1}{2} Z^2 \ln Z$.

$F_1 = -\frac{1}{12} \ln Z$.

We can now apply this general result to the light branes above – when they give rise to (half) hypermultiplets.

(a) D0-branes: $Z = \frac{2\pi i n}{g_s}$ (n → n 0-branes) $\Rightarrow F_g^{(D0)} = \chi_g \times \frac{\chi(M)}{2} \times \sum_{n \neq 0 \in \mathbb{Z}} \left(\frac{2\pi i n}{g_s} \right)^{2-2g}$.

$\chi(M)$ → $(h^{11} - h^{2,1})$ → Witten index for the QM.

This will turn out to be the contr. from constant maps. $\int_{M_g} c_{g-1}^3$.

$$= \frac{\chi(M)}{2} \times \chi_g \times (-1)^{g-1} \times \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} g_s^{2g-2}$$

(b) Single D2-brane: bd. to D0. $Z = \frac{2\pi(A + i n)}{g_s}$ (wrapped on an isolated genus zero surface)

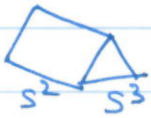
$$F_g^{(D2)} = -\frac{\chi_g}{(2\pi)^{2g-2}} \times g_s^{2g-2} \times \sum_{n \in \mathbb{Z}} (A + i n)^{2g-2}.$$

→ multiple covers.

A → Kahler parameter.

$$= -\frac{\chi_g}{(2g-3)!} \times g_s^{2g-2} \times \sum_{n=1}^\infty n^{2g-3} e^{-2\pi n A}$$

This simple case is relevant to the simplest CY_3 - the conifold (S^2)



. Putting both terms together (there is a single $D2$ brane bound to $D0$)

$$F_g = g_s^{2g-2}\left[(-1)^{g-1}\chi_g \cdot \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} - \frac{\chi_g}{(2g-3)!}\underbrace{\sum_{n=1}^{\infty}}_{\text{G.W. invariants.}} n^{2g-3} e^{-2\pi n A}\right]$$

$(g>1)$

(Here $\chi(M)=2$, and there is a single hypermult.)

Also $$F_0 = \frac{1}{g_s^2}\left[-\zeta(3) + \frac{\pi^2}{6}A + (\,)\ldots - \frac{A^3}{12} + \underbrace{\sum_{n=1}^{\infty}\frac{e^{-2\pi n A}}{n^3}}_{\text{multicover factor.}}\right]$$

(Candelas et. al.)

$$F_1 = \frac{A}{24} + \frac{1}{12}\ln(1-e^{-2\pi A}) \quad - \quad \text{BCOV.}$$

reproduces previously known results.

There is a nice way to resum the all-genus part. fn.

$$F(g_s,t) = \frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{(\sinh\frac{m g_s}{2})^2}\left[1+e^{-2\pi m t}\right].$$

$$= -\sum_{n=1}^{\infty} n \ln\left[(1-q^n)(1-zq^n)\right] \quad (q=e^{-g_s},\ z=e^{-2\pi t})$$

i.e. $$e^{-F(g_s,t)} = \prod_{n=1}^{\infty}\underbrace{(1-q^n)^n}_{\text{MacMahon fn counting 3d partitions}}(1-zq^n)^n.$$

Related to a BPS index in 5d.

More generally, when D2 branes (bound to D0) wrap on a cont. family of higher genus R.S. and there are multiple bound states, then the no. of BPS states is different from the above. Moreover, they can ~~have~~ be in higher spin massive hypermultiplets. The general structure is

$$\left[(\tfrac{1}{2},0)\oplus 2(0,0)\right]\otimes\sum_{j_1,j_2}\underbrace{N_{j_1 j_2}}_{\in\mathbb{Z}}(j_1,j_2).$$

$N_{j_1 j_2}$ ⟶ need to be figured out for each CY - Spectrum of BPS particles ~~in the~~ for the 5d theory. ($\mathbb{R}^4\times S^1$)

The piece $\left[(\tfrac{1}{2},0)\oplus 2(0,0)\right]\otimes(j_1,j_2)$ gives a contribution to the R_+^2 term in the eff. action in a straightforward way. Firstly, the F_+ does not couple to j_2 and gives only a free particle det. $\sim(-1)^{2j_2}(2j_2+1)$ (degeneracy). So we can think of this as

$\left[(\tfrac{1}{2},0)\oplus 2(0,0)\right]\otimes(j_1,0)$. The contribution of this to the

R_+^2 term is that of a particle of spin (j_1) to the the vacuum amplitude in a general Schwinger calculation.

We can actually write $N_{j_1}\left[(\tfrac{1}{2},0)\oplus 2(0,0)\right]\otimes(j_1,0) = \sum n_r\,[I_r]$

$\hookrightarrow = \sum_{j_2}(-1)^{2j_2}(2j_2+1)N_{j_1 j_2}$ $\qquad [I_r] \to \left[\tfrac{1}{2}\oplus 2(0)\right]^{\otimes r}$.

