Practical QCD at colliders

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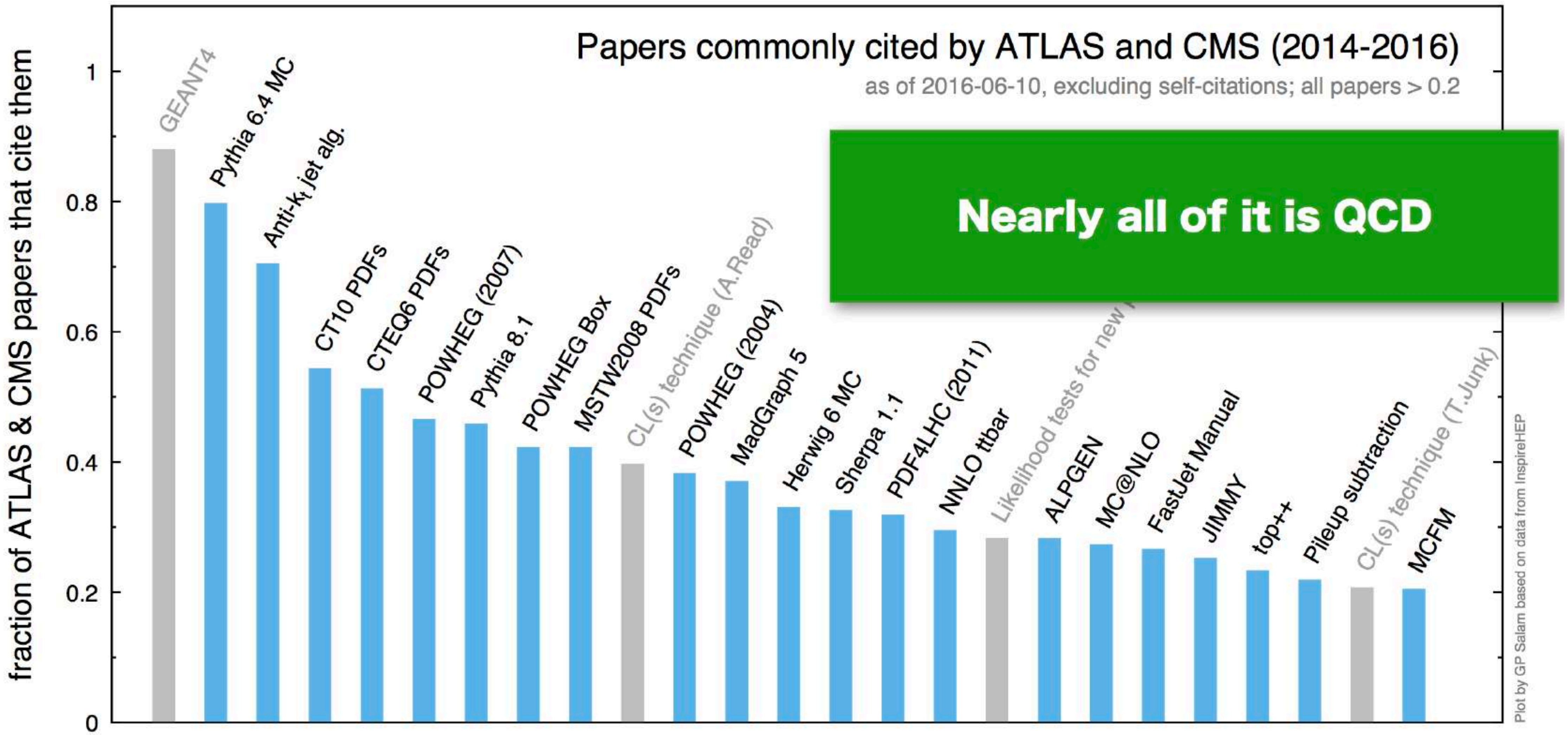
5th Lecture

Joint ICTP-SAIFR school on Particle Physics — June 2018

Today

Today I want to cover briefly two big areas:

- jets
- Monte Carlos



Both are ubiquitous at the LHC!

Where do jets enter ?

Essentially everywhere at colliders!

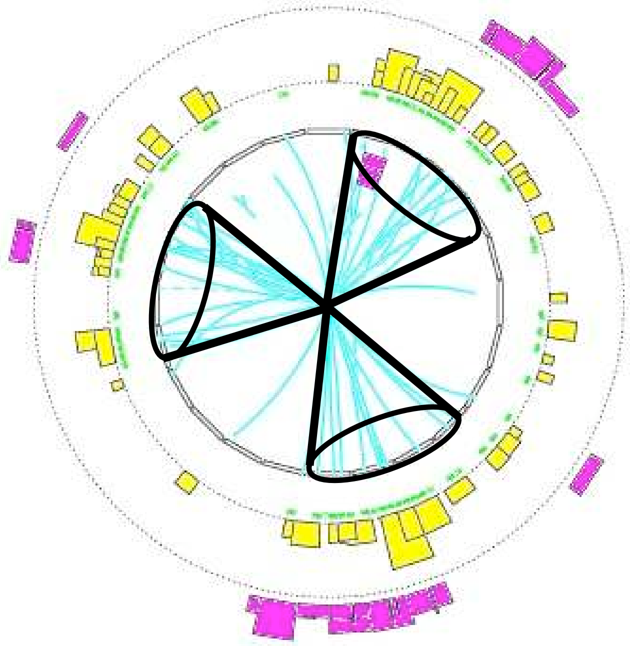
Jets are an essential tool for a variety of studies:

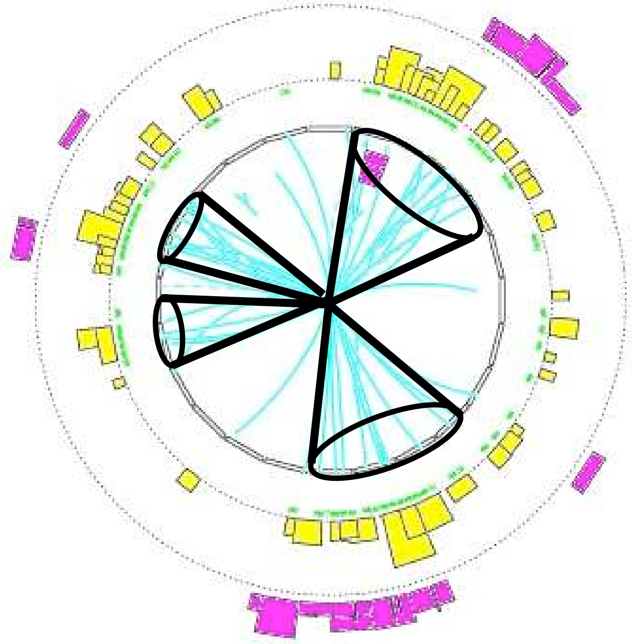
- top reconstruction
- mass measurements
- most Higgs and NP searches
- general tool to attribute structure to an event
- instrumental for QCD studies, e.g. inclusive-jet measurements
  $\Rightarrow$ important input for PDF determinations

Jets

Jets provide a way of projecting away the multiparticle dynamics of an event ⇒ leave a simple quasi-partonic picture of the hard scattering

The projection is fundamentally ambiguous ⇒ jet physics is a rich subject



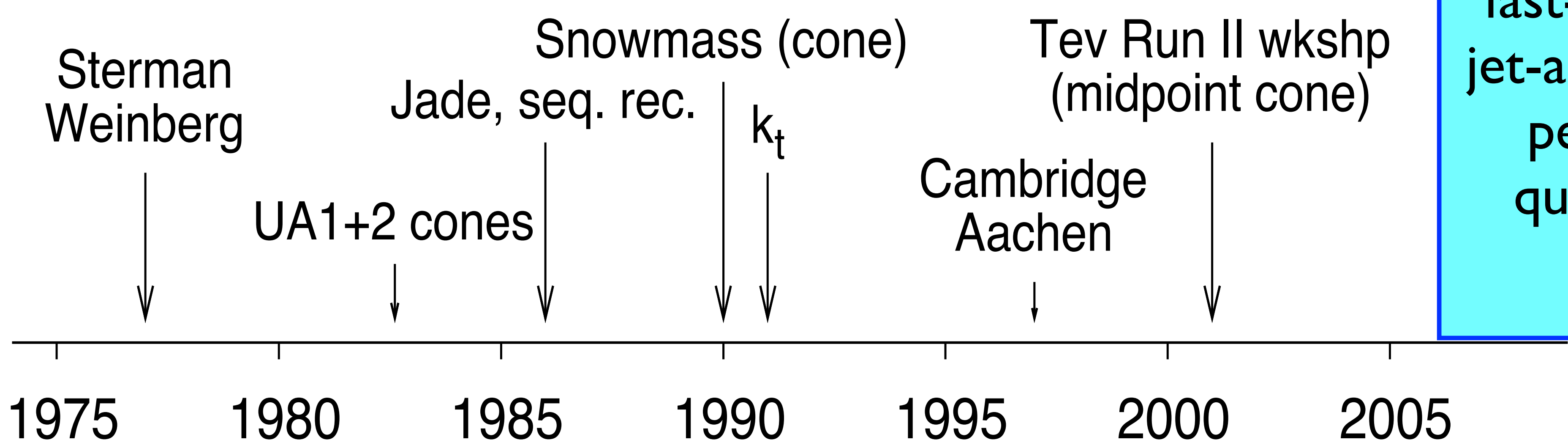


Ambiguities:

1) Which particles should belong to a same jet ?

