Aspects of superstring perturbation theory.

- old subject.
- new developments: dealing with (IR) divergences.

L.1. Motavation + general idea
L.2: Details

Simplest example: Virasoro-Shapiro amplitude:

4-tachyon amplitude with incoming momenta p_1, p_2, p_3, p_4

$$\propto \int d^2z \prod_{i=2}^{4} |z-z_i|^{\alpha' p_1 \cdot p_i} \prod_{i<j=2}^{4} |z_i - z_j|^{2+\alpha' p_i \cdot p_j}$$

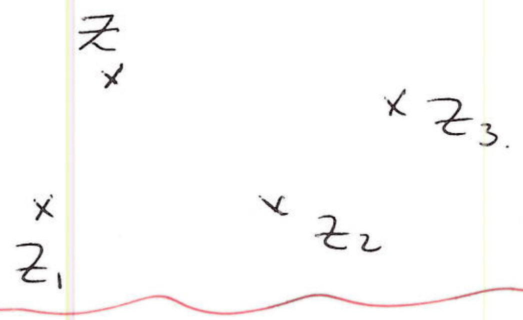
Set $\alpha'=1$ from now on.

z integral diverges for

$$p_1 \cdot p_i \leq -2.$$

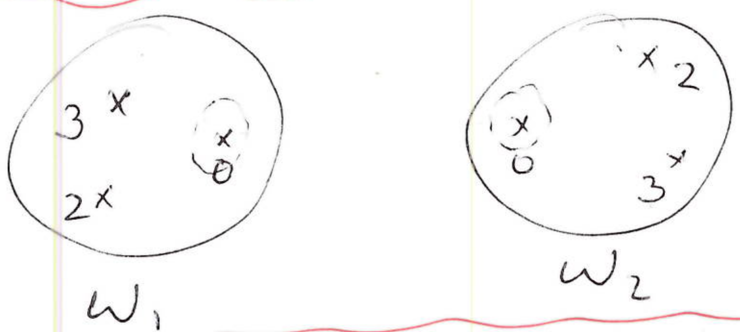
Usual route: Define by analytic continuation.

However we shall try to understand the divergence better.



$z \to z_1, z_2$ or z_3.

Represent a sphere with 4-punctures as 'plumbing fixture' of 2 spheres each with 3-punctures. (at 0, 2, 3 say).



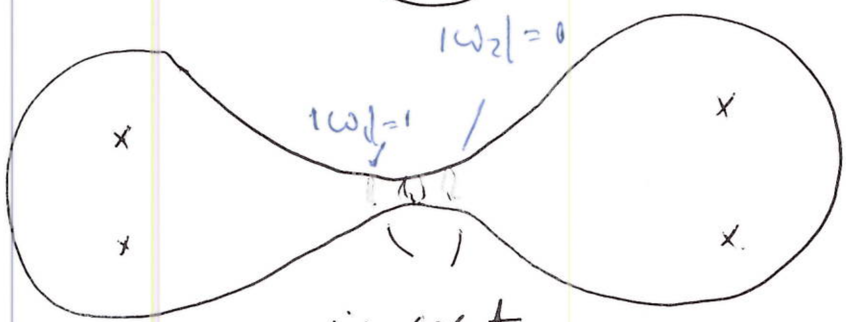
Identify w_1 around 0 and w_2 around 0 via

$$w_1 w_2 = q \;\left(= e^{-(s+i\theta)}\right)$$

$\Rightarrow$ a sphere with four punctures.

w_1 coordinate: $w_1 = 2, 3, \frac{q}{2}, \frac{q}{3}$

$q \to 0$ $(s \to \infty)$ $\Rightarrow$ two punctures come together.



insert complete set of states.

$|\phi_m\rangle\langle\phi_m| q^{L_0} \bar{q}^{\bar{L}_0} |\phi_n\rangle\langle\phi_n|$

$\int d^2 z \to \int ds\, d\theta$ (a) insert $b_0 \bar{b}_0$ between $\langle\phi_m|$ $|\phi_n\rangle$ (b) take each v. ob. $c\bar{c}e^{ip_i\cdot X}$

$$\int_0^\infty ds \int_0^{2\pi} d\theta\, e^{-s(L_0+\bar{L}_0)} e^{-i\theta(L_0-\bar{L}_0)}$$

covers part of integration region.

$$\sim \frac{1}{L_0+\bar{L}_0} \delta_{L_0,\bar{L}_0}$$

Origin of divergence: If the intermediate state has $L_0+\bar{L}_0 \leq 0$, integral diverges.

$|\phi\rangle = c_1 \bar{c}_1 |p_i+p_j\rangle$.

$\Leftrightarrow$ precisely correspond to $p_i\cdot p_j \leq -2$.

Analytic continuation

$\equiv$ replacing the integral by $(L_0+\bar{L}_0)^{-1}\delta_{L_0,\bar{L}_0}$ even for $L_0+\bar{L}_0 \leq 0$.

Note: 1 For a state of momentum k,

$$L_0 + \bar{L}_0 = \frac{k^2}{2} + \frac{m^2}{2}$$

m → mass of the state.

$L_0 + \bar{L}_0 = 0 \Rightarrow k^2 + m^2 = 0 \Rightarrow$ on-shell condition.

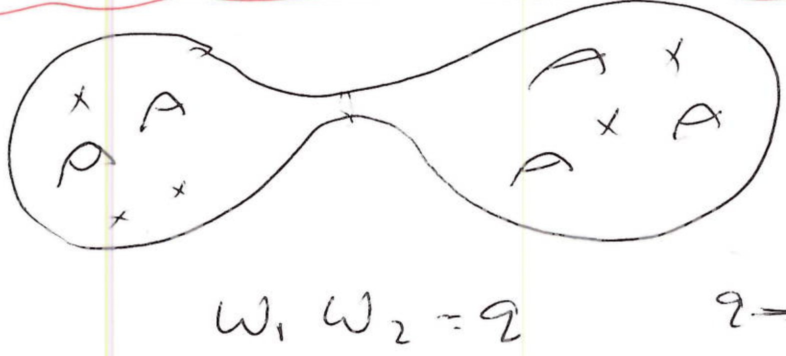
2 $\frac{1}{L_0 + \bar{L}_0} \sim \frac{1}{k^2 + m^2} \Rightarrow$ propagator!

