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Exercises:

3) Isothermal sphere

Assume that the DM velocity distribution is given by a Maxwell-Boltzmann distribution with constant velocity dispersion

i.e $$f(E) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{-E/\sigma^2}$$

where $E = +\phi(r) + \frac{1}{2}v^2$, $\rho_1 = \text{const.}$

with $\phi(r)$ gravitational potential & v^2 the DM velocity.

Then we have

$$\rho(r) = \int_0^\infty 4\pi v^2 dv\, f(v) = \rho(\phi(r))$$

Use this form of $\rho(r)$ in the Poisson equation to obtain $\phi(r)$ & $\rho(r)$:

$$\Delta\phi(r) = +4\pi G_N \rho(r)$$

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Solution:

$$\rho(\phi) = 4\pi \int_0^\infty v^2 dv \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{\frac{1}{2}v^2+\phi}{\sigma^2}} =$$

$$= 4\pi e^{-\frac{\phi(r)}{\sigma^2}} \rho_1 \int_0^\infty \frac{v^2 dv}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}}$$

Redefine $x = \frac{v^2}{2\sigma^2}$ $dx = \frac{dv\, v}{\sigma^2}$

$$\rho(\phi) = 4\pi\rho_1 \frac{e^{-\frac{\phi(r)}{\sigma^2}}}{(2\pi)^{3/2}} \int_0^\infty \sqrt{2x}\, dx\, e^{-x}$$

$$= \rho_1 e^{-\frac{\phi(r)}{\sigma^2}} \frac{\cancel{4\pi}^{2}}{\cancel{2\pi}(2\pi)^{1/2}} \cdot \left(\frac{\pi}{2}\right)^{1/2} =$$

$$= \rho_1 e^{-\frac{\phi(r)}{\sigma^2}}$$

N.B. $\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

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So we have the Poisson's equation

$$\Delta \phi(r) = 4\pi G_N \rho_1 \, e^{-\frac{\phi(r)}{\sigma^2}}$$

Assuming spherical symmetry, we have

$$\Delta \to \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) + \text{angular}$$

i.e using as variable $\rho(r)$ with

$$\phi(r) = -\sigma^2 \ln\left(\frac{\rho(r)}{\rho_1}\right)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\ln\left(\frac{\rho(r)}{\rho_1}\right)\right) = -\frac{4\pi G_N}{\sigma^2}\rho(r)$$

i.e.

$$\cancel{\frac{r^2}{r^2}}\frac{d^2}{dr^2}\ln(\rho(r)) + \frac{2\cancel{r}}{r^{\cancel{2}}}\frac{d}{dr}\ln(\rho(r)) = -\frac{4\pi G_N \rho(r)}{\sigma^2}$$

i.e. $$\frac{d}{dr}\left(r^2 \frac{d}{dr}\ln\left(\frac{\rho(r)}{\rho_1}\right)\right) = -\frac{4\pi G_N r^2}{\sigma^2}\rho(r)$$

Ansatz: $$\rho(r) = \frac{\sigma^2}{2\pi G_N r^2}$$

$$\frac{d\ln(\rho(r))}{dr} = \frac{d}{dr}\left[-2\ln(r) + \ln\left(\frac{\sigma^2}{2\pi G_N}\right)\right] = -\frac{2}{r}$$

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So this does solve the equation

$$\frac{d}{dr}\left(r^2\left(-\frac{2}{r}\right)\right) = -2\frac{d}{dr}r = -2 = -2$$

So we obtain now for $\phi(r)$

$$\phi(r) = -\sigma^2 \ln\left(\frac{\rho(r)}{\rho_1}\right) =$$

$$= -\sigma^2 \ln\left(\frac{\sigma^2}{2\pi G_N \rho_1 r^2}\right) =$$

$$= -\sigma^2 \ln\left(\frac{\sigma^2}{2\pi G_N \rho_1}\right) + 2\sigma^2 \ln(r)$$

But this result is singular for $r \simeq 0$
so usually one adds a constant:

$$\rho(r) = \frac{\rho_1}{1 + \left(r/a\right)^2} \qquad a = \frac{\sigma}{\rho_1}\frac{1}{\sqrt{2\pi G_N}}$$

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Exercise 1 : FRW expansion

a) Continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \text{for } p = w\rho$$

$$\Rightarrow \frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a}(1+w) = 0$$

i.e. $\rho(a) \simeq \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$

i.e. in red-shift $\rho(z) = \rho_0 (1+z)^{3(1+w)}$

b) $$H^2(t) + \frac{\kappa}{a^2(t)} = \frac{8\pi G_N}{3}\rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

i.e.

$$\dot{a}^2(t) = -\kappa + \frac{8\pi G_N}{3}\rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} a^2$$

positive root for expansion!

$$\dot{a}(t) = \pm \sqrt{\frac{8\pi G_N}{3}\frac{\rho_0}{a_0^{-3(1+w)}} a^{-1-3w} - \kappa}$$

$$\int \frac{da}{\sqrt{\frac{8\pi G_N}{3}\frac{\rho_0 a_0^{3(1+w)}}{a^{1+3w}} - \kappa}} = \int dt$$

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Generically difficult to find a solutio
Relatively easy for Radiation Dominance
$w = \frac{1}{3}$ so that

$$\dot{a} = \pm \sqrt{\frac{8\pi G_N \rho_0}{a_0^{-4}} a^{-2} - x} =$$

$$= \pm \underbrace{\sqrt{8\pi G_N \rho_0 a_0^4}}_{A^{-1}} a^{-1} \left(1 - A^2 x a^2\right)^{1/2}$$

$$\int \frac{A a \, da}{\left(1 - A^2 x a^2\right)^{1/2}} = \int dt$$

$$\frac{1}{2A} \frac{1}{\sqrt{|x|}} \int \frac{dx}{(1-x)^{1/2}} = \int dt$$

$$-\frac{1}{A\sqrt{|x|}} \sqrt{1 - A^2 x a^2} = t - t_0$$

Separate the cases depending on sign!

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$x > 0$ closed universe!

Maximal scale factor for

$$A^{-2} a^{-2} = x \quad \text{i.e.} \quad a^2_{max} = \frac{1}{x A^2}$$

$$\Rightarrow a_{max} = \frac{1}{\sqrt{x}\, A}$$

So rescale a by a_{max} & define

$$y = \frac{a}{a_{max}} = \sqrt{x} A a \quad \& \; 0 \leq y \leq 1$$

$$dy = \sqrt{x} A \, da$$

$$\frac{dy}{dt} = \sqrt{x} A \frac{da}{dt} = \sqrt{x} A \sqrt{A^{-2} a^{-2} - x}$$

$$= \sqrt{x} \sqrt{a^{-2} - A^2 x} =$$

$$= \sqrt{x} \sqrt{\frac{x A^2}{y^2} - x A^2} = x A \sqrt{\frac{1}{y^2} - 1}$$

$$= x A \frac{1}{y} \sqrt{1 - y^2}$$

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So we get

$$\int_{y(t_i)}^{y(t)} \frac{y\,dy}{xA\sqrt{1-y^2}} = \int dt$$

$$\frac{1}{2xA}\int_{y(t_i)}^{y(t)} \frac{dy^2}{\sqrt{1-y^2}} = t - t_i$$

$$\frac{1}{2xA}\left[-\sqrt{1-y^2}\cdot 2\right]_{y(t_i)}^{y(t)} = t - t_i$$

$$\frac{1}{xA}\left[\sqrt{1-y^2(t_i)} - \sqrt{1-y^2(t)}\right] = t - t_i$$

Take as initial condition

$y(t_i) = 0$ for $t_i = 0$ & obtain

$$\frac{1}{xA}\left[1 - \sqrt{1-\frac{a^2(t)}{a^2_{max}}}\right] = t - \cancel{t_i}$$

$$\frac{1}{xA}\left[1 - \sqrt{1 - xA^2a^2(t)}\right] = t - \cancel{t_i}$$

$$\xrightarrow[x \to 0]{} \frac{1}{xA}\left[\cancel{1} - \cancel{1} + \frac{1}{2}xA^2a^2(t)\right] = t - \cancel{t_i}$$