The contribution of I_{r+1} in the Schwinger computation is

$$F^{(r)}(Z) = \int_\epsilon^\infty \frac{ds}{s}\left(2\sinh\frac{s}{2}\right)^{2r-2} e^{-sZ/F}$$

Thus for D2 branes (bound to D0) w/ $A = \sum n_j t_j$

we get.

$$F^{(r)}(t,\lambda) = \sum_{k>0}\frac{1}{k}\left(2\sinh\frac{k\lambda}{2}\right)^{2r-2} \wedge e^{-2\pi k A} \qquad (\lambda = g_s F)$$

The total spectrum is this

$$F(\lambda,\{t\}) = \sum_{k,\{n_j\},\, r\geq 0} n_r^{\{n_j\}}\,\frac{1}{k}\left(2\sinh\frac{k\lambda}{2}\right)^{2r-2} \wedge e^{-2\pi k(\sum n_j t_j)}$$

This is a non-trivial reorganisation of the all genus partition fn. in terms of a new set of integer invariants $[n_r^{\{n_j\}}]$ which determine the $\alpha_{\{n_j\}}$. The n_r being BPS indices are easier to calculate. Note that I_{r+1} contributes to F_g only for $g\geq r$.

Open-Closed String Duality + Topological Strings

Gauge-String duality (or AdS/CFT correspondence) is believed to be a reflection of an open-closed string duality, where

$\sum_{\text{holes}}$  $\Big|_A \longleftrightarrow$  $\Big|_B$ (fixed # of handles)

The string backgd. A w/ D-branes can be equivalently viewed as the pure closed string background B, after taking into account, the back reaction of the D-branes.

One usually looks at a certain limit of this duality where the LHS decouples from the closed strings and reduces to a field theory of N D-branes - a gauge theory. Then we have.

$\sum_{\text{holes}}$  $\Big|$ Gauge Theory A. $\longleftrightarrow$  $\Big|$ Closed String B. (fixed # of handles in B = fixed order of $1/N$ expansion in A)

Note both LHS + RHS are autonomously defined.

We don't have too many examples where we understand this duality explicitly (especially for finite N + λ = 't Hooft coupling). This is why the topological string examples of this are valuable.

The statement:

OPEN		CLOSED
N Top. D-branes on A-model on T^*S^3 (wrapping S^3) - deformed conifold	$\longleftrightarrow$	A-model top. strings on S^2 resolved conifold.
$\equiv$ $U(N)$ Chern-Simons theory on S^3 		
$\lambda = \frac{2\pi N}{k+N}$	$\longleftrightarrow$	$t = i\lambda$.
λ/N	$\longleftrightarrow$	g_s.

As we have seen, the $U(N)$ Chern-Simons theory is exactly what the open topological string gives you (no need to take a limit). Thus LHS is a well defined 3d QFT. - in fact exactly solvable (Witten). The RHS is similarly well defined and as we have seen,

quite computable explicitly at all genus. The dictionary entries are understandable: (a) the 't Hooft coupling measures the size of the closed string backgd. ($\lambda' \sim R_{AdS}$) and (b) $g_s \sim 1/N$. ($g_s \sim g_{YM}^2 = \lambda/N$)
The flop in the geometry is also reminiscent of what happens in AdS/CFT.
<u>CHECKS:</u> The partition fn. as well as Wilson loops (for different topologies and different repns.) are computable in the Chern-Simons theory.

$$Z(k,N) = \int [\mathcal{D}A]\, e^{i\frac{k}{2\pi} S_{cs}[A]} \qquad S_{cs}[A] = \int_{S^3} \mathrm{Tr}\left[A dA + \tfrac{2}{3}A^3\right]$$

$$= e^{\frac{i\pi}{8}N(N-1)} \times \frac{1}{(N+k)^{N/2}} \sqrt{\frac{N+k}{N}} \prod_{j=1}^{N-1} \left\{ 2\sin\frac{j\pi}{N+k} \right\}^{N-j}$$

The free energy $F(k,N)$ or $F(N,\lambda)$ has a large N expansion.

$$F(N,\lambda) = \sum_{g=0}^{\infty} \left(N/\lambda\right)^{2-2g} F_g(\lambda)$$

and what do we see!

$$F_g(t)_{=i\lambda} = (-1)^{g-1} \frac{\chi_g\, 2\zeta(2g-2)}{(2\pi)^{2g-2}} - \frac{\chi_g}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-nt}. \qquad (g>1)$$

$$F_0(t) = -\zeta(3) + \frac{\pi^2}{6}t + i(m+1/4)\pi t^2 - \frac{t^3}{12} + \sum_{n=1}^{\infty} \frac{e^{-nt}}{n^3}.$$

$$F_1(t) = \frac{1}{24}t + \frac{1}{12}\ln(1-e^{-t}).$$

This is exactly what we got from the Schwinger computation of D2 + D0 branes for the conifold which captures the $F_g(t)$ of this topological closed string background.
Similarly, many other checks for. Wilson loops, Extended to other 3d spaces like Lens spaces etc.
Can try to prove this duality using an underlying LGM description
– generalised also to superstring where open topological string computes superpot. terms of IIA on $CY_3 \times \mathbb{R}^4$ w/ N D6 branes wrapping 3-cycle (e.g. S^3) and filling $\mathbb{R}^4$. $\Rightarrow \mathcal{N}=1$ SYM + S vars
$F_{g,n}(S,t) \int d^4x\, d^4\theta\, [Nh\, S^{h-1}]\, W^{2g}$.
For $g=0$, terms like $\int d^4x\, \mathrm{Tr} F^2 (\bar{\lambda}_\alpha \lambda_\alpha)^{h-2}$. coeff of x^{2g+h} ($S = \mathrm{Tr}\, W_\alpha^2$ $(W_\alpha = \lambda_\alpha + \ldots)$)