2) How does recombine the particle momenta to give the jet-momentum?

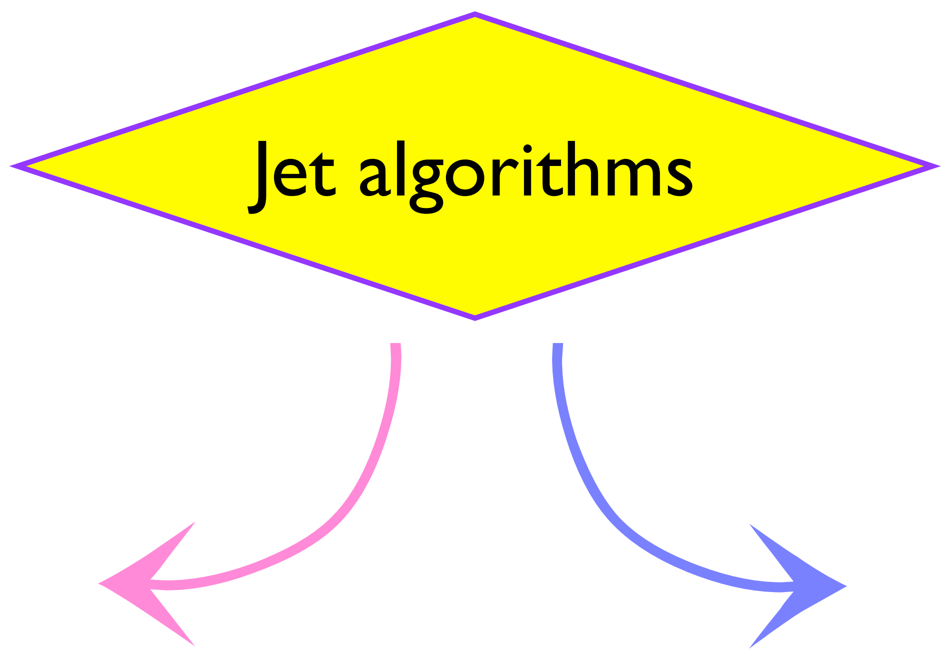
Jet developments



fast-kt, SISCone, anti-kt, jet-areas, jet-flavour, non-perturbative effects, quality measures, jet-substructure ...

Two broad classes of jet algorithms

Today many extensions of the original Sterman-Weinberg jets.
Modern jet-algorithms divided into two broad classes



Cone type	Sequential
(UA1,JetCLU, Midpoint, SISCone..)	(kt-type, Jade, Cambridge/Aachen...)
top down approach: cluster particles according to distance in coordinate-space	bottom up approach: cluster particles according to distance in momentum-space
Idea: put cones along dominant direction of energy flow	Idea: undo branchings occurred in the PT evolution

Jet requirements

Snowmass accord

FERMILAB-Conf-90/249-E
[E-741/CDF]

Toward a Standardization of Jet Definitions

Several important properties that should be met by a jet definition are [3]:

1. Simple to implement in an experimental analysis;
2. Simple to implement in the theoretical calculation;
3. Defined at any order of perturbation theory;
4. Yields finite cross section at any order of perturbation theory;
5. Yields a cross section that is relatively insensitive to hadronization.

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Other desirable properties:

- flexibility
- few parameters
- fast algorithms
- transparency
- ...

Inclusive k_t/Durham-algorithm

Catani et. al '92-'93; Ellis&Soper '93

Inclusive algorithm:

1. For any pair of final state particles i,j define the distance

$$d_{ij} = \frac{\Delta y_{ij}^2 + \Delta\phi_{ij}^2}{R^2} \min\{k_{ti}^2, k_{tj}^2\}$$

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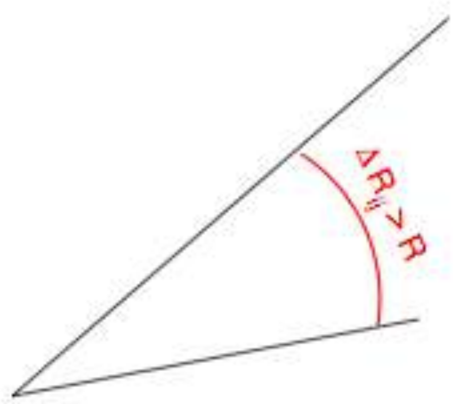
3. Find the smallest distance. If it is a d_{ij} recombine i and j into a new particle ($\Longrightarrow$ recombination scheme); if it is d_{iB} declare i to be a jet and remove it from the list of particles

NB: if $\Delta R_{ij} \equiv \Delta y_{ij}^2 + \Delta\phi_{ij}^2 < R$ then partons (ij) are always recombined, so R sets the minimal interjet angle



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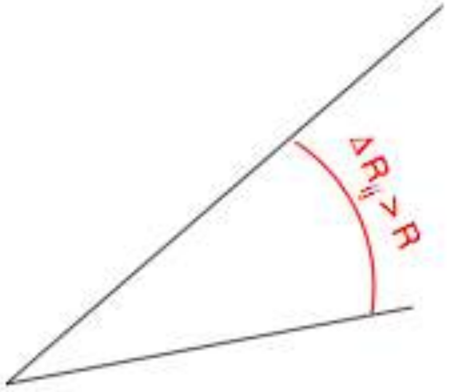

4. repeat the procedure until no particles are left

Exclusive k_t/Durham-algorithm

Inclusive algorithm gives a variable number of jets per event, according to the specific event topology

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Exclusive version: run the inclusive algorithm but stop when either

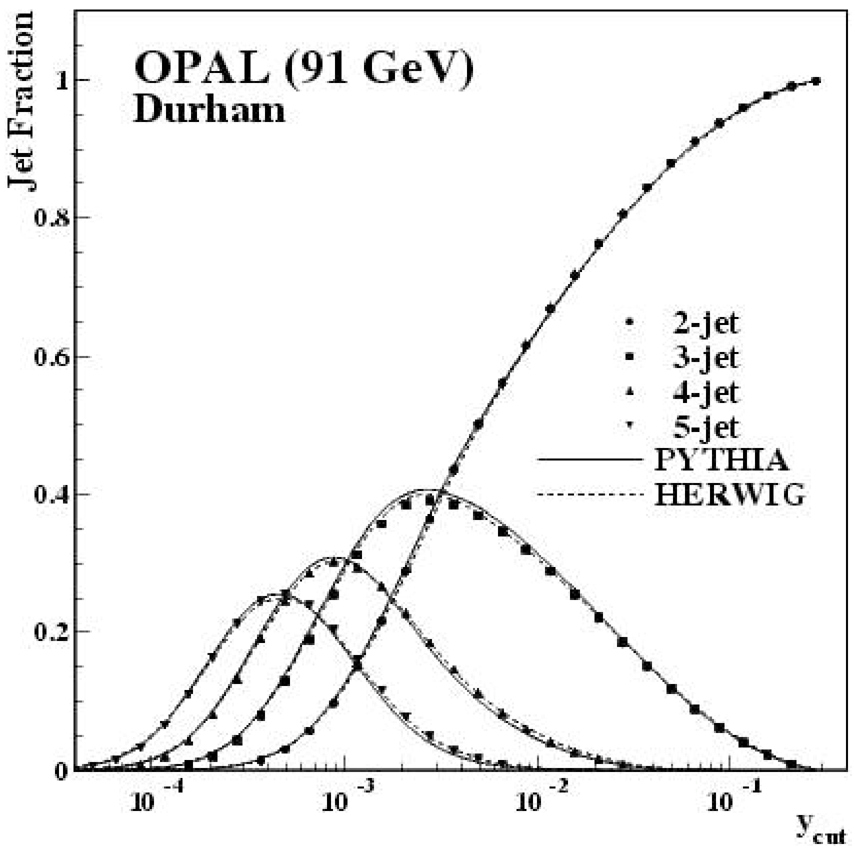
- all d_{ij}, $d_{iB} > d_{cut}$ or
- when reaching the desired number of jets n

k_t/Durham-algorithm in e^+e^-

k_t originally designed in e^+e^-, most widely used algorithm in e^+e^- (LEP)

$$y_{ij} = 2\min\{E_i^2, E_j^2\}\,(1 - \cos\theta_{ij}^2)$$

- can classify events using y_{23}, y_{34}, y_{45}, y_{56} ...
- resolution parameter related to minimum transverse momentum between jets



