SAIFR - ICTP Summer School 2018 Early Universe Laura Covi

Exercise Sheet 4: Thermal Universe & Isothermal sphere

Exercise 1

Consider a stable particle decoupling from the thermal bath and compute its final abundance $Y_X = n_X/s$ by solving approximately the Boltzmann equation:

$$\frac{dY_X}{dx} = -\frac{s(x)\langle\sigma v\rangle_x}{xH(x)}(Y_X^2 - Y_{X,eq}^2), \tag{1}$$

where $x = \frac{m_X}{T}$, $s = s(m_X)x^{-3}$ is the entropy density and $H(T) = H(m_X)x^{-2}$ the Hubble parameter during radiation domination, i.e. $s(m_X) = \frac{2\pi^2}{45} g_S m_X^3$, $H(m_X) = \frac{\pi}{3}\left(\frac{g_\rho}{10}\right)^{1/2} \frac{m_X^2}{M_P}$ with $g_S \sim g_\rho$ counting the relativistic degrees of freedom. (This equation can be obtained from the original Boltzmann equation in the slides by doing the redefinitions above).

a) Estimate the temperature x_f, when the particle density starts deviating from equilibrium by the approximation

$$n_{X,eq}(x_f)\langle\sigma v\rangle_{x_f} = H(x_f)\ , \tag{2}$$

taking the temperature expansion

$$\langle\sigma v\rangle_x \simeq a + bx^{-1} + \dots\ , \tag{3}$$

and the non-relativistic Maxwell-Boltzmann distribution for the equilibrium number density, i.e. $n_{X,eq}(x) = \frac{g}{\pi^2} m_X^3 x^{-3} e^{-x}$.

b) Give an estimate of x_f for $a = \frac{\alpha^2}{m_X^2}, b = 0$ with $\alpha \sim 0.01$ and for the maximal annihilation cross-section allowed by the unitarity bound, i.e. $a = \frac{16\pi}{m_X^2}$.

c) Solve the Boltzmann equation after x_f by neglecting the term $Y_{X,eq}^2$ in the eq. (1) for $x > x_f$.

d) Match the solution to $Y_{X,eq}(x)$ at x_f and discuss the dependence of the final result from $Y_{X,eq}(x_f)$ and x_f.

Exercise 2

Assume that the (non-relativistic) Dark Matter velocity distribution is given by a Maxwell-Boltzmann distribution with constant velocity dispersion, i.e.

$$f(E) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-E/\sigma^2}$$

where $E = \frac{1}{2}m\vec{v}^2 + \Phi(r)$, ρ_0 is a normalization constant, $\Phi(r)$ the local gravitational potential and $m, \vec{v}$ the Dark Matter mass and velocity.

a) Determine then $\rho(\Phi(r))$ and insert this into the Poisson equation

$$\Delta\Phi(r) = 4\pi G_N \rho(r) \ .$$

to find a differential equation for the DM density $\rho(r)$:

b) Show that the corresponding form for $\rho(r)$ is

$$\rho(r) = \frac{\sigma^2}{2\pi m G_N} \frac{1}{r^2} \ .$$

c) To avoid the singularity at $r = 0$, where in any case the baryon density is not negligible, modify this solution to

$$\rho(r) = \frac{\rho_0}{1 + \left(\frac{r}{r_0}\right)^2}$$

where $r_0 = \frac{\sigma}{\rho_0} \frac{1}{\sqrt{2\pi m G_N}}$ is the typical radius where $\rho_b \sim \rho$. How does then the gravitational potential grow in the centre ?