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$$\Rightarrow \frac{1}{2} A\, a^2(t) = t - \cancel{t_i}$$

i.e. $$a(t) = \sqrt{\frac{2t}{A}} = \sqrt{\frac{2}{A}}\, t^{1/2} \quad \text{ok!}$$

Now consider $x < 0$, then one has

$$\dot{a} = \pm A^{-1} a^{-1} \left(1 + |x| A^2 a^2\right)^{1/2}$$

so for the positive solution one has

$$\int \frac{da\, aA}{\sqrt{1 + |x| A^2 a^2}} = \int dt$$

Rescale again $y = \sqrt{|x|}\, A\, a(t)$

$$\frac{1}{A\left(\sqrt{|x|}\right)^2} \int \frac{y\, dy}{\sqrt{1 + y^2}} = \int dt$$

$$\frac{1}{2xA} \int \frac{dy^2}{\sqrt{1+y^2}} = \frac{1}{\cancel{2}xA}\left[\sqrt{1+y^2(t)} - \sqrt{1+y^2(t_i)}\right]$$

$$= t - t_{in} \Rightarrow \frac{1}{\cancel{2}xA}\left(\sqrt{1+|x|A^2a^2} - 1\right) = t$$

same as before!

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Consider now $w = 0$

$$\dot{a} = \pm\sqrt{\underbrace{8\pi G_N \rho_0 a_0^3}_{A_M^{-2}}\, a^{-1} - x} =$$

$$= \pm A_M^{-1} a^{-1/2} \sqrt{1 - x A_M^2 a(t)}$$

$$\int \frac{A_M a^{1/2}\, da}{\sqrt{1 - x A_M^2 a(t)}} = t - t_{in}$$

Try out substitution

$$\xi^2 = |x| A_M^2 a(t) \qquad a^{1/2} = \frac{\xi}{\sqrt{x} A_M}$$

$$2\xi d\xi = |x| A_M^2 da$$

$$\int \frac{\cancel{A_M} \frac{\xi}{\sqrt{x}} \frac{2\xi}{|x| A_M^2} d\xi}{\sqrt{1 - \xi^2}} = t - t_{in}$$

$$\frac{2}{x^{3/2} A_M^2} \int \frac{\xi^2 d\xi}{\sqrt{1 - \xi^2}} = t - t_{in}$$

Again consider different cases

$x > 0$ closed universe, we have

$$\int \frac{\xi^2 d\xi}{\sqrt{1-\xi^2}} = -\frac{1}{2}\xi\sqrt{1-\xi^2} + \frac{\mathrm{ArcSin}[\xi]}{2}$$

Instead for $x < 0$ open universe

$$\int \frac{\xi^2 d\xi}{\sqrt{1+\xi^2}} = \frac{1}{2}\xi\sqrt{1+\xi^2} - \frac{\mathrm{ArcSinh}[\xi]}{2}$$

So we finally get

$$\frac{1}{x^{3/2}A_M^2}\left\{\begin{array}{l} -\xi\sqrt{1-\xi^2} + \mathrm{ArcSin}[\xi] \\ \xi\sqrt{1+\xi^2} - \mathrm{ArcSinh}[\xi] \end{array}\right\}_0^{\sqrt{x}A_M a^{1/2}} = t - t_{in}$$

i.e. $x > 0$

$$-\frac{1}{x^{3/2}A_M^2}\left[\sqrt{x}A_M a^{1/2}\sqrt{1-xA_M^2 a(t)} - \mathrm{ArcSin}[\sqrt{x}A_M a^{1/2}]\right] = t - t_{in}$$

$$t = -\frac{a^{1/2}}{xA_M}\sqrt{1-xA_M^2 a(t)} + \frac{1}{x^{3/2}A_M^2}\mathrm{ArcSin}[\sqrt{x}A_M a^{1/2}]$$

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Consider now the limit $x \to 0$

$$t = \frac{-a^{1/2}}{x A_M}\left(1 - \frac{1}{2} x A_M^2 a(t)\right) + O($$

$$+ \frac{1}{x^{3/2} A_M^2}\left(\sqrt{x}\, A_M a^{1/2} + \frac{1}{6} x^{3/2} A_M^3 a^{3/}\right.$$

$$= a^{1/2}\left(\cancel{-\frac{1}{x A_M} + \frac{1}{x A_M}}\right) +$$

$$+ a^{3/2}\left(\frac{1}{2} A_M + \frac{1}{6} A_M\right) = \frac{2}{3} A_M a$$

OK

So for $x < 0$ instead

$$t = \frac{1}{|x| A_M} a^{1/2}(t) \sqrt{1 + |x| A_M^2 a(t)} -$$

$$- \frac{1}{|x|^{3/2} A_M^2} \operatorname{ArcSinh}\left[|x|^{1/2} A_M a^{1/2}(t)\right]$$

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Exercise 4

c) $u_k'' - \frac{a''}{a} u_k = 0$

$u_k = a(t)$ is a solution evidently!

To obtain the 2nd solution consider the Wronskian:

$$W = u_k'^{(1)} u_k^{(2)} - u_k^{(1)} u_k^{(2)'}$$

so we have

$$W' = u_k''^{(1)} u_k^{(2)} + \cancel{u_k^{(1)'} u_k'^{(2)}} - \cancel{u_k'^{(1)} u_k'^{(2)}}$$
$$- u_k^{(1)} u_k''^{(2)} =$$
$$= \frac{a''}{a} \left(u_k^{(1)} u_k^{(2)} - u_k^{(1)} u_k^{(2)} \right) = 0$$

So the Wronskian is konstant & we have therefore

$$W = a' u_k^{(2)} - a u_k^{(2)'} \quad \text{konst.}$$

$$\Rightarrow u_k'^{(2)} = \frac{a'}{a} u_k^{(2)} - \frac{W}{a}$$

first order differential equation!