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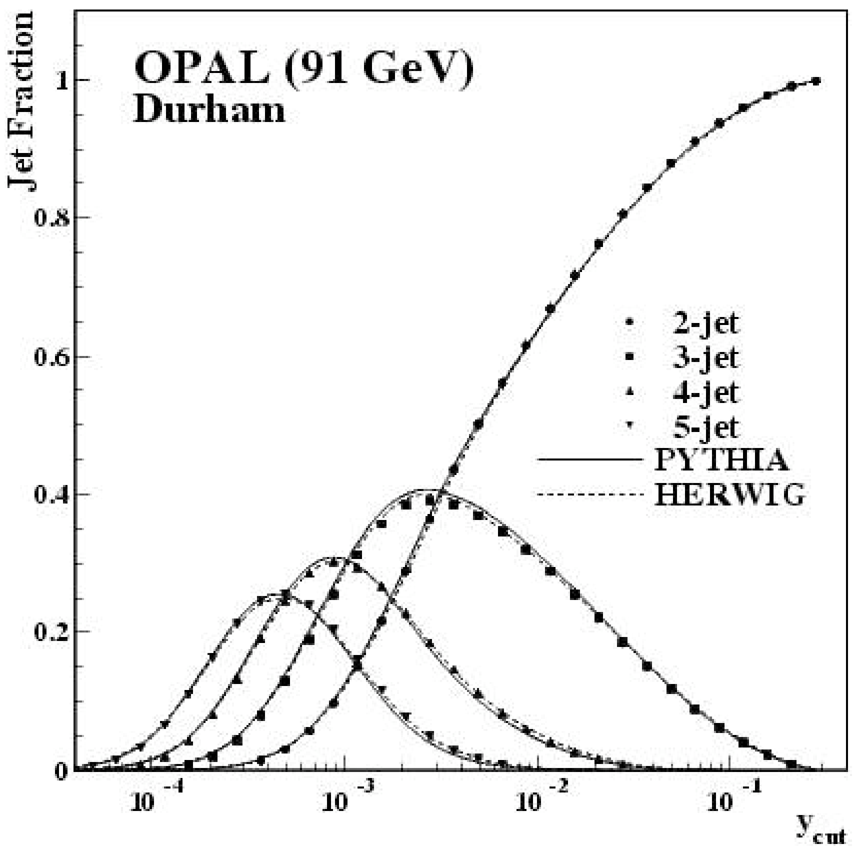

Satisfies fundamental requirements:

1. Collinear safe: collinear particles recombine early on
2. Infrared safe: soft particles do not influence the clustering sequence

⇒ collinear + infrared safety important: it means that cross-sections can be computed at higher order in pQCD (no divergences)!

The CA and the anti-k_t algorithm

The Cambridge/Aachen: sequential algorithm like k_t, but uses only angular properties to define the distance parameters

$$d_{ij} = \frac{\Delta R_{ij}^2}{R^2} \qquad d_{iB} = 1 \qquad \Delta R_{ij}^2 = (\phi_i - \phi_j)^2 + (y_i - y_j)^2$$

Dotshitzer et. al '97; Wobisch & Wengler '99

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The anti-kt algorithm: designed not to recombine soft particles together

$$d_{ij} = \min\{1/k_{ti}^2, 1/k_{tj}^2\}\Delta R_{ij}^2/R^2 \qquad d_{iB} = 1/k_{ti}^2$$

Cacciari, Salam, Soyez '08

Recombination schemes in e^+e^-

Given two massless momenta p_i and p_j how does one recombine them to build p_{ij} ? Several choices are possible.

Most common ones:

1. E-scheme $p_{ij} = p_i + p_j$
2. E_0-scheme $\vec{p}_{ij} = \vec{p}_i + \vec{p}_j$ $\quad E_{ij} = |\vec{p}_{ij}|$
3. P_0-scheme $E_{ij} = E_i + E_j$ $\quad \vec{p}_{ij} = \frac{E_{ij}}{|\vec{p}_i + \vec{p}_j|}(\vec{p}_i + \vec{p}_j)$

E_0/P_0-schemes give massless jets, along with the idea that the hard parton underlying the jet is massless

E-scheme give massive jets. Most used in recent analysis.

Recombination schemes in hh

Most common schemes:

- E-scheme (as in e+e-)
- p_t, p_t^2, E_t, E_t^2 schemes
  - first preprocessing, i.e. make particles massless, rescaling the 3-momentum in the E_t, E_t^2 schemes or the energy in the p_t, p_t^2 schemes
  - then define

$$p_{t,ij} = p_{t,i} + p_{t,j}$$

$$\phi_{ij} = \left(w_i \phi_i + w_j \phi_j\right) / \left(w_i + w_j\right)$$

$$y_{ij} = \left(w_i y_i + w_j y_j\right) / \left(w_i + w_j\right)$$

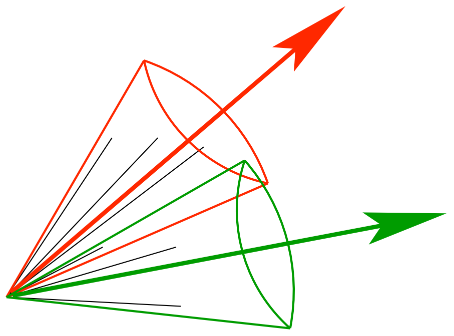
where the weights w_i are p_{ti} for the p_t, E_t schemes and p_{ti}^2 for the p_t^2 and E_t^2 schemes

NB: *a jet-algorithm is fully specified only once all parameters and the recombination scheme is specified too*

Cone algorithms

1. A particle i at rapidity and azimuthal angle $(y_i, \Phi_i) \subset$ cone C iff

$$\sqrt{(y_i - y_C)^2 + (\phi_i - \phi_C)^2} \leq R_{\text{cone}}$$



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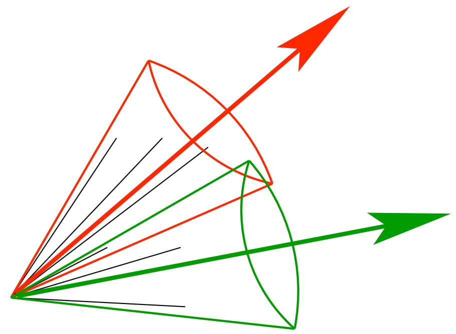
2. Define

$$\bar{y}_C \equiv \frac{\sum_{i \in C} y_i \cdot p_{T,i}}{\sum_{i \in C} p_{T,i}} \qquad \bar{\phi}_C \equiv \frac{\sum_{i \in C} \phi_i \cdot p_{T,i}}{\sum_{i \in C} p_{T,i}}$$



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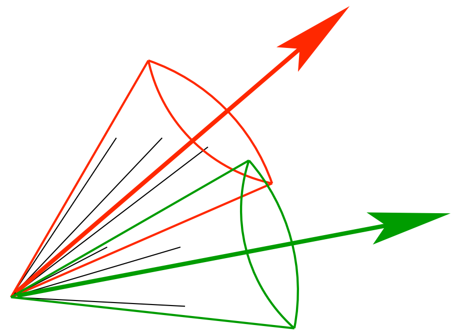

3. If weighted and geometrical averages coincide $(y_C, \phi_C) = (\bar{y}_C, \bar{\phi}_C)$
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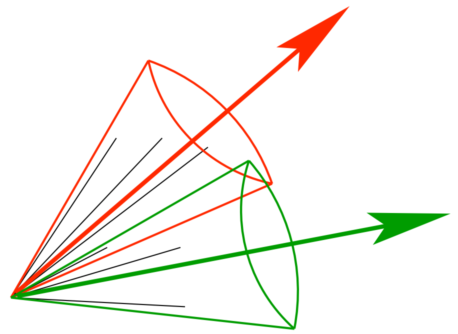

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4. Stable cones can overlap. Run a split-merge on overlapping jets: merge jets if they share more than an energy fraction f, else split them and assign the shared particles to the cone whose axis they are closer to.
   Remark: too small f (<0.5) creates hugh jets, not recommended

Cone algorithms

- The question is where does one start looking for stable cone ?
- The direction of these trial cones are called seeds
- Ideally, place seeds everywhere, so as not to miss any stable cone
- Practically, this is unfeasible. Speed of recombination grows fast with the number of seeds. So place only some seeds, e.g. at the (y, Φ)-location of particles.