A general string amplitude

$$\int_{M_{g,n}} \langle \quad \rangle$$

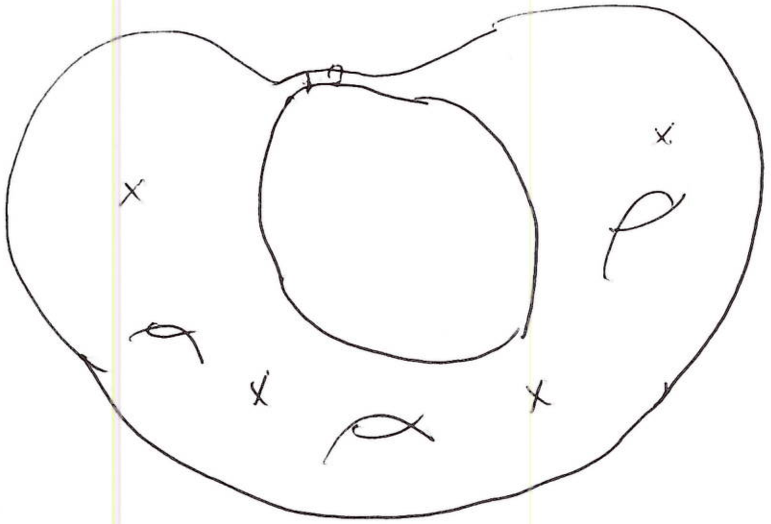
$M_{g,n}$ → Moduli space of genus g Riemann surface with n-punctures.

All divergences ⟷ degenerate ~~region~~ of Riemann surface.



$w_1 w_2 = q \qquad q \to 0$

→ separating type.



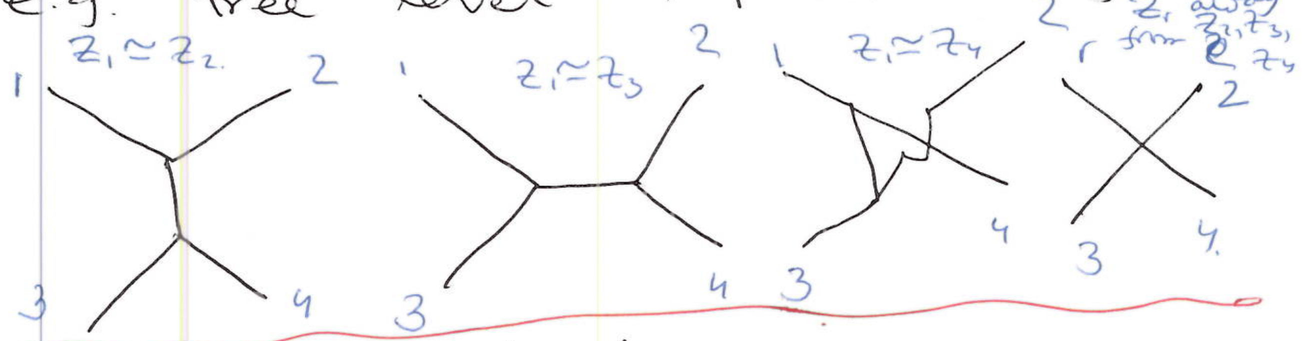
→ non-separating type.

String field theory:

A field theory whose Feynman diagrams reproduce string scattering amplitudes.

In any given order, string field theory amplitude = sum of Feynman diagrams.

e.g. tree level 4-point amplitude



Represent propagators by

$$\propto \delta_{L_0,\bar{L}_0}\frac{1}{L_0+\bar{L}_0} \propto \int_0^\infty e^{-s(L_0+\bar{L}_0)} \int_0^{2\pi} e^{i\theta(L_0-\bar{L}_0)}$$

Each Feynman diagram

⇒ integration over part of

$M_{g,n}$ ("Triangulation" of $M_{g,n}$)

$\{s_i, \theta_i\}$ ~~integration~~ ⇒ coordinates of $M_{g,n}$.

Sum of Feynman diagrams

⇒ integration over $M_{g,n}$.

Advantage: If we represent propagators ~~as~~ as $\frac{1}{L_0+\bar{L}_0}\delta_{L_0,\bar{L}_0}$,

the analytic continuation

is automatic.

(no further analytic continuation is needed).

~~[illegible]~~ However there are more ~~[illegible]~~ important reasons why SFT is essential.

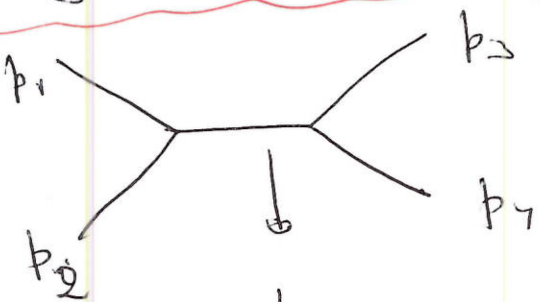
Virasoro-Shapiro

Return to ~~Veneziano~~ amplitude:

$$\int d^2z \prod_{i=2}^{4} |z-z_i|^{p_1\cdot p_i} \prod_{i<j=2}^{4} |z_i-z_j|^{2+p_i\cdot p_j}$$

If $p_1\cdot p_i = -2-2n$ for any i, the amplitude genuinely has poles.

→ cannot be removed by analytic continuation. $L_0+\bar{L}_0=0$ for some state



$$\frac{1}{(p_1+p_2)^2+M^2}$$

$= 0 \Rightarrow$ poles.