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The homogeneous solution is

$$u_k'^{(2)} = \frac{a'}{a} u_k^{(2)} \quad \text{again } u_k^{(2)} \sim a$$

but the inhomogeneous solution can be obtained by variation of the constant

$$u_k^{(2)} = c(t)\, a(t)$$

$$u_k'^{(2)} = c'(t)\, a(t) + c(t)\, a'(t)$$

so we get

$$c'(t)\, a(t) + \cancel{c(t) a'(t)} = \cancel{a'(t) c(t)} - W a^{-1}(t)$$

$$\Rightarrow c'(t) = - W a^{-2}(t)$$

$$c(t) = \int dt \left(-W \frac{1}{a^2(t)} \right) = $$

$$= -W \int \frac{dt'}{a^2(t')}$$

$$\Rightarrow u_k^{(2)} = -W a(t) \int_{t_{in}}^{t} \frac{dt'}{a^2(t')}$$

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Exercise 1:

Show that

$$\frac{g}{E}\int d^3p\, \hat{\mathcal{L}}\left[f(t,E)\right] =$$

$$= \int d^3p\, g\left[\frac{\partial f}{\partial t} - H\frac{\bar{p}^2}{E}\frac{\partial f}{\partial E}\right] =$$

$$= \frac{dn(t)}{dt} + 3H\, n(t)$$

for $n(t) = g\int d^3p\; f(t,E)$ finite

i.e.

assuming $f(t,E) \xrightarrow[E\to\infty]{} 0$

Answer:

$$g\int d^3p\, \frac{\partial f}{\partial t} = \frac{d}{dt}\, g\int d^3p\, f(t,E) = \frac{dn(t)}{dt}$$

$$g\int d^3p\left[-H\frac{\bar{p}^2}{E}\frac{\partial f}{\partial E}(t,E)\right] =$$

$$= \frac{4\pi g}{(2\pi)^3}(-H)\int_0^\infty p^2 dp\; \frac{p^2}{E}\frac{\partial f}{\partial E}(t,E)$$

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Change integration variable to E

$$E^2 = p^2 + m^2 \Rightarrow 2E\,dE = 2p$$

$$\Rightarrow \frac{4\pi g}{(2\pi)^3}(-H)\int_m^\infty dE\, p^3 \frac{\partial f}{\partial E}(t,E)$$

$$= \frac{4\pi g}{(2\pi)^3}(-H)\int_m^\infty dE \left[\frac{\partial}{\partial E}(p^3 f) - f\right.$$

$$= \frac{4\pi g}{(2\pi)^3}(-H)\left\{\underbrace{\left[p^3 f\right]_m^\infty}_{\text{vanishes on boundaries!}} - \int_m^\infty dE\, f\, 3p\right.$$

$$= \frac{4\pi g}{(2\pi)^3} 3H \int_m^\infty dE\, f\, p^2 \frac{dp}{dE} =$$

$$= \frac{4\pi g}{(2\pi)^3} 3H \int_0^\infty dp\, p^2 f(t,E) =$$

$$= 3H\, g \int d^3\vec{p}\, f(t,E) = 3H\, n(t)$$

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Exercise 2: Freeze-out with the unitarity cross-section

$$\sigma^{u}_{J} = \frac{4\pi(2J+1)}{(2s_p+1)^2} \frac{(1-\eta_J^2)}{\bar{p}_i^2}$$

η_J = elastic ~~x~~cross-section

J = angular momentum of the partial wave

s_p = particle spin

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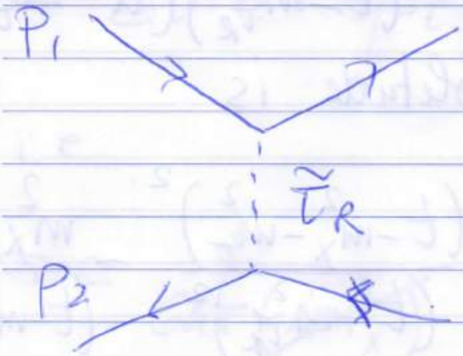
3. Exercise: compute the cross-section for $\tilde{B} + \tilde{B} \to \tau_R + \bar{\tau}_R$ via $\tilde{\tau}_R$



Vertices

$$i\sqrt{2}\,\frac{e}{\cos\theta_W}\,P_R$$

$$i\sqrt{2}\,\frac{e}{\cos\theta_W}\,P_L$$

The amplitudes are

$$\mathcal{A}_t = -i\,\frac{2e^2}{\cos^2\theta_W}\,\frac{\bar{u}_\tau(k_1)P_R u_\chi(p_1)\,\bar{v}_\chi(p_2)P_L v_\tau(k_2)}{t - m^2_{\tilde{\tau}_R}}$$

$$\mathcal{A}_u = i\,\frac{2e^2}{\cos^2\theta_W}\,\frac{\bar{u}_\tau(k_1)P_R u_\chi(p_2)\,\bar{v}_\chi(p_1)P_L v_\tau(k_2)}{u - m^2_{\tilde{\tau}_R}}$$

Amplitudes squared:

$$|\mathcal{A}_t|^2 = \frac{e^4}{\cos^4\theta_W}\left(\frac{t - m_\chi^2 - m_\tau^2}{t - m^2_{\tilde{\tau}_R}}\right)^2$$

$$|\mathcal{A}_u|^2 = \frac{e^4}{\cos^4\theta_W}\left(\frac{u - m_\chi^2 - m_\tau^2}{u - m^2_{\tilde{\tau}_R}}\right)^2$$

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Interference gives

$$\mathcal{A}_t \mathcal{A}_u^* = -\frac{e^4}{\cos^4\theta_w} \frac{m_\chi^2 (s-2m_\tau^2)}{(t-m_{\tilde{\tau}_R}^2)(u-m_{\tilde{\tau}_R}^2)}$$

So the total amplitude is

$$|\mathcal{M}|^2 = \frac{2e^4}{\cos^4\theta_w}\left[\frac{(t-m_\chi^2-m_\tau^2)^2}{(t-m_{\tilde{\tau}_R}^2)^2} - \frac{m_\chi^2(s-2}{(t-m_{\tilde{\tau}_R}^2)(}\right.$$

Now consider $v_\chi \to 0$ limit

$$t \simeq -m_\chi^2 \beta_\tau^2 \qquad s = 4m_\chi^2$$

$$|\mathcal{M}|^2 = \frac{2e^4}{\cos^4\theta_w}\left[\frac{(m_\chi^2(1+\beta_\tau^2)+m_\tau^2)^2}{(m_\chi^2\beta_\tau^2+m_{\tilde{\tau}_R}^2)^2} + (-1)\right.$$

$$2\frac{m_\chi^2(4m_\chi^2-2m_\tau^2)}{4m_\chi^2+2(m_{\tilde{\tau}_R}^2-m_\chi^2-m_\tau^2)} \frac{1}{m_\chi^2\beta_\tau^2+m_{\tilde{\tau}_R}^2}$$

Now we have

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s}\frac{1}{|P_{1cm}|^2}|\mathcal{M}|^2$$

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with $|p_{1cm}|^2 = \frac{s}{4} - m_X^2 = \frac{s}{4}\beta_X^2$

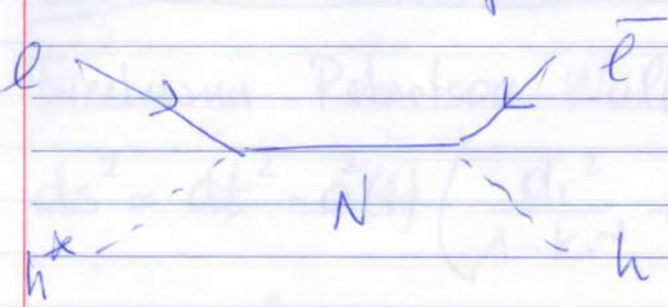
& $\Delta t \simeq s\beta_\tau\beta_X = 4m_X^2\beta_\tau\beta_X$

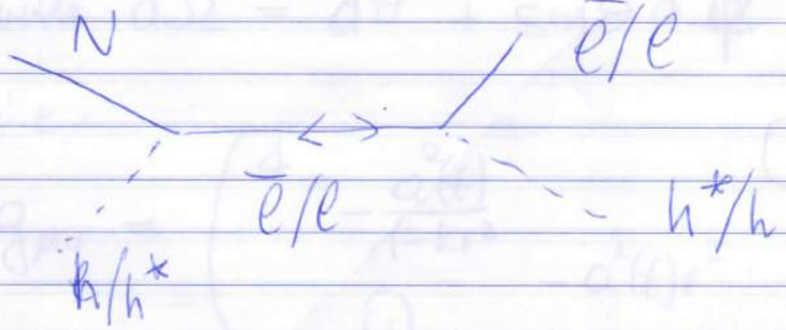
i.e.