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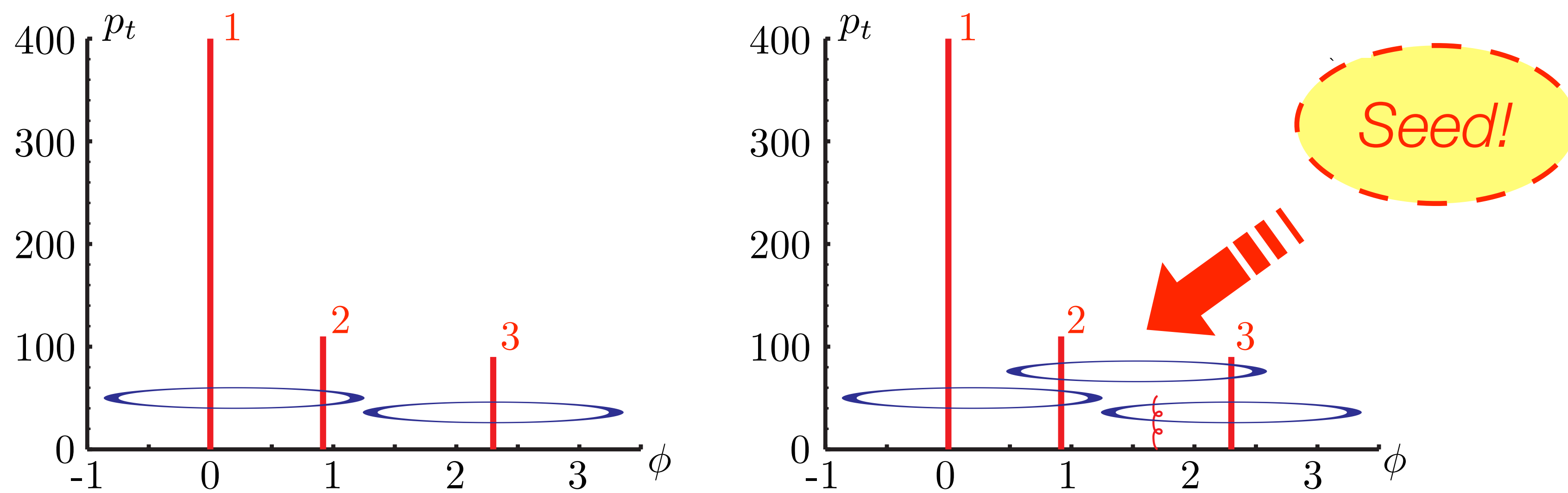
Seeds make cone algorithms infrared unsafe

Jets: infrared unsafety of cones



3 hard ⇒ 2 stable cones   3 hard + 1 soft ⇒ 3 stable cones

Soft emission changes the hard jets ⇒ algorithm is IR unsafe

<u>Midpoint algorithm</u>: take as seed position of emissions and midpoint between two emissions (postpones the infrared satefy problem)

Seedless cones

Solution:
use a seedless algorithm, i.e. consider all possible combinations of particles as candidate cones, so find all stable cones [⇒ jets]

Blazey '00

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The problem:
clustering time growth as $N2^N$. So for an event with 100 particles need 10^{17} ys to cluster the event ⇒ prohibitive beyond PT (N=4,5)

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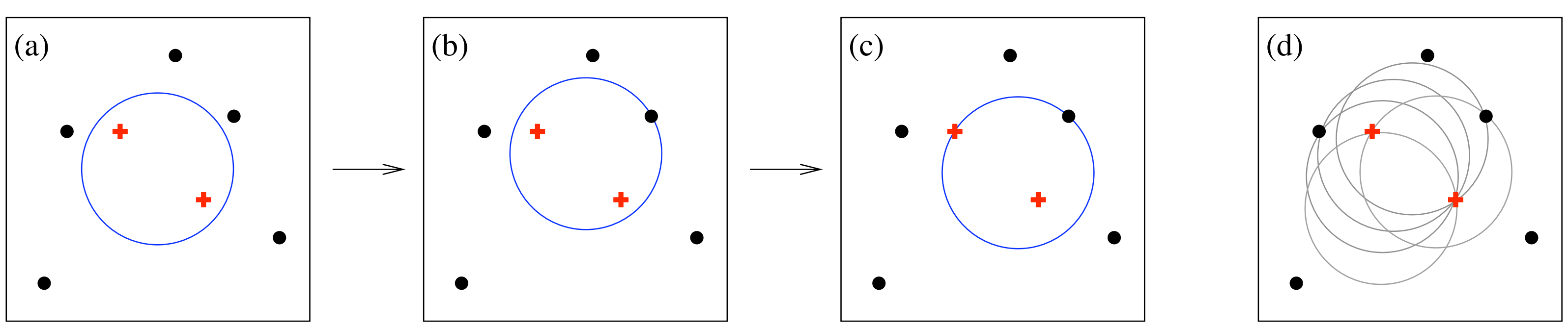
Better solution:

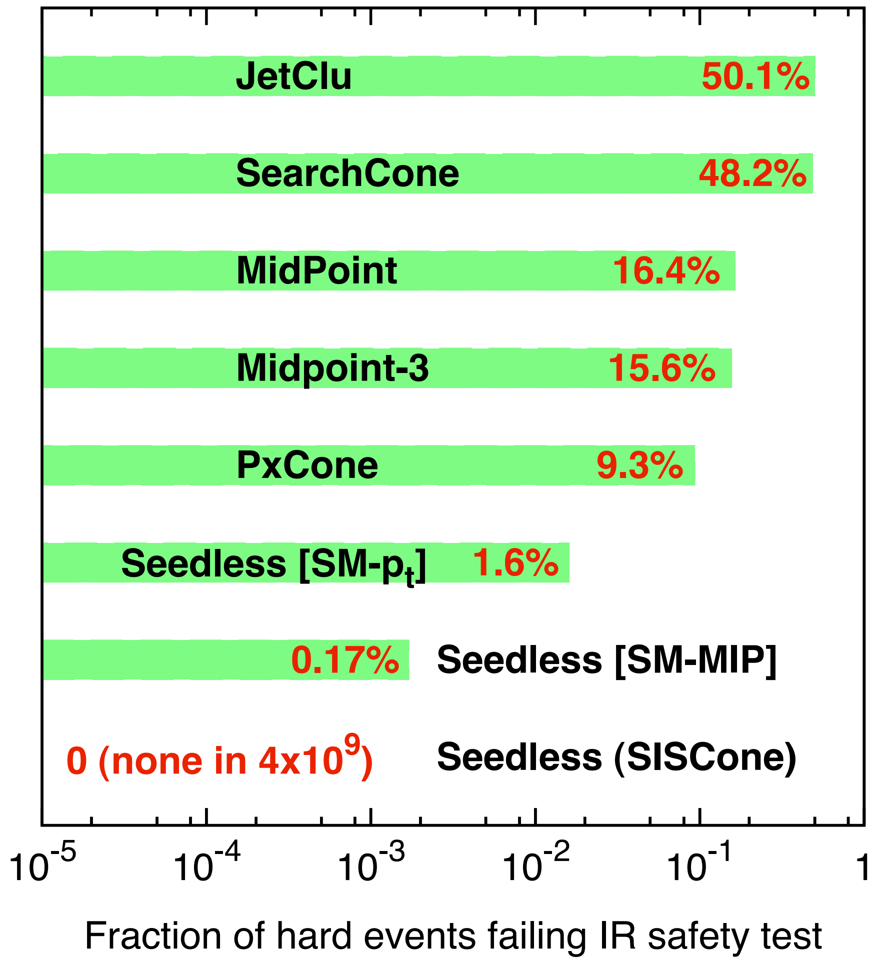
SISCone recasts the problem as a computational geometry problem, the identification of all distinct circular enclosures for points in 2D and finds a solution to that ⇒ $N^2 \ln N$ time IR safe algorithm