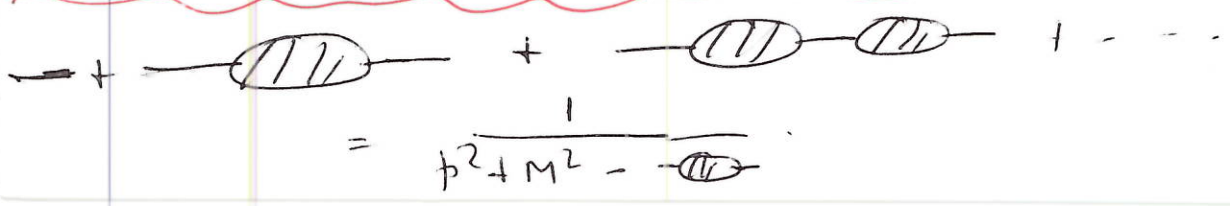
Amplitude $= \infty$.

What do we do?

In QFT, M^2 is renormalized to

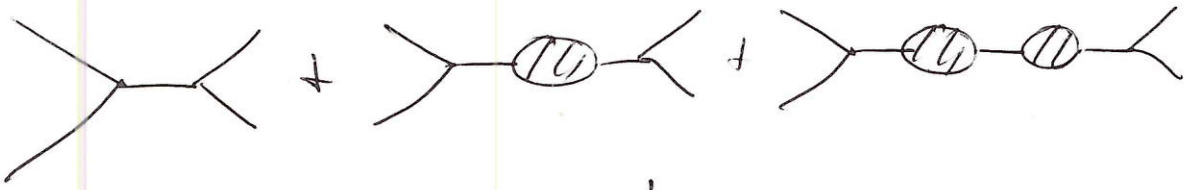
$$M^2 - \Sigma_R - i\,\Sigma_I$$

self energy correction



The imaginary part of ~~...~~ Σ prevents poles for real external momenta.

- Requires resumming a class of diagrams.



$+ \cdots$

Usual world-sheet description does not give a way to sum over subset of diagrams, and/or organising the sum.

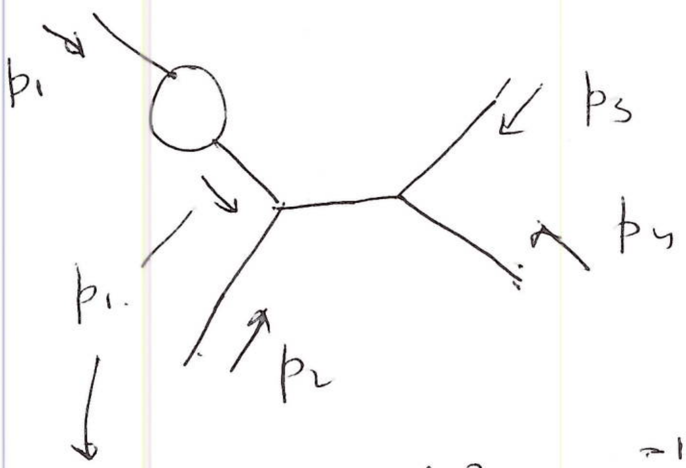
String field theory does.

In general, whenever a field theory amplitude diverges due to vanishing internal propagator, string theory amplitudes will have divergence

→ not removable by analytic continuation.

Example 2:

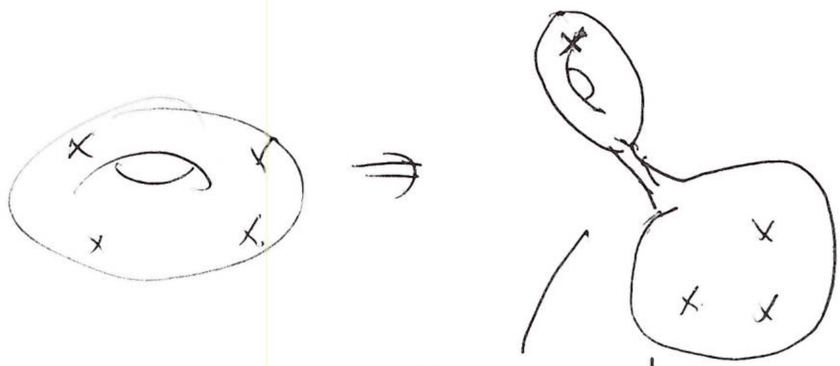
Consider a field theory diagram:



Propagator $(p_1^2+m^2)^{-1} = \infty$ since $p_1^2+m^2=0$

tree level

String theory:



Divergent.

In field theory we shall resum self energy insertion on external legs & replace $p_1^2+m^2=0$ by $p_1^2+m_{phys}^2=0$. ← LSZ.

→ Not possible in string world-sheet description.

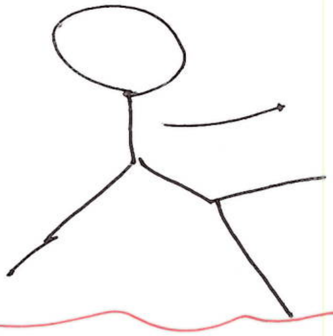
$p_1^2 + m^2 = 0$ is part of requirement of BRST invariant vertex op.

$\rightsquigarrow$ not changeable.

Possible in SFT.

Example. 3

Massless field:



$\frac{1}{k^2}\Big|_{k=0} = \infty$

$\Rightarrow$ tadpole divergence.

In field theory this means 1PI action has a term $c\,\phi$

constant field.

$\phi = 0$ is no longer a vacuum.

$\frac{\delta S}{\delta \phi}$ has term $\propto c$ at $\phi = 0$

Remedy: Look for new minimum.

Not possible in string world-sheet theory

$\Phi = 0 \Rightarrow$ Conformally invariant world-sheet theory.

$\rightarrow$ essential for the formulation of the theory.

Possible in SFT.

This suggests that a systematic way to formulating perturbative string theory is via string field theory.