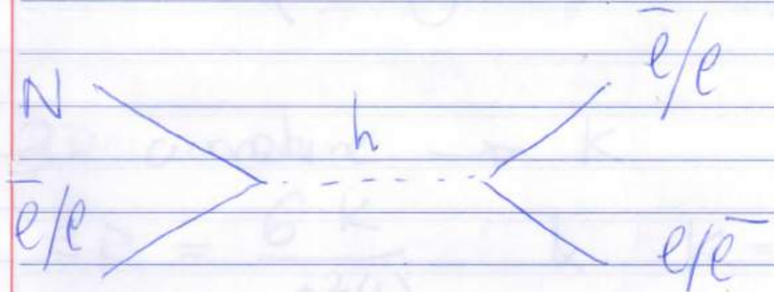
$$\sigma \simeq \frac{1}{64\pi\cancel{(4m_X^2)}}\frac{1}{m_X^2\beta_X^{\cancel{2}}}\cancel{4m_X^2}\beta_\tau\cancel{\beta_X}|\mathcal{M}|^2$$

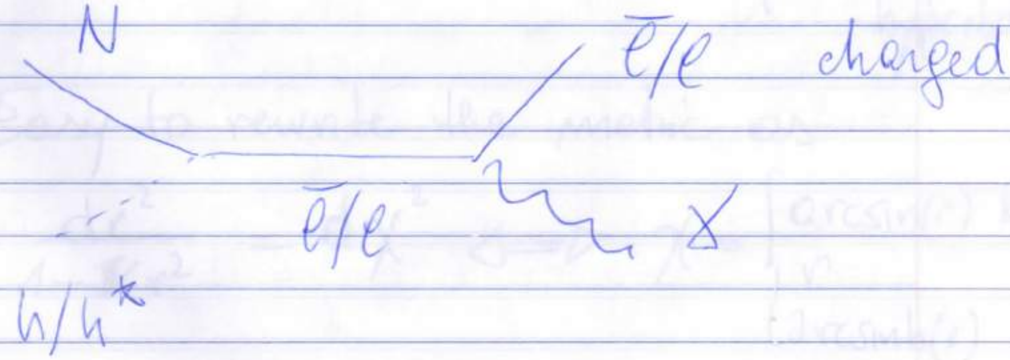
$$= \frac{1}{64\pi}\frac{|\mathcal{M}|^2}{m_X^2}\frac{\beta_\tau}{\beta_X}$$

Wash-out diagrams









SAIFR-ICTP School 2018

Lemaître-Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

i.e.

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -\frac{a^2(t)}{1-kr^2} & & \\ & & -a^2(t)r^2 & \\ 0 & & & -a^2(t)r^2\sin^2\theta \end{pmatrix}$$

3D curvature $\rightarrow k$

$$^3R = \frac{6k}{a^2(t)} \quad \& \quad k = \begin{cases} +1 & \text{closed hypersphere} \\ 0 & \text{flat (eucl.)} \\ -1 & \text{open hyperbolic} \end{cases}$$

Easy to rewrite the metric as

$$\frac{dr^2}{1-kr^2} = d\chi^2 \iff \chi = \begin{cases} \arcsin(r) & k=1 \\ r & 0 \\ \operatorname{arcsinh}(r) & -1 \end{cases}$$

Moreover conformal time: $dt^2 = a^2(\eta)d\eta^2$

$$\Rightarrow ds^2 = a^2(\eta)\left(d\eta^2 - d\chi^2 - \begin{pmatrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{pmatrix} d\Omega^2\right)$$

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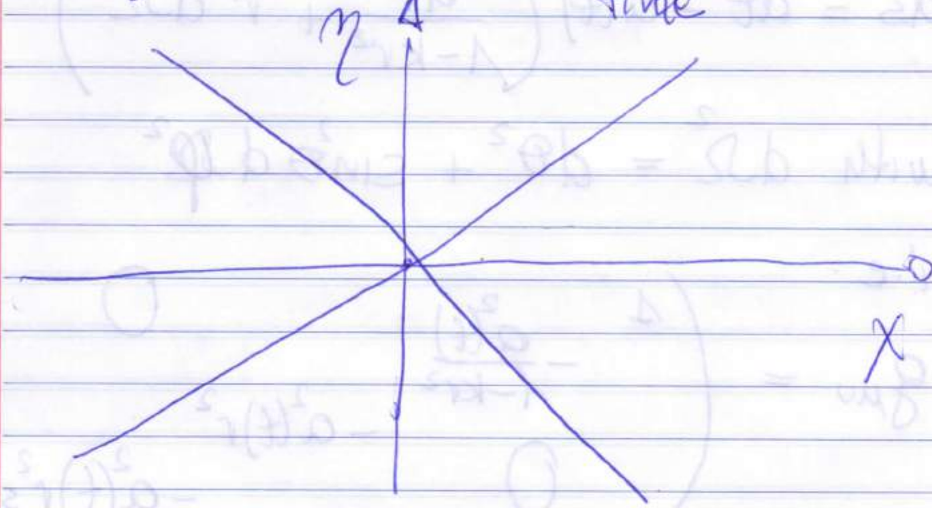
In the conformal convention, light travels along straight line, i.e.

$d\eta = d\chi$ "light-cone in conformal time"



i.e. $r = \begin{cases} \sin(\eta) \\ \eta \\ \sinh(\eta) \end{cases}$ radial trajectory!

LFRW metric is locally conformal to the Minkowski metric up to the conformal factor $a^2(\eta)$

Light trajectory $d\eta = d\chi$ i.e.