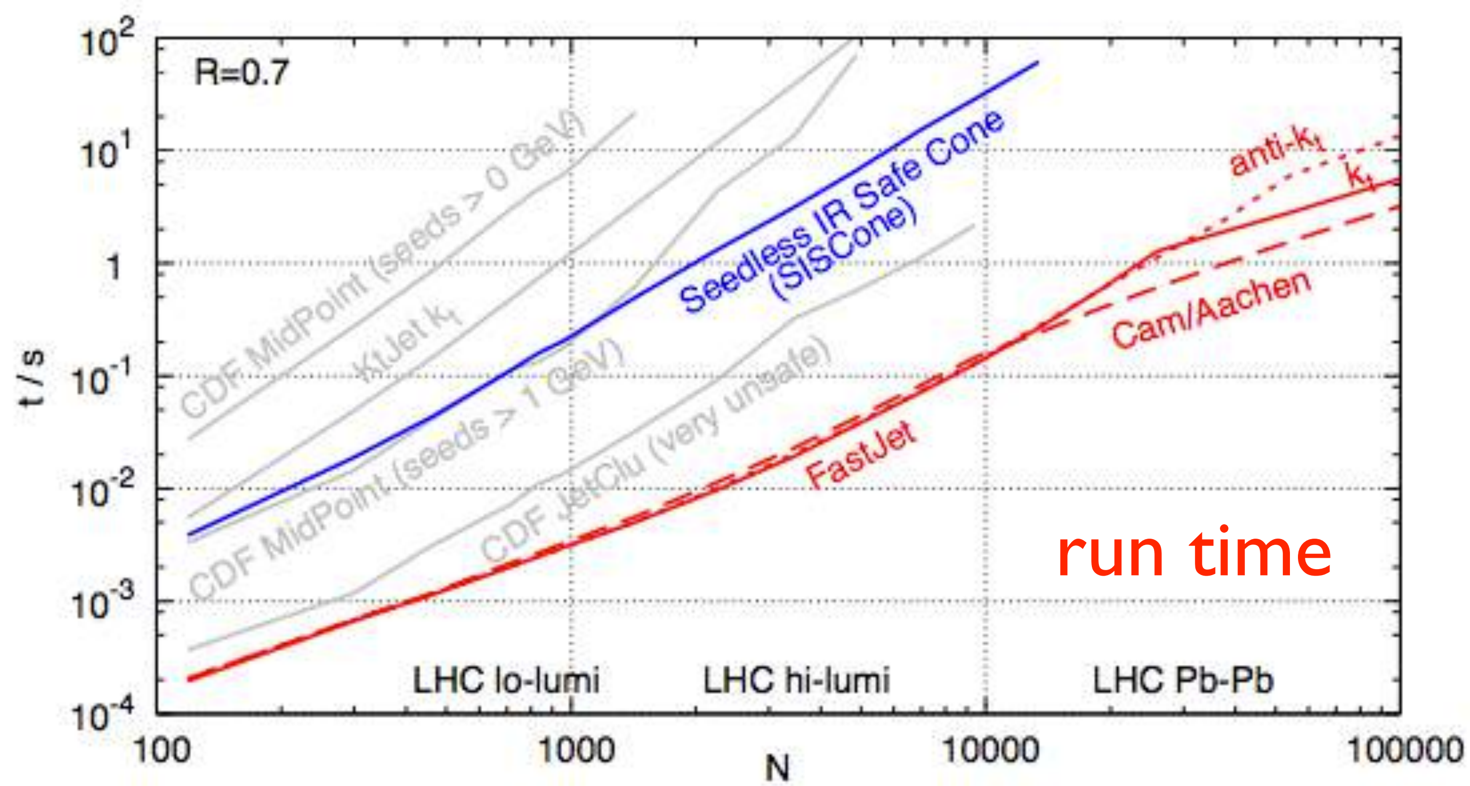


Salam, Soyez '07

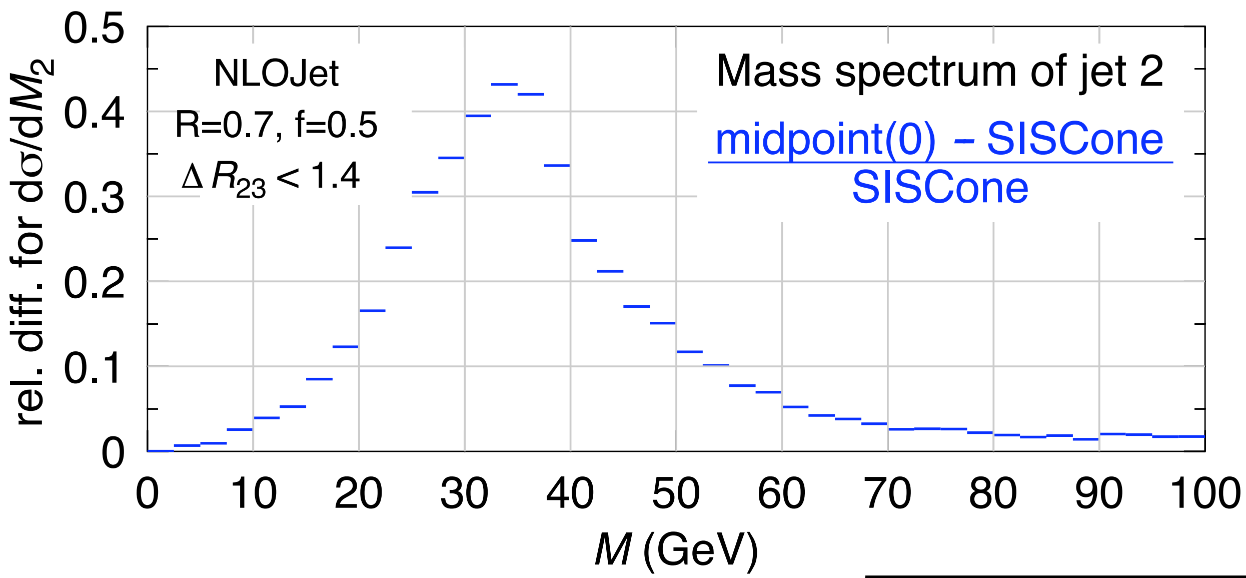
IR safety test & time comparisons

IR safety test: take a random hard event, add very soft emissions, count the number of times the hard jets change due to soft emissions