Disadvantage: In SFT we have to sum over many Feynman diagrams instead of having a single ~~Feynman~~ expression in a given loop order.

Strategy: Use usual string perturbative theory, but fall back on SFT whenever we are in trouble.

Strategy for constructing SFT
~ work backward

1 Generalize on-shell amplitudes to off-shell amplitude.

2 "Triangulate" ~~Divide~~ $M_{g,n}$ ~~into~~ ~~subsp~~ so that each component can be regarded as 1 Plumbing fixture of Riemann surfaces of lower genus / lower no. of punctures. 2 Or an elementary vertex not having any degeneration.

3 Read out the propagators and vertices from the above description.

4 Write down a "gauge fixed" action whose Feynman rules give same vertices and propagators.

5 Write down a gauge invariant action whose gauge fixing leads to the action in step 4 & correct physical state condition.

Strategy for construction of SFT

(1) Generalize on-shell amplitude → ~~off~~ shell amplitude.

(2) Find a field theory that reproduces these off-shell amplitudes.

(3) Find a gauge invariant field theory whose gauge fixing gives this field theory.

Bosonic string theory (Zwiebach 92)

World-sheet:

Matter CFT: $c_L = 26$, $c_R = 26$

Ghost CFT: $b, c, \bar{b}, \bar{c}$

$c_L = -26 \quad c_R = -26$

$b(z) = \sum b_n z^{-n-2}, \quad c(z) = \sum_n c_n z^{-n+1}$

$\bar{b}(\bar{z}) = \sum \bar{b}_n \bar{z}^{-n-2}, \quad \bar{c}(\bar{z}) = \sum \bar{c}_n \bar{z}^{-n+1}$

$|0\rangle$: $SL(2,\mathbb{C})$ invariant vacuum.

$b_n|0\rangle = 0 = \bar{b}_n|0\rangle$ for $n \geq -1$

$c_n|0\rangle = 0 = \bar{c}_n|0\rangle$ for $n \geq 2$

$b_0^{\pm} = b_0 \pm \bar{b}_0, \quad c_0^{\pm} = \frac{1}{2}(c_0 \pm \bar{c}_0), \quad L_0^{\pm} = L_0 \pm \bar{L}_0$

$L_n, \bar{L}_n$: Total Virasoro generators (matter + ghost)

$\mathcal{H}$: Hilbert space of states satisfying:

$$b_0^- |A\rangle = 0, \qquad L_0^- |A\rangle = 0.$$

~~On-shell state~~: Ghost no:

$|0\rangle$: 0, matter : 0.

$b, \bar{b}$: -1, $c, \bar{c}$: 1

Physical states (on-shell):
states in $\mathcal{H}$ of ghost no. 2,
satisfying

$$Q_B |A\rangle = 0, \qquad A \equiv |A\rangle + Q_B |\Lambda\rangle$$

Q_B : BRST charge

$\sum c_n L_{-n} + \sum \bar{c}_n \bar{L}_{-n}$ + ghost terms.

Off-shell string state: Arbitrary state in $\mathcal{H}$.

Given $|A_1\rangle, \ldots |A_n\rangle \in \mathcal{H}$, we want to define off-shell amplitudes of $A_1, \ldots A_n$.

*

Note on-shell condition:

$Q_B |A\rangle = 0$. $\quad |A\rangle = Q_B|\Lambda\rangle + |A)$

$|A)$ can be taken to be dimension 0-primary.

→ Correlation fn. on $\Sigma_{g,n}$ (Riemann surface of genus g and n-punctures) independent of choice of world-sheet metric.

Off-shell amplitude of $|A_1\rangle, \ldots |A_n\rangle \in \mathcal{H}$ depends on world-sheet metric

→ Need a way to parametrize world-sheet metric.

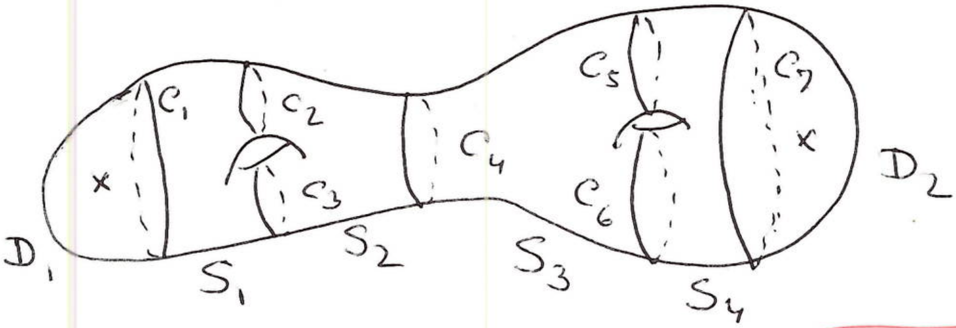
Take $\Sigma_{g,n}$ and divide it into:

① n disks $D_1, \ldots D_n$

② $2g-2+n$ spheres $S_1, \ldots S_{2g-2+n}$ each with 3-holes

glued across $3g-3+2n$ circles.

* $g=2,\ n=2$



$n=2, \quad 2g-2+n=4$

$3g-3+2n=7$

w_a: local coordinates on D_a s.t.
$\hat{a}$'th puncture is at $w_a=0$.

z_i: coordinates on S_i

On common boundaries:

$z_i = f_{ia}(w_a), \quad z_i = F_{ij}(z_j)$

Capture full information on $\Sigma_{g,n}$.