$\eta_{obs} - \eta_{em} = \chi_{obs} - \chi_{em}$

i.e. the conformal period is constant:

$$\Delta\eta = \frac{\Delta t_{em}}{a(t_{em})} = \frac{\Delta t_{obs}}{a(t_{obs})}$$

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$$\Rightarrow \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} = 1+z$$

$t \to \eta \to z$ through

$$\boxed{\frac{dz}{1+z} = -H(t)\,dt = -\frac{da}{a}}$$

Natural way to measure $a(t)$! $H(t)$

Einstein's equation for (FRW-metric

$$R^\alpha{}_\beta - \frac{1}{2} R\, \delta^\alpha{}_\beta = 8\pi G_N T^\alpha{}_\beta + \Lambda \delta^\alpha{}_\beta$$

Derive the Riemann tensor & then the Ricci tensor & scalar: first start with the Christoffel symbols:

$$\Gamma^\alpha_{\gamma\beta} = \frac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\delta\beta}}{\partial x^\gamma} - \frac{\partial}{\partial x^\delta} g_{\gamma\beta}\right)$$

$$R^\alpha{}_\beta = g^{\alpha\gamma}\left(\frac{\partial \Gamma^\delta_{\gamma\beta}}{\partial x^\delta} - \frac{\partial \Gamma^\delta_{\gamma\delta}}{\partial x^\beta} + \Gamma^\delta_{\gamma\beta}\Gamma^\sigma_{\delta\sigma} - \Gamma^\sigma_{\gamma\delta}\Gamma^\delta_{\beta\sigma}\right)$$

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So for a diagonal $g_{\mu\nu}$ we get

$$\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2} g^{\alpha\gamma} \frac{\partial g_{\gamma\gamma}}{\partial x^{\beta}} + \frac{1}{2} g^{\alpha\beta} \frac{\partial g_{\beta\beta}}{\partial x^{\gamma}}$$

$$- \frac{1}{2} g^{\alpha\alpha} \frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}}$$

So the only non-zero components are

$$\Gamma^{0}_{ii} = -H g_{ii} \quad \Gamma^{i}_{0i} = \Gamma^{i}_{i0} = H$$

& time independent pieces

$$\Gamma^{i}_{jj} = -\frac{1}{2} g^{ii} \frac{\partial g_{jj}}{\partial x^{i}} = -\frac{1}{2} e^{ii} \frac{\partial e_{jj}}{\partial x^{i}}$$

$$\Gamma^{i}_{ij} = \Gamma^{i}_{ji} = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x^{j}} = \frac{1}{2} e^{ii} \frac{\partial e_{ii}}{\partial x^{j}}$$

Where $e_{ii} = -\frac{g_{ii}}{a^2(t)} = \left(\frac{1}{1-kr^2}, r^2, r^2 s\right.$

i.e $\Gamma^{r}_{rr} = -\frac{kr}{1-kr^2}$

$$\Gamma^{r}_{\theta\theta} = -r(1-kr^2) \quad \Gamma^{r}_{\varphi\varphi} = -r \sin^2\theta (1-k$$

$$\Gamma^{\theta}_{rr} = \Gamma^{0}_{\theta\theta} = \Gamma^{\varphi}_{ii} = 0$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta \cos\theta$$

$$\Gamma^{\theta}_{\theta r} = \frac{1}{r} = \Gamma^{\varphi}_{\varphi r} \qquad \Gamma^{\varphi}_{\varphi\theta} = \frac{1}{\operatorname{tg}\theta}$$

Plugging these into the Ricci tensor we get

$$R^{0}_{\ 0} = g^{00}\left(\cancel{\frac{\partial \Gamma^{\delta}_{00}}{\partial x^{\delta}}} - \frac{\partial \Gamma^{\delta}_{0\delta}}{\partial x^{0}} + \right.$$

$$\left. + \cancel{\Gamma^{\delta}_{00}\Gamma^{\sigma}_{\delta\sigma}} - \Gamma^{\sigma}_{0\delta}\Gamma^{\delta}_{0\sigma} \right) =$$

$$= g^{00}\left(-\frac{\partial \Gamma^{i}_{0i}}{\partial x^{0}} - \Gamma^{i}_{0i}\Gamma^{i}_{0i} \right) =$$

$$= -3\left(\dot{H} + H^{2}\right)$$

Similarly

$$R^{i}_{\ i} = -\left(\dot{H} + 3H^{2} + \frac{2k}{a^{2}}\right)$$

Then the Ricci curvature is

$$R = -6\left(\dot{H} + 2H^{2} + k/a^{2}\right)$$

combination of spatial & time-like curvature!

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So the 0-0 component of Einstein's e

$$R^0{}_0 - \frac{1}{2}R = 3\left(H^2 + \frac{k}{a^2}\right)$$

& i-i component: $R^i{}_i - \frac{1}{2}\delta^i{}_i R$:

$$-\dot{H} - 3H^2 - \frac{2k}{a^2} + 3\dot{H} + 6H^2 + 3\frac{k}{a^2}$$

$$= 2\dot{H} + 3H^2 + \frac{k}{a^2}$$

but $\dot{H} = \frac{\ddot{a}}{a} - H^2$

$$\Rightarrow 2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}$$

So we get

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3} T^0{}_0$$

$$2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} = 8\pi G_N T^i{}_i$$

i.e.

$$\frac{\ddot{a}}{a} = 4\pi G_N \left(T^i{}_i - \frac{1}{3}T^0{}_0\right)$$

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Perfect fluid: $T^0_{\ 0} = \rho$

$T^i_{\ i} = -p$

i.e. $H^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho$

$$\frac{\ddot{a}}{a} = 4\pi G_N\left(-p - \frac{1}{3}\rho\right) =$$

$$= -\frac{4\pi G_N}{3}\left(\rho + 3p\right) =$$

$$= -\frac{4\pi G_N}{3}\rho\left(1 + 3\frac{p}{\rho}\right)$$

negative acceleration if $\rho, p > 0$!

Need $\frac{p}{\rho} < -\frac{1}{3}$ to get positive $\ddot{a}$!

$\hookrightarrow$ inflation!

Friedmann's eq. for $k = 0$

$$H^2 = \frac{8\pi G_N}{3}\rho = H_0^2 \Omega_\rho$$

For many components

$$H = H_0\sqrt{\sum_i \Omega_i (1+z)^{3(1+w)}}$$

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Particle Horizon (past lightcone)

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a(t)}$$

so for $a(t) = t^{\alpha}$ one obtain

$$\chi_p(\eta) = \int_0^t t^{-\alpha} dt = \frac{1}{-\alpha+1} t^{-\alpha+1}$$

$$= \frac{1}{1-\frac{2}{3(1+w)}} t^{1-\frac{2}{3(1+w)}}$$

for $\alpha = \frac{2}{3(1+w)}$

So we see that ~~the~~ the distance from the initial time $t=0$ is

$$d_p(t) = a(t)\chi_p(t) = \frac{t}{1-\frac{2}{3(1+w)}} \begin{cases} 3t & w=0 \\ 2t & w=\frac{1}{3} \end{cases}$$

In general

$$\chi_p(t) = \int_{t_i}^{t} \frac{da}{\dot{a}\,a(t)} = \int_{a_i}^{a} \frac{da}{Ha^2}$$

$$H(a) \sim a^{-\frac{3}{2}(1+w)} \qquad Ha^2 = a^{\frac{1}{2}-\frac{3}{2}w}$$

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i.e.

$$\int_{a_i}^{a} a^{\frac{3}{2}w-\frac{1}{2}}\, da = \frac{1}{\frac{3}{2}w+\frac{1}{2}}\left(a^{\frac{3}{2}w+\frac{1}{2}} - a_i^{\frac{3}{2}w+\frac{1}{2}}\right)$$

always finite for $w > -\frac{1}{3}$!

Cosmological perturbation theory

Need to expand both field(s) and metric beyond the background (homogeneous/isotropic) level:

$$\phi = \varphi + \delta\varphi$$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$$

We have in general

$$\delta g_{00} = 2a^2 \Phi$$

$$\delta g_{0i} = a^2\left(\partial_i B + S_i\right)$$

$$\delta g_{ij} = a^2\big(2\psi\,\delta_{ij} + 2\partial_i\partial_j E + \partial_j F_i + \partial_i F_j + h_{ij}\big)$$

We have in total 10 d.o.f.