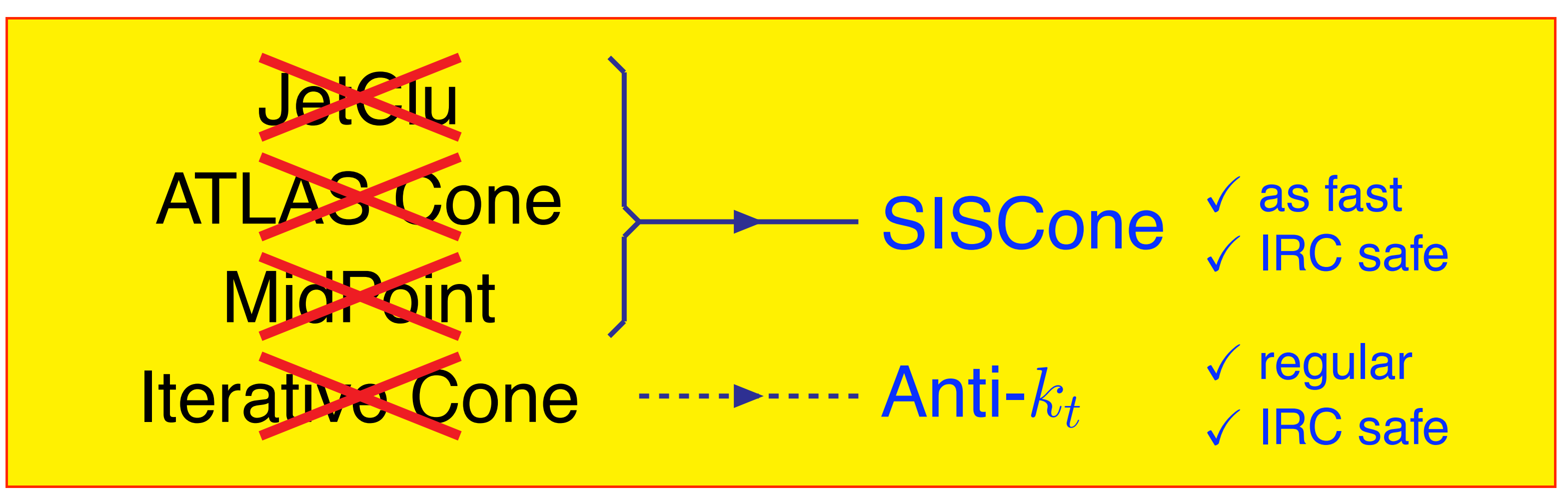
Physical impact of infrared unsafety



Up to 40% difference in mass spectrum

IR-unsafety is an issue at the LHC

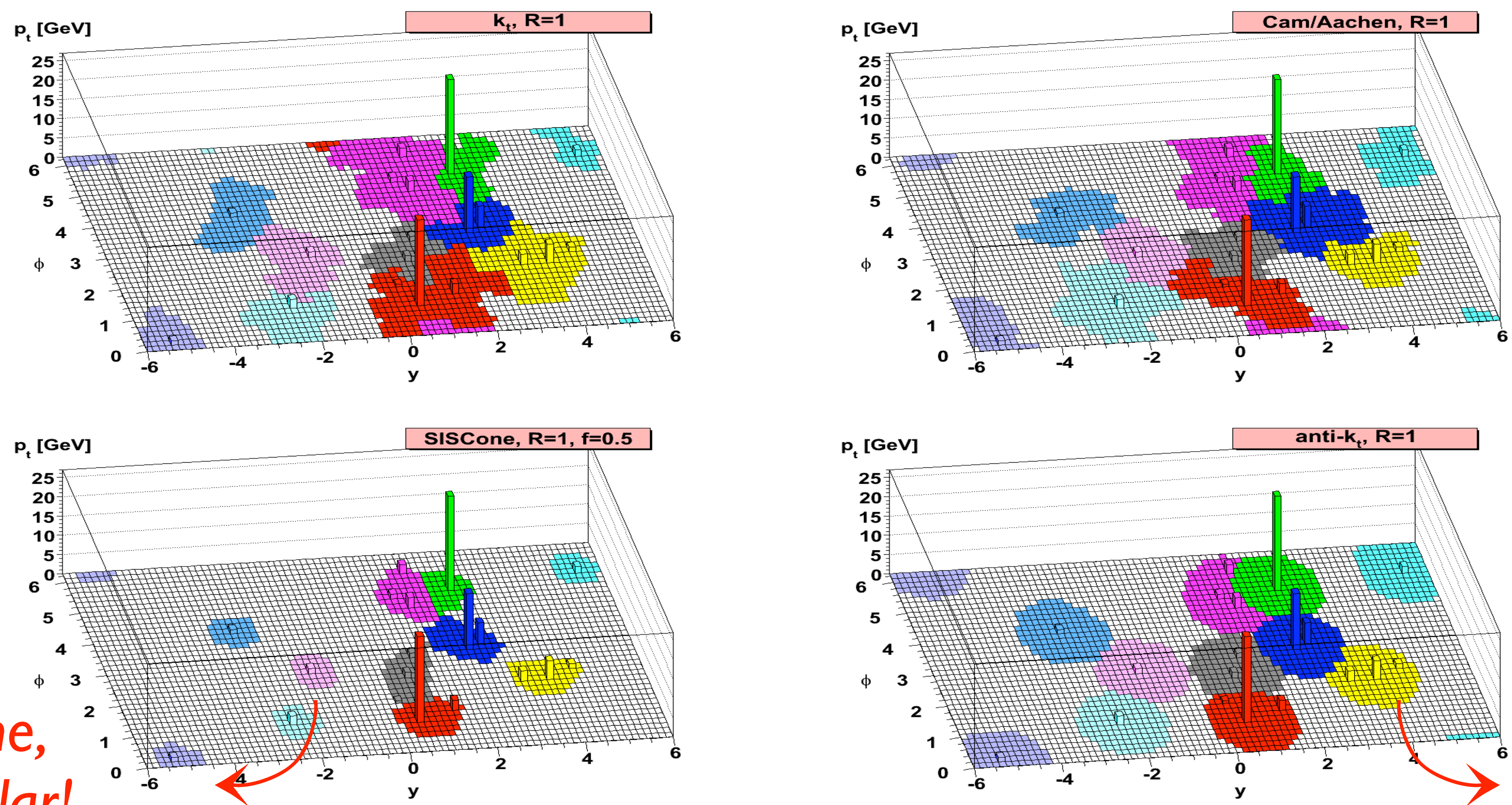
Observable	1st miss cones at	Last meaningful order
Inclusive jet cross section	NNLO	NLO
3 jet cross section	NLO	LO (NLO in NLOJet)
$W/Z/H$ + 2 jet cross sect.	NLO	LO (NLO in MCFM)
jet masses in 3 jets	LO	none (LO in NLOJet)



If you don't want theoretical efforts to be wasted!

Jet area

Given an infrared safe, fast jet-algorithm, can define the jet area A as follows: fill the event with an infinite number of infinitely soft emissions uniformly distributed in η-φ and make A proportional to the # of emissions clustered in the jet



NB: cone, not circular!

NB: new anti-kt

What jet areas are good for

jet-area ≡ catching area of the jet when adding soft emissions

⇒ use the jet area to formulate a simple area based subtraction of pile-up events

1. cluster particle with an IR safe jet algorithm
2. from *all jets* (most are pile-up ones) in the event define the median

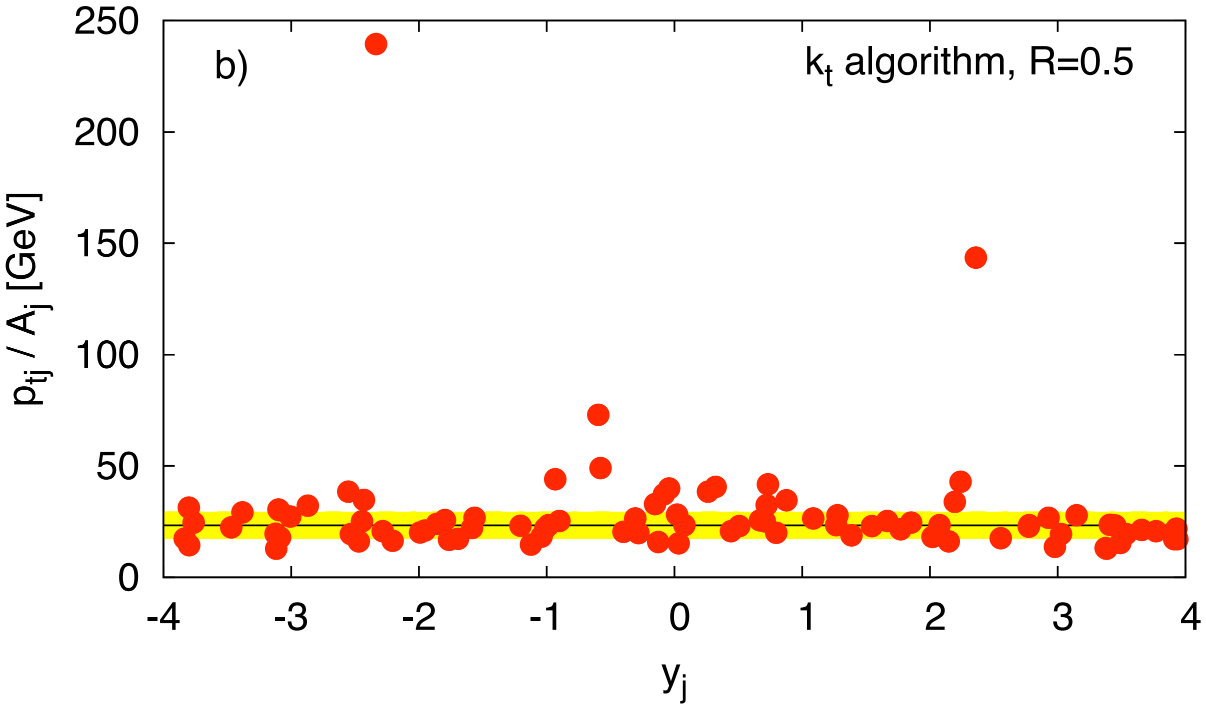
$$\rho = \frac{p_{t,j}}{A_j}$$

3. the median gives the typical p_t/A_j for a given event
4. use the median to subtract off dynamically the soft part of the soft events

$$p_j^{\mathrm{sub}} = p_j - A_j \rho$$

Pileup = generic p-p interaction (hard, soft, single-diffractive...) overlapping with hard scattering