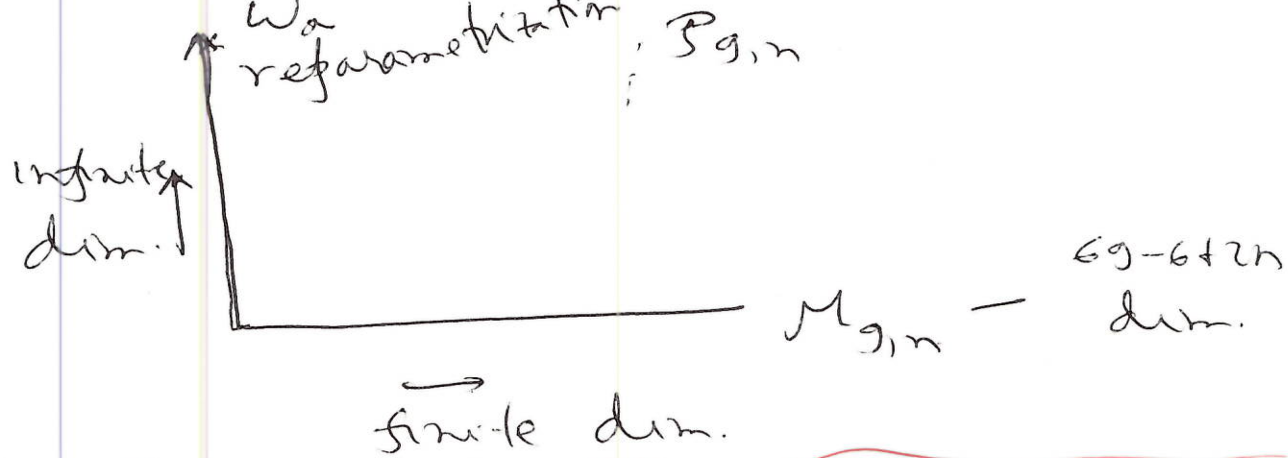
(Moduli space of $\Sigma_{g,n}$)
* $\hat{M}_{g,n}$: space of $\{f_{ia}\}, \{F_{ij}\}$ keeping $w_a=0$ fixed.
modulo reparametrization of $z_i, \hat{w_a}$

choose metric on D_a's: $|dw_a|^2$

$\Rightarrow$ Off-shell amplitudes are not invariant under w_a reparametrization except $w_a \to e^{i\alpha} w_a$

$P_{g,n}$: space of $\{f_{ai}\}$, $\{F_{ij}\}$ and $w_a \to e^{i\alpha_a} w_a$ modulo z_i reparametrization.

$\Rightarrow$ fiber bundle over $M_{g,n}$.



<u>Tangent space of</u> $P_{g,n}$.

infinitesimal deformation

e.g. $f_{ia} \to f_{ia} - \delta f_{ia}$.

For fixed w_a, this changes z_i

$$z_i^{new} = f_{ia}(w_a) - \delta f_{ia}(w_a)$$
$$= z_i^{old} - \delta f_{ia}\left(f_{ia}^{-1}(z_i^{old})\right)$$

$v(z_i) \equiv \delta f_{ia}\left(f_{ia}^{-1}(z_i)\right)$ is an infinitesimal holomorphic vector field on Σ defined around the overlap circle of D_a and S_i.

Given $|A_1\rangle, \cdots |A_n\rangle \in \mathcal{H}$, introduce a p-form $\omega_p(A_1, \cdots A_n)$ on $\mathcal{P}_{g,n}$ $\forall\, p$ as follows:

$$\omega_0^{(g,n)}(A_1, \cdots A_n) = (2\pi i)^{-(3g-3+n)} \langle \prod_i \otimes A_i \rangle_{\Sigma_{g,n}}$$

inserted using the local coordinates of $\Sigma_{g,n}$ w_i, i.e. using the metric $|dw_i|^2$ near i-th puncture.

a point in $\mathcal{P}_{g,n}$.

correlation fn. on $\Sigma_{g,n} \in \mathcal{P}_{g,n}$.

To specify $\omega_p(A_1, \cdots A_n)$ we need to give the result of its contraction with p arbitrary tangent vectors $V^{(1)}, \cdots V^{(p)}$ of $\mathcal{P}_{g,n}$.

$V^{(i)} \leftrightarrow$ vector field $v^{(i)}$ on $\Sigma_{g,n}$ around some overlap circle $C^{(i)}$.

$$b[V^{(i)}] \equiv \oint_{C^{(i)}} b(u)\, v^{(i)}(u)\, du + \oint_{C^{(i)}} \overline{b(u)}\, \overline{v^{(i)}(u)}\, d\bar{u}$$

$\oint$ includes $\frac{1}{2\pi i}$ factors.

$$\omega_p^{(g,n)}(A_1, \cdots A_n)\,[V^{(1)}, \cdots V^{(b)}]$$

contractions

$$= (2\pi i)^{-(3g-3+n)} \left\langle b[V^{(1)}] \cdots b[V^{(b)}] \prod_{i=1}^{n} A_i \right\rangle_{\Sigma_{g,n}}$$

$\omega_p^{(g,n)}$ satisfies:

$$\omega_p^{(g,n)}(Q_B A_1, A_2, \cdots A_n) + (-1)^{n_{A_1}} \omega_p(A_1, Q_B A_2, A_3, \cdots A_n) + \cdots + (-1)^{n_{A_1} + \cdots n_{A_{n-1}}} \omega_p^{(g,n)}(A_1, \cdots A_{n-1}, Q_B A_n)$$

$$= (-1)^p \, d\omega_{p-1}^{(g,n)}(A_1, \cdots A_n)$$

Off-shell amplitude:

① Choose a section $S_{g,n}$ of $\mathcal{P}_{g,n}$.

② Off-shell amplitude.

$$= \int_{S_{g,n}} \omega_{6g-6+2n}^{(g,n)}(A_1, \cdots A_n)$$

If $Q_B|A_i\rangle = 0 \;\forall i$, the result is independent of $S_{g,n}$ (except possibly boundary terms)

In order that this has field theory interpretation, we need to put restriction on $S_{g,n}$.

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1. $S_{g,n}$ is symmetric under permutations of $1, \cdots n$.
(When punctures are exchaged, local coordinates also get exchaged).