4 scalars : Φ, B, ψ, E

2 Vectors : S_i, F_i (transverse $\vec{\nabla}\cdot\vec{S}=0$, $\vec{\nabla}\cdot\vec{F}=0$)

$\hookrightarrow$ 4 d.o.f.

1 Tensor : h_{ij} with $h^i_i = 0$

& $\partial_i h^i_j = 0$

traceless & transverse

$\Rightarrow$ in total 2 d.o.f

6 - 1 - 3 symmetric 3x2 !

But we can exploit the gauge freedom

i.e. diffeomorphism invariance of GR

to gauge away 2 scalars & 1 vector

$$x^\mu \rightarrow x^{\mu'} = x^\mu + \delta\xi^\mu$$

$$\delta g_{\mu\nu} \rightarrow \delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \partial_\gamma g_{\mu\nu}\xi^\gamma - g_{\gamma\nu}\partial_\mu\xi^\gamma - g_{\mu\delta}\partial_\nu\xi^\delta$$

Fix $\delta\xi^0$ & $\delta\xi^i = \delta\xi^i_\perp + \partial^i\xi$

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to set to zero 2 scalars & 1 vector

Scalar gauge invariant combinations:

$$\begin{cases} \Phi = \phi - \frac{1}{a}\left[a\left(B - E'\right)\right]' \\ \Psi = \psi + \frac{a'}{a}\left(B - E'\right) \end{cases}$$

Longitudinal gauge (conformal Newtonian)

$B = E = 0$ so the scalar perturbation become

$$ds^2 = a^2\left((1+2\phi)\,d\eta^2 - (1-2\psi)\,\delta_{ij}\,dx^i dx^j\right)$$

↳ "Newtonian potential"

Einstein's equations:

$$\Delta\psi - 3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) = 4\pi G_N a^2\,\delta T^0_0$$

$$\partial_i\left(\psi' + \mathcal{H}\phi\right) = 4\pi G_N a^2\,\delta T^0_i$$

$$\Big[\psi'' + \mathcal{H}\left(2\psi + \Phi\right)' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{2}\vec{\nabla}^2\left(\phi - \psi\right)\Big]\delta_{ij} - \frac{1}{2}\partial_i\partial_j\left(\Phi - \psi\right) = -4\pi G_N a^2\,\delta T^i_j$$

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$i \neq j$ usually $\overline{\delta T^i_j} = 0$

$\Rightarrow \partial_i \partial_j (\phi - \psi) = 0$

i.e. $\phi = \psi$

$\overline{\delta T^0_0} = \delta\bar{\varepsilon}$

$\overline{\delta T^i_j} = -\delta\bar{p}\,\delta^i_j$

$\overline{\delta T^0_i} = \frac{1}{a}(\varepsilon_0 + p_0)\,\partial_i \delta u$

Moreover $\delta\bar{p} = c_s^2 \delta\bar{\varepsilon} + \tau \delta S$

$$\Rightarrow \Phi'' + 3(1+c_s^2)\mathcal{H}\Phi' - c_s^2 \vec{\nabla}\Phi$$
$$+ \left(2\mathcal{H}' + (1+c_s^2)\mathcal{H}^2\right)\Phi =$$
$$= 4\pi G_N a^2 \tau \delta S$$

Coupled system: $\Phi, \delta\varphi, \psi$

Gauge invariant combinations for $\psi = \Phi$:

$u \equiv \frac{\Phi}{4\pi \dot{\varphi}}$ $\qquad v \equiv a^2\left(\delta\varphi + \frac{\dot{\varphi}}{\mathcal{H}}\Phi\right) \equiv aR$

curvature pert.

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Then we obtain for v the e.o.m.

$$v'' - c_s^2 \vec{\nabla}^2 v - \frac{z''}{z} v = 0$$

where $z = \frac{a\phi'}{H}$

Gravitational waves: simpler as there is no source! Giving for

$$h^i_j = \int d^3k \; h^\alpha_k(\eta) \, e_\alpha{}^i{}_j \, e^{ikx}$$

$$h''_k + \left(k^2 - \frac{2}{\eta^2}\right) h_k = 0$$

"free field" in de Sitter!

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Thermodynamics

thermal equilibrium $\nearrow$ kinetic equilibrium (elastic scattering) $\hookrightarrow$ Fermi-Dirac, Bose-Einstein

thermal equilibrium $\searrow$ chemical equilibrium (inelastic scattering)

Basic quantity: phase space distribution $f(\vec{p}, \vec{x})$

homogeneous/isotropic universe implies $f(\vec{p}, \vec{x}) \to f(\vec{p})$ at the background level!

In general we have

$$n = g \int d^3\vec{p}\, f(\vec{p})$$

$$\rho = g \int d^3\vec{p}\, E(\vec{p})\, f(\vec{p})$$

$$P = g \int d^3\vec{p}\, \frac{|\vec{p}|^2}{3E}\, f(\vec{p})$$

relativistic fluid: $E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$

i.e. very relativistic gives

$$E = |\vec{p}| + \frac{1}{2}\frac{m^2}{|\vec{p}|}$$

$$\Rightarrow \frac{|\vec{p}|^2}{3E} \simeq \frac{E}{3} \quad \text{i.e.} \quad p = \frac{\rho}{3}$$

non-relativistic instead

$$E = m + \frac{1}{2}\frac{|\vec{p}|^2}{m}$$

i.e. $\rho = m\,n$

$$p = g\int d^3p\, \frac{|\vec{p}|^2}{3m} f(\vec{p}) \simeq 0$$

In kinetic equilibrium

$$f_{F/B}(\vec{p}) = \frac{1}{e^{\frac{E-\mu}{T}} \pm 1}$$

$\mu \rightarrow$ chemical potential related to the number of particles minus antiparticles & to a conserved charge

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relativistic case $\to$ conserved quantum number, i.e. baryon number.

non-relativistic case $\to$ number of particle or mass conservation

Indeed we have then

$$n_p - n_{\bar{p}} = \frac{g}{2\pi^2}\int_m^\infty dE\,(E^2-m^2)^{1/2}E\ \times$$

$$\times\left(f_p(E) - f_{\bar{p}}(E)\right) =$$

$$= \frac{g}{2\pi^2}\int_m^\infty dE\,E\,(E^2-m^2)^{1/2}\frac{e^{\frac{E+\mu}{T}}\cancel{\pm 1} - e^{\frac{E-\mu}{T}}\cancel{\mp 1}}{\left(e^{\frac{E-\mu}{T}}\pm 1\right)\left(e^{\frac{E+\mu}{T}}\pm 1\right)}$$

$$= \frac{g}{2\pi^2}\int_m^\infty dE\,E\,(E^2-m^2)\,\frac{e^{E/T}\left(e^{\frac{\mu}{T}} - e^{-\frac{\mu}{T}}\right)}{e^{\frac{2E}{T}} \pm e^{E/T}\left(e^{\frac{\mu}{T}}+e^{-\frac{\mu}{T}}\right)+1}$$

$$= \frac{g}{2\pi^2}\,2\sinh\left(\frac{\mu}{T}\right)\int_m^\infty dE\,E\sqrt{E^2-m^2}\ \times$$

$$\times\ \frac{1}{E^{E/T}+e^{-E/T}\pm 2\cosh\left(\frac{\mu}{T}\right)}$$

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$$\xrightarrow[T \gg m,\mu]{} \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu}{T}\right)+\left(\frac{\mu}{T}\right)^3\right]$$

$$\xrightarrow[T \ll m,\mu]{} 2g\left(\frac{mT}{2\pi}\right)^{3/2}\sinh\left(\frac{\mu}{T}\right)e^{-m/T}$$