Sample 2 TeV mass reconstruction



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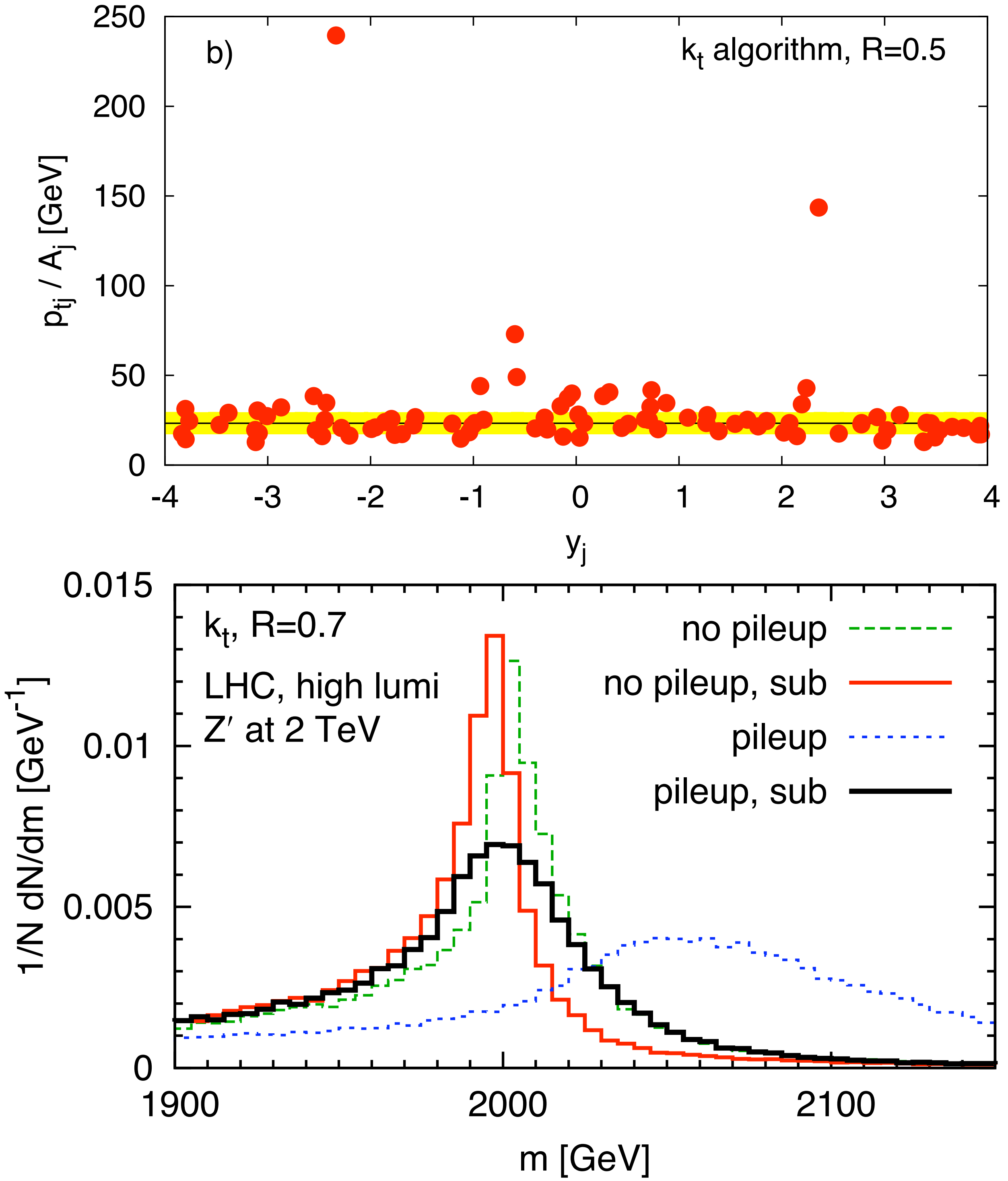


Cacciari et al. '07

Quality measures of jets

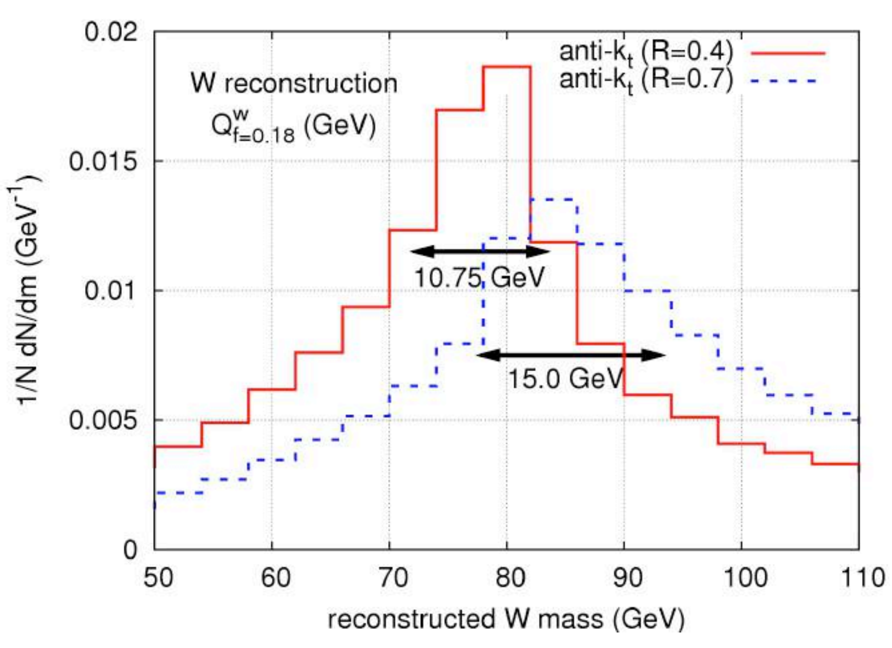
Suppose you are searching for a heavy state ($H \to gg$, $Z' \to qq$, ...)

The object is reconstructed through its decay products
$\Rightarrow$ *Which jet algorithm (JA) is best ? Does the choice of R matter?*

Define: $Q^w_f(JA, R) \equiv$ *width of the smallest mass window that contains a fraction f of the generated massive objects*

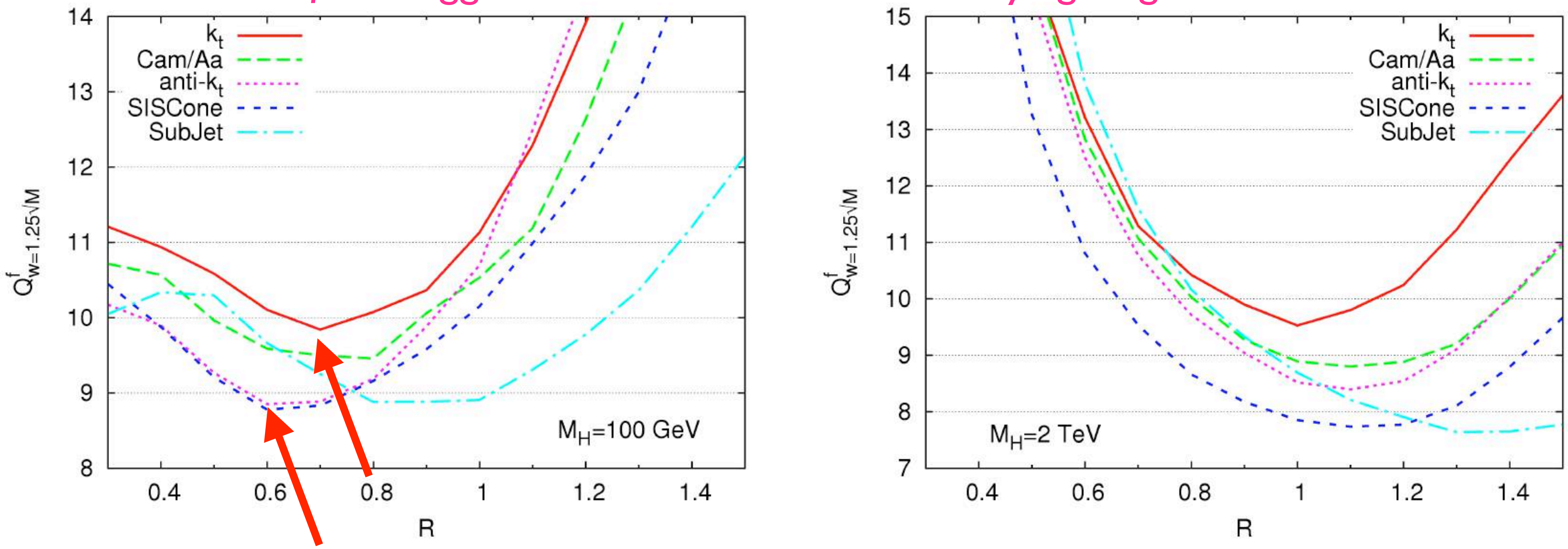
- good algo $\Leftrightarrow$ small $Q^w_f(JA, R)$
- ratios of $Q^w_f(JA,R)$: mapped to ratios of effective luminosity (with same $S/\sqrt{B}$)

$$\mathcal{L}_2 = \rho_{\mathcal{L}} \mathcal{L}_1 \qquad \rho_{\mathcal{L}} = \frac{Q_z^f(JA_2, R_2)}{Q_z^f(JA_1, R_1)}$$



Quality measures: sample results

NB: Here "fake Higgs" = narrow resonance decaying to gluons



- At 100GeV: use a Tevatron standard algo (k_t, R=0.7) instead of best choice (SISCone, R=0.6 $\Rightarrow$ lose $\rho_{\mathcal{L}} = 0.8$ in effective luminosity