If needed, we can take weighted measure of sections. e.g.

$$S_{g,n} = \frac{1}{N} \sum_{i=1}^{N} S_{g,n}^{(i)}$$

$$\int_{S_{g,n}} \omega^{(g,n)}_{6g-6+2n} = \frac{1}{N} \sum_{i=1}^{n} \int_{S_{g,n}^{(i)}} \omega^{(g,n)}_{6g-6+2n}$$

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2. Plumbing fixture:

Take $\Sigma_{g_1,n_1} \in S_{g_1,n_1}$, $\Sigma_{g_2,n_2} \in S_{g_2,n_2}$

Take one puncture from each & glue using

$$w_1 w_2 = q = e^{-s+i\theta}$$

$$0 \leq s < \infty, \quad 0 \leq \theta < 2\pi.$$

$\to$ An element of $P_{g_1+g_2,\, n_1+n_2-2}$ (equipped with local coords).

Requirement: $S_{g_1+g_2,\, n_1+n_2-2}$ must contain these elements of $\mathcal{P}_{g_1+g_2,\, n_1+n_2-2}$

Plumbing fixture of S_{g_1,n_1} & S_{g_2,n_2}

$6g_1-6+2n_1$ parameters.

$6g_2-6+2n_2$ parameters.

$+\,2$ (from s, θ)

$\Rightarrow\ 6(g_1+g_2)-6+2(n_1+n_2-2)$ parameters

$\parallel$

dim. of $S_{g_1+g_2,\, n_1+n_2-2}$.

$\Rightarrow$ gives a codimension 0 subspace of $S_{g_1+g_2,\, n_1+n_2-2}$

no ~~fixed~~ determined.

For the rest we have a choice..

Must hold for all (g,n).

Systematic algorithm:

1. start with $S_{0,3} \equiv R_{0,3}$
→ a 3-punctured sphere with specific choice of local coordinates around punctures.

$SL(2,\mathbb{C})$ exchanging the puncture locations must exchange the local coordinates.
(symmetry restriction)

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2. Plumbing fixture of $S_{0,3}$ with $S_{0,3}$ generates a family of 4-puncture spheres equipped with local coordinates at punctures.

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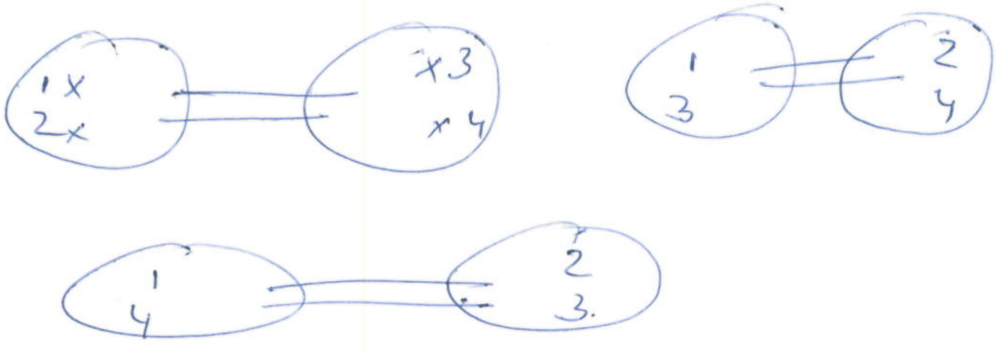
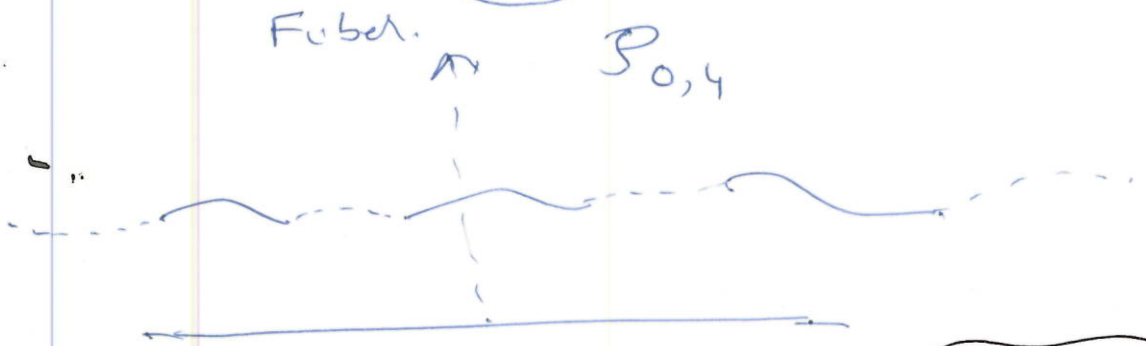
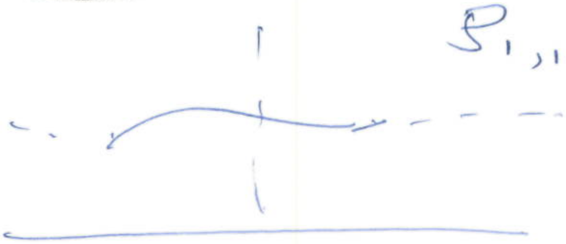
s, t and u-channel.





"Fill the gap" by $\mathcal{R}_{0,4} \subset \mathcal{P}_{0,4}$.

Similarly glue 2 points on $\Sigma_{0,3} \in \mathcal{S}_{0,3}$ to get a family of $\Sigma_{1,1}$.



Fill the gap with $\mathcal{R}_{1,1}$.

Now take $\mathcal{R}_{0,3} \equiv \mathcal{S}_{0,3}$, $\mathcal{R}_{0,4}$, $\mathcal{R}_{1,1}$ and glue in all possible ways to construct subspaces of $\mathcal{P}_{g,n}$ for higher g, n.