Total radiation density:

$$\rho_{rad} = T^4\left(\sum_B \frac{\pi^2}{30} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_F \frac{\pi^2}{30} g_i \left(\frac{T_i}{T}\right)^4\right)$$

$$= \frac{\pi^2}{30} g_* T^4$$

i.e

$$H^2(T) = \frac{8\pi G_N}{3}\frac{\pi^2}{30} g_* T^4 \quad \text{RADIATION DOMINANCE}$$

$$\Rightarrow H(T) \simeq \sqrt{\frac{\pi^2 g_*}{90}}\frac{T^2}{\tilde{M}_p} \simeq \overbrace{\frac{\pi}{3}\left(\frac{g_*}{10}\right)^{1/2}}^{1.086\ (\gamma e^{\pm}, \nu_i)}\frac{T^2}{\tilde{M}_p}$$

$$= \frac{1}{2t} \quad \text{for RD}$$

i.e. $t = \frac{1}{2H} \Rightarrow t \sim 1.455\,\mathrm{s}\left(\frac{1\,\mathrm{MeV}}{T}\right)^2$

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Entropy:

2nd law of thermodynamics:

$$T dS = dE + p dV = d(\rho V) + p dV$$
$$= d\left[(\rho+p) V\right] - V dp$$

i.e.

$$dS = \frac{1}{T} d\left[(\rho+p) V\right] - \frac{V}{T} dp$$

but $dp = \frac{\rho+p}{T} dT$ from

$$\frac{dS}{dT dV} = \frac{dS}{dV dT}$$

$$\Rightarrow dS = d\left[\frac{(\rho+p)V}{T}\right]$$

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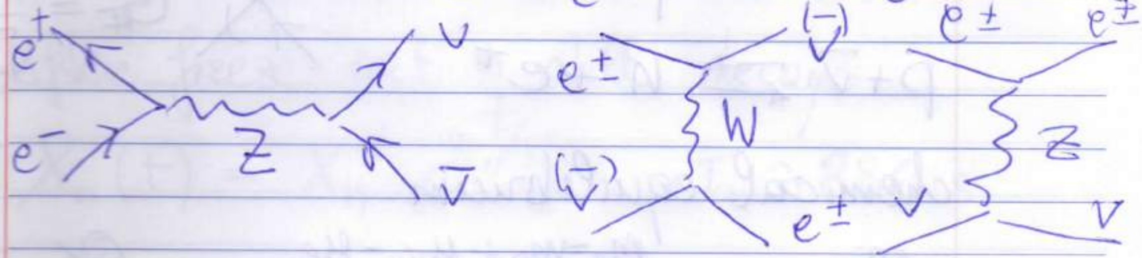
Neutrino decoupling:

$e^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_e$ $\quad$ $e^\pm + \overset{(-)}{\nu}_e \leftrightarrow e^\pm + \overset{(-)}{\nu}_e$



Fermi theory: $\sigma_{e\nu} \sim \frac{\alpha_W^2}{M_Z^4} s$

$\Rightarrow \langle \sigma_{e\nu} v \rangle \sim \frac{\alpha_W^2 T^2}{M_Z^4}$

$\Gamma \sim \sigma_{e\nu} \cdot n_e \propto \frac{\alpha_W^2}{M_Z^4} T^5$

to be compared with

$$H = \frac{\pi}{3} \sqrt{\frac{g_*}{10}} \frac{T^2}{\tilde{M}_p}$$

$$\frac{\Gamma}{H} \sim \frac{\alpha_W^2 T^3 \tilde{M}_P}{M_Z^4} \sim 1 \Rightarrow T \sim \left(\frac{M_Z^4}{\alpha_W^2 \tilde{M}_P} \right)^{1/3}$$

$M_Z = 91$ GeV $\quad \tilde{M}_p = 2.4 \cdot 10^{18}$ GeV $\quad \alpha_W \sim \frac{1}{29}$

$$T_{\nu\, dec} \sim \left(\frac{0.9^4 \cdot 29^2 \cdot 10}{2.4} \right)^{1/3} \text{MeV} \simeq 3 \text{ MeV}$$

Real 1.5 MeV

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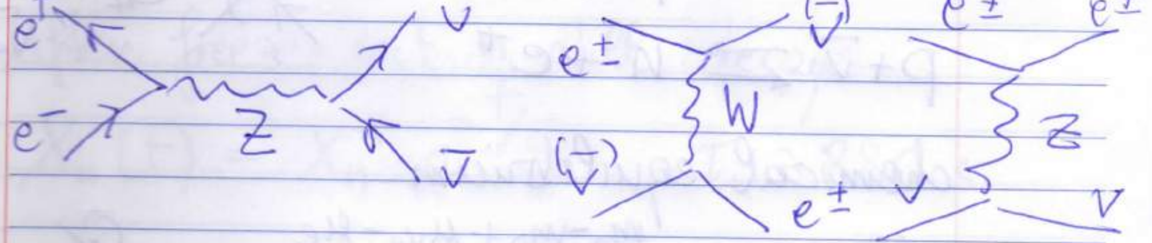
Neutrino decoupling:

$e^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_e \qquad e^\pm + \overset{(-)}{\nu_e} \leftrightarrow e^\pm + \overset{(-)}{\nu_e}$



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$\Rightarrow \langle \sigma_{e\nu} v \rangle \sim \frac{\alpha_W^2}{M_Z^4} T^2$

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$T_{\nu\,dec} \sim \left(\frac{0.9^4 \cdot 29^2 \cdot 10}{2.4} \right)^{1/3}$ MeV $\simeq 3$ MeV

Real 1.5 MeV

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Nucleons decoupling

$n + \nu \rightleftharpoons p + e^-$

$p + \bar{\nu} \rightleftharpoons n + e^+$

$G_F = \frac{\pi \alpha_W}{\sqrt{2} M_W^2}$

chemical equilibrium

$$\frac{n_n^{eq}}{n_p^{eq}} = e^{-\frac{m_n - m_p + \mu_{\nu_e} - \mu_e}{T}} \simeq e^{-Q/T}$$

with $Q \sim \Delta m \sim 1.29$ MeV

How long do neutrons track equilibrium?

$$\Gamma_{n\nu} \sim \frac{1+3g_A^2}{2\pi^3} G_F^2 \frac{45\zeta(5)}{2} T_\nu^5 \left(1 + \frac{7\pi^4}{45 \cdot 30}\left(\frac{Q}{T_\nu}\right) + \dots\right.$$

$$\sim 1.63\, s^{-1} \left(\frac{T_\nu}{Q}\right)^3 \left(0.25 + \frac{T_\nu}{Q}\right)^2 \sim H$$

$\Rightarrow T_{\nu *} \simeq 0.84$ MeV

i.e.

$$\frac{n_n^{eq}}{n_p^{eq}}(T_{\nu *}) \simeq e^{-1.536} \sim 0.21$$

highly sensitive to g_* & T_ν in $H(T)$!