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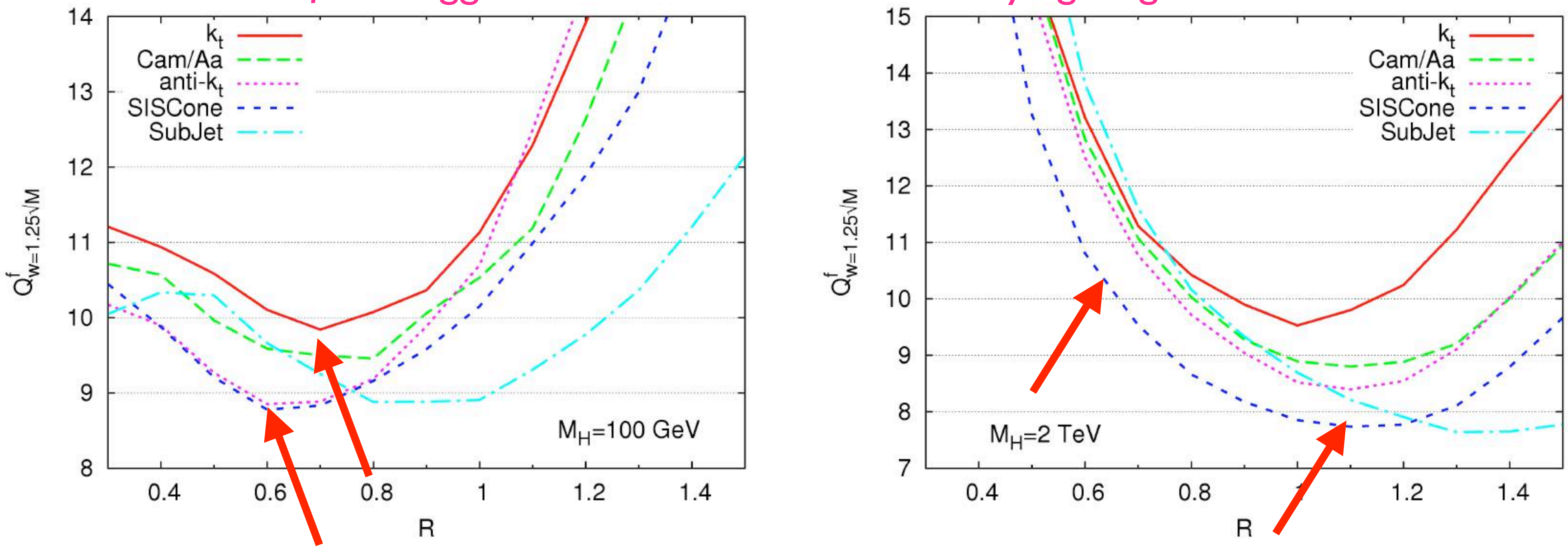

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- At 2 TeV: use $M_{Z'}$=100GeV best choice (or k_t) instead SIScone, R=1.1 $\Rightarrow$ lose $\rho_{\mathcal{L}} = 0.6$ in effective luminosity

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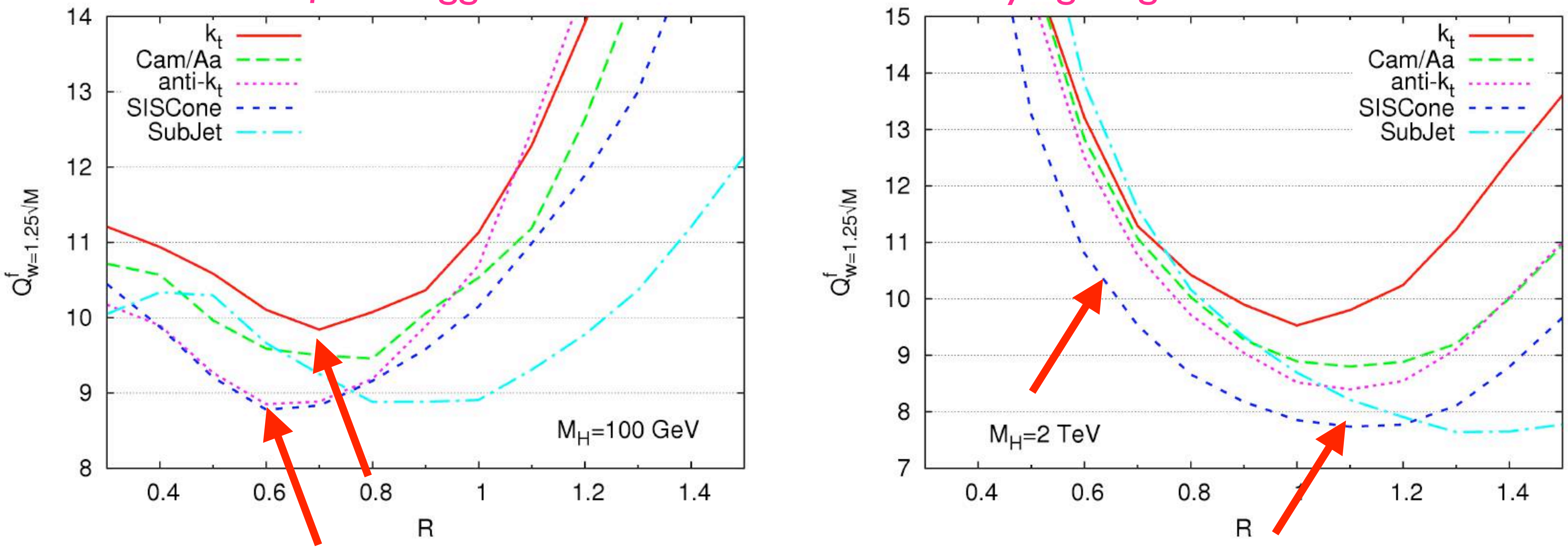

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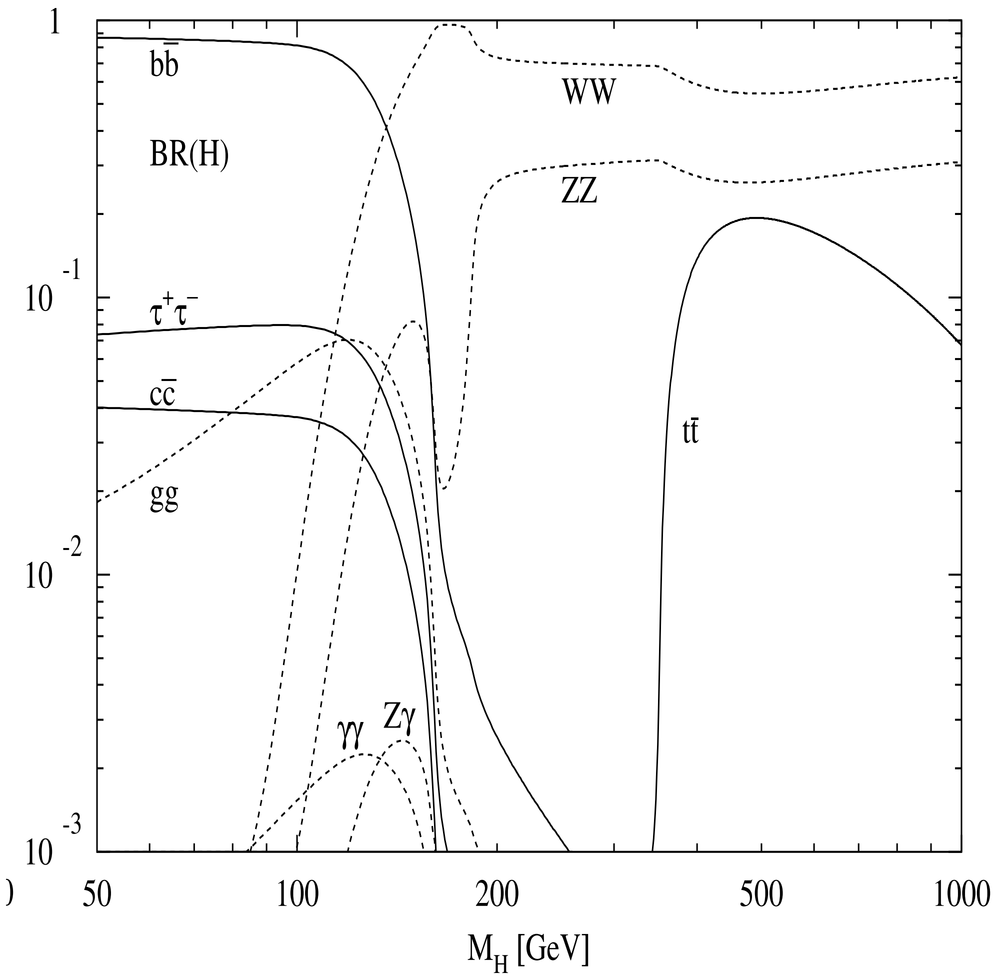