Fill the gap by $\mathcal{R}_{g,n}$.

$$\{A_1, \ldots A_n\} = \sum_g (g_s)^{2g-2} \int_{R_{g,n}} \omega^{(g,n)}_{6g-6+2n}(A_1, \ldots A_n)$$

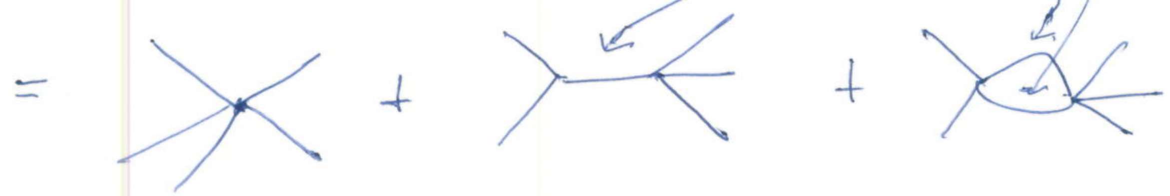
~~of order g_s^{2g}~~

$\to$ elementary vertex with external states $A_1, \ldots A_n$.

Full off-shell amplitude:

$$\sum_g (g_s)^{2g} \int_{S_{g,n}} \omega^{(g,n)}_{6g-6+2n}(A_1, \ldots A_n)$$

Plumbing fixture



$+ \cdots\cdots$

Propagator: Earlier we argued

$$\int ds\, d\theta\, e^{-s(L_0+\bar{L}_0)}\, e^{i\theta(L_0-\bar{L}_0)}$$

$$\sim \frac{1}{L_0+\bar{L}_0}\, \delta_{L_0,\bar{L}_0}$$

Now we can make it more concrete: from $b[V^{(\alpha)}]$'s associated with $\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}$

$$ds \wedge d\theta\, (b_0+\bar{b}_0)(b_0-\bar{b}_0)\, e^{-s(L_0+\bar{L}_0)}$$

~~$b_0 = \oint dz$~~ $$e^{i\theta(L_0-\bar{L}_0)} \propto b_0 \bar{b}_0 \frac{1}{L_0+\bar{L}_0} \delta_{L_0,\bar{L}_0}$$

$b_0 = \oint dw_i\, w_i\, b(w_i)$, $\bar{b}_0 = \oint d\bar{w}_i\, \bar{w}_i\, \bar{b}(\bar{w}_i)$

$i = 1, 2$ for $w_1 w_2 = q$

String field theory

String field: An arbitrary state $|\Psi\rangle \in \mathcal{H}$.

If $|\varphi_r\rangle$ is a basis of $\mathcal{H}$, then

$$|\Psi\rangle = \sum_r \psi_r |\varphi_r\rangle$$

ψ_r: dynamical variables.

$$\simeq \int d^D k \sum_n \psi_n(k)\, |\varphi_n(k)\rangle$$

$\sum_n$: discrete sum; k: momentum label.

$\psi_n(k)$ are Fourier transform of string fields.

Action:

$$\frac{1}{g_s^2}\left[\frac{1}{2}\langle\Psi| c_0^- c_0^+ L_0^+ |\Psi\rangle + \sum_n \frac{1}{n!}\{\Psi^n\}\right]$$

$b_0^+|\Psi\rangle = 0$

$$\{\Psi^n\} \equiv \{\underbrace{\Psi\,\Psi \cdots \Psi}_{n \text{ times}}\}$$

This gives the gauge fixed action

Gauge invariant action: (BV formalism)

$$\frac{1}{g_s^2}\left[\frac{1}{2}\langle\psi|c_0^- Q_B|\psi\rangle + \sum_n \frac{1}{n!}\{\psi^n\}\right]$$

eom $Q_B|\psi\rangle + \cdots = 0$

~~$b_0^+|\psi\rangle = 0$~~ Gauge invariance:

$$\delta|\psi\rangle = Q_B|\Lambda\rangle + \text{non-linear terms}$$

$b_0^+|\psi\rangle = 0$ gauge $\Rightarrow$ gauge fixed action.

Type II / heterotic string theory

Focus on heterotic for simplicity (Type II is more of the same)

World-sheet SCFT:

Matter: $\underbrace{c_L = 26,\ c_R = 15}$ — NS, R; GSO odd, GSO even.

Ghost: $\underbrace{b, c}_{(c_L, c_R) = (0, -26)}$, $\underbrace{\bar{b}, \bar{c}}_{(c_L, c_R) = (-26, 0)}$

Commuting: β, γ: $(c_L, c_R) = (0, 11)$

Bosonization:

$\beta = \partial\xi\, e^{-\phi} \qquad \gamma = \eta\, e^{\phi}$

(ξ, η) : fermionic $\qquad \phi$: bosonic

Ghost no: $e^{q\phi}: 0$, $\xi: -1$, $\eta: 1$

$\beta: -1$, $\gamma: 1$

Picture no: $e^{q\phi}: q$, $\xi: 1$, $\eta: -1$

$\beta, \gamma: 0$.

On genus g surface, need

Total picture no. $= 2g-2$

Total ghost no. $= -(6g-6)$

for non-vanishing correlator.

NS sector vertex op $\sim e^{q\phi}$ $\qquad q \in \mathbb{Z}$.

GSO even for q even $\qquad$ GSO odd for q odd

R-sector vertex ope $\sim e^{(2n-1)\frac{\phi}{2}}$, $n \in \mathbb{Z}$.

GSO even for n even $\qquad$ GSO odd for n odd.

ξ, η : GSO even.

Mode expansion of ξ, η:

$$\xi(z) = \sum \xi_n z^{-n} \qquad \eta(z) = \sum_n \eta_n z^{-n-1}$$

Small Hilbert space

$\Rightarrow$ states annihilated by η_0

Vertex of ~~mode~~ has no ξ without derivative.

$\mathcal{H}$: Hilbert space of GSO even (matter + ghost) states in small Hilbert space satisfying

$$L_0^- |s\rangle = 0, \quad b_0^- |s\rangle = 0$$

$\mathcal{H}_n$: subspace of $\mathcal{H}$ carrying picture no. n.