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$$X_n^* \simeq \frac{n_n^*}{n_n^* + n_p^*} \simeq 0.158 + 0.005\,(N_\nu - 3)$$

After freeze-out n still decays

$$X_n(t) = X_n^*\, e^{-t/\tau_n} \qquad \tau_n \simeq 886\,s$$

Deuterium bottleneck:

$$p + n \rightleftarrows D\,\gamma$$

This is the starting point since all other heavier nuclei are sintetized from D

$$D + D \rightleftarrows {}^3He + n$$
$$\rightleftarrows T + p \qquad X_D = \frac{2n_D}{n_N}$$

very large n_γ suppresses the equilibrium value of D $\Rightarrow$ low D abundance!

$$\frac{n_\gamma}{n_D} \sim 10^{10} \frac{1}{\eta_{10} X_D}\left(\frac{B_D}{T}\right)^2 e^{-B_D/T}$$

$B_D \sim 2.23$ MeV

< 1 only @ $T \sim 0.06$ MeV !

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Boltzmann equation

How do we describe particles that are not in thermal equilibrium? We solve the corresponding Boltzmann equation, i.e.

$$\hat{\mathcal{L}}[f] = C[f]$$

Liouville operator Collision integral

In GR we can write $\hat{\mathcal{L}}$ in covariant form as

$$\hat{\mathcal{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$$

FRW metric in cartesian coordinates: only non-zero Γ's are

$$\Gamma^0_{ii} = -H g_{ii} \qquad \Gamma^i_{0i} = \Gamma^i_{i0} = H$$

So we obtain

$$\hat{\mathcal{L}} = p^0 \partial_0 - \vec{p} \cdot \vec{\nabla} - \Gamma^0_{ii} p^i p^i \frac{\partial}{\partial p^0} - 2 \Gamma^i_{0i} p^0 p^i \frac{\partial}{\partial p^i} =$$

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$$= p^0 \partial_0 - \bar{p}\cdot\vec{\nabla} - H\vec{p}^{\,2}\frac{\partial}{\partial p^0} - 2Hp^0\bar{p}\frac{\partial}{\partial \bar{p}}$$

Now assume that f depends only on $|\bar{p}|$ and t, i.e. $f(E,t)$

Then we have

$$\hat{\mathcal{L}} = E\frac{d}{dt} - H\vec{p}^{\,2}\frac{\partial}{\partial E}$$

i.e.

$$\frac{df}{dt} - H\vec{p}^{\,2}\frac{\partial f}{\partial E} = \frac{1}{E}C[f]$$

but now let us integrate in $\int d^3p\, g$ to obtain an equation for n:

$$g\int d^3p\left[\frac{df}{dt} - H\vec{p}^{\,2}\frac{\partial f}{\partial E}\right] =$$

$$= g\frac{4\pi}{(2\pi)^3}\int_m^\infty pE\,dE\left[\frac{df}{dt} - H\frac{\bar{p}^2}{E}\frac{\partial f}{\partial E}\right]$$

$$= \frac{\partial n}{\partial t} - H\frac{g}{2\pi^2}\int_m^\infty dE\left(\frac{\partial}{\partial E}(p^3 f) - 3p^2\frac{\partial p}{\partial E}f\right)$$

$$= \frac{\partial n}{\partial t} - \cancel{H\frac{g}{2\pi^2}\left[p^3 f\right]_m^\infty} + 3H\frac{g}{2\pi^2}\int_0^\infty p^2 dp\, f$$

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$$\Rightarrow \frac{1}{E}\int \hat{\mathcal{L}} \Rightarrow \frac{\partial n}{\partial t} + 3Hn$$

this describes the dilution of a number density due to the expansion

$$\frac{dn}{n} = -3\frac{da}{a} \quad \text{i.e. } n \propto a^{-3}$$

So we obtain the BE for n

$$\frac{\partial n_X}{\partial t} + 3H n_X = g\int \frac{d^3p}{E} C\left[f_X(E,t)\right]$$

The collision integral contains all the possible scattering processes that change the number of particles i.e. starting at lowest order in the coupling & phase space:

→ decays (if possible)

$X \to i + j$

→ annihilations $2 \to 2$

$X + \bar{X} \to i + j$

+ higher order processes, but <u>not</u> elastic scatterings!

e.g. for annihilations

$$C[f_x] = -\int d\Pi_{\bar{x}}\, d\Pi_i\, d\Pi_j\, \delta^4(p_x + p_{\bar{x}} - p_i - p_j)$$

$$\Big[|\mathcal{M}(x+\bar{x} \to ij)|^2 f_x f_{\bar{x}} (1 \pm f_i)(1 \pm f_j) - |\mathcal{M}(ij \to x\bar{x})|^2 f_i f_j (1 \pm f_x)(1 \pm f_{\bar{x}}) \Big]$$

where $d\Pi_i = \dfrac{g_i\, d^3 p_i}{(2\pi)^3\, 2E_i}$ invariant volume element!

Simplifications:

* CP conservation gives
$$|\mathcal{M}(x+\bar{x} \to ij)|^2 = |\mathcal{M}(ij \to x+\bar{x})|^2 = |\mathcal{M}|^2$$

* ij particles are in equilibrium, but diluted enough that we can use the Maxwell-Boltzmann distribution, then
$$f_i f_j \equiv f_x^{eq} f_{\bar{x}}^{eq}$$

Moreover neglect $\pm f_i$ terms w.r.t. 1!

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$$\Rightarrow \frac{dn_x}{dt} + 3Hn_x = -\frac{1}{2}\int d\Pi_x \, d\Pi_{\bar{x}} \left(f_x f_{\bar{x}} - f_x^{eq} f_{\bar{x}}^{eq}\right) \times$$

$$\times \underbrace{\int d\Pi_i \, d\Pi_j \, |\mathcal{M}|^2 \delta^4(p_x + p_{\bar{x}} - p_i - p_j)}_{\sigma(x+\bar{x} \to ij)\, v}$$

Moreover taking $f_x \propto f_x^{eq}$

$\hookrightarrow$ kinetic equilibrium assumption

i.e. $f_x = \frac{n_x}{n_x^{eq}} f_x^{eq}$ gives

$$\frac{dn_x}{dt} + 3Hn_x = -\langle \sigma(x+\bar{x} \to ij) v \rangle \times \left(n_x n_{\bar{x}} - n_x^{eq} n_{\bar{x}}^{eq}\right)$$

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Standard reheating

Inflaton decays at time Γ_φ^{-1} & produces a thermal bath of particles instantaneously, i.e.

$\Gamma_\varphi \sim H(T_{RH})$ gives

$$3\Gamma_\varphi^2 M_{pe}^2 \equiv \frac{\pi^2}{30} g_* T_{RH}^4$$

i.e. $$T_{RH} = \left(\frac{\pi^2}{90} g_*\right)^{-1/4} \left(\Gamma_\varphi M_{pe}\right)^{1/2}$$

$$T_{RH} = \left(\frac{3}{\pi}\right)^{1/2} \left(\frac{g_*}{10}\right)^{-1/4} \left(\Gamma_\varphi M_{pe}\right)^{1/2}$$