NS sector off-shell state $\in \mathcal{H}_{-1}$

R sector off-shell states $\in \mathcal{H}_{-1/2}$

For genus g amplitude of m NS and n R-sector states:

Needed picture no: $2g-2$

We have $-m - \frac{n}{2}$

Deficiency: $2g-2+m+\frac{n}{2}$.

Compensated by inserting as many picture changing operators (PCO's).

$$\mathcal{X}(y) = \{Q_B, \xi(y)\}$$

$$= c\partial\xi + e^{\phi} T_F - \tfrac{1}{4}\partial\eta\, e^{2\phi} b - \tfrac{1}{4}\partial(\eta\, e^{2\phi} b)$$

super stress tensor in matter SCFT

$\mathcal{X}(y)$ is in small Hilbert space.

Location of PCO's: extra data needed for defining off-shell amplitudes.

$\mathcal{P}_{g,m,n}$: Base $\mathcal{M}_{g,m+n}$ $\Rightarrow$ Includes sum over spin structure. locally $\mathcal{M}_{g,m+n}$

Fiber: Local coordinates at $m+n$ punctures and location of $2g-2+m+\frac{n}{2}$ PCO's.

$\omega_p^{(g,n)}$: constructed as before.

Additional ingradient: What is the contraction of $\omega_p^{(g,n)}$ with one or more $\frac{\partial}{\partial y_i}$?

New tangent vectors.

Replace $\chi(y_i)$ by $-\partial \xi(y_i)$.

With this $\omega_p^{(g,n)}$ satisfies the old identity:

$$\omega_p^{(g,n)}(Q_B A_1, A_2, \cdots A_n) + \cdots$$
$$= (-1)^p \, d\omega_{p-1}^{(g,n)}(A_1, \cdots A_n)$$

Choose sections $S_{g,m,n}$ consistent with plumbing fixture and proceed as before

Plumbing fixture:

$$\Sigma_{g_1,m_1,n_1} \times \Sigma_{g_2,m_2,n_2} \xrightarrow[\text{gluing}]{NS} \Sigma_{g_1+g_2,m_1+m_2-2,n_1+n_2}$$

# of PCO's:

$$2g_1 - 2 + m_1 + \frac{n_1}{2} + 2g_2 - 2 + m_2 + \frac{n_2}{2}$$

vs. $$2(g_1+g_2) + (m_1+m_2-2) + \frac{n_1+n_2}{2}$$

$\to$ same.

induced PCO on $\Sigma_{g_1+g_2,m_1+m_2-2}$ from Σ_{g_1,m_1,n_1} & Σ_{g_2,m_2,n_2}

R-sector gluing:

$$\Sigma_{g_1,m_1,n_1} \times \Sigma_{g_2,m_2,n_2} \to \Sigma_{g_1+g_2,\, m_1+m_2,\, n_1+n_2-2}$$

# of PCO's:

$$2g_1-2+m_1+\frac{n_1}{2} + 2g_2-2+m_2+\frac{n_2}{2}$$

vs. $2(g_1+g_2-2)+m_1+m_2+\frac{n_1+n_2-2}{2}$

$\downarrow$

1 more.

$\Rightarrow$ 1 PCO missing after plumbing fixture.

Where to put it?

Consistent procedure:

$\oint$ For $w_1 w_2 = -2$

insert $\oint \frac{dw_1}{w_1} X(w_1) = \oint \frac{dw_2}{w_2} X(w_2)$

$\equiv X_0$

$\rightsquigarrow$ average over sections instead of a single section.

$\Rightarrow$ Consistent off-shell amplitude with field theory interpretation.

Off-shell amplitude to SFT

NS - sector propagator: $\frac{b_0^+ b_0^-}{L_0^+} \delta_{L_0, \bar{L}_0}$

R - sector propagator: $\frac{b_0^+ b_0^-}{L_0^+} \mathcal{X}_0 \delta_{L_0, \bar{L}_0}$

Makes hard to construct action.

$1/\mathcal{X}_0$ is non-local.

$\mathcal{X}_0$ has kernel.

Solution: Take string field

$|\Psi\rangle \in \mathcal{H}_{-1} + \mathcal{H}_{-1/2}$

+ an additional string field

$|\tilde{\Psi}\rangle \in \mathcal{H}_{-1} + \mathcal{H}_{-3/2}$

$$S = \Big[-\frac{1}{2 g_s^2} \langle \tilde{\Psi} | c_0^- c_0^+ L_0^+ \mathcal{G} | \tilde{\Psi} \rangle$$

$$+ \frac{1}{g_s^2} \langle \tilde{\Psi} | c_0^- c_0^+ L_0^+ | \Psi \rangle$$

$$+ \sum_n \frac{1}{n!} \{ \Psi^n \} \Big] g_s^{-2}$$

Note: Interaction term involves Ψ and not $\tilde{\Psi}$.

$\mathcal{G} = $ I in NS sector
$\mathcal{X}_0$ in R sector.

Ex. check propagator in $\Psi\Psi$ sector is

$$b_0^+ b_0^- \frac{1}{L_0+\bar{L}_0} \delta_{L_0,\bar{L}_0} \mathcal{G}.$$

$$\begin{pmatrix} -\mathcal{G} & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & \mathcal{G} \end{pmatrix}$$

Ψ : interacting field.

$\mathcal{G}|\tilde{\Psi}\rangle - |\Psi\rangle$: free field $\to$ decouples.

Gauge invariant action:

$$\frac{1}{g_s^2}\Big[-\frac{1}{2}\langle\tilde{\Psi}|c_0^- Q_B \mathcal{G}|\tilde{\Psi}\rangle$$
$$+ \langle\tilde{\Psi}|c_0^- Q_B|\Psi\rangle + \sum_n \frac{1}{n!}\{\Psi^n\}\Big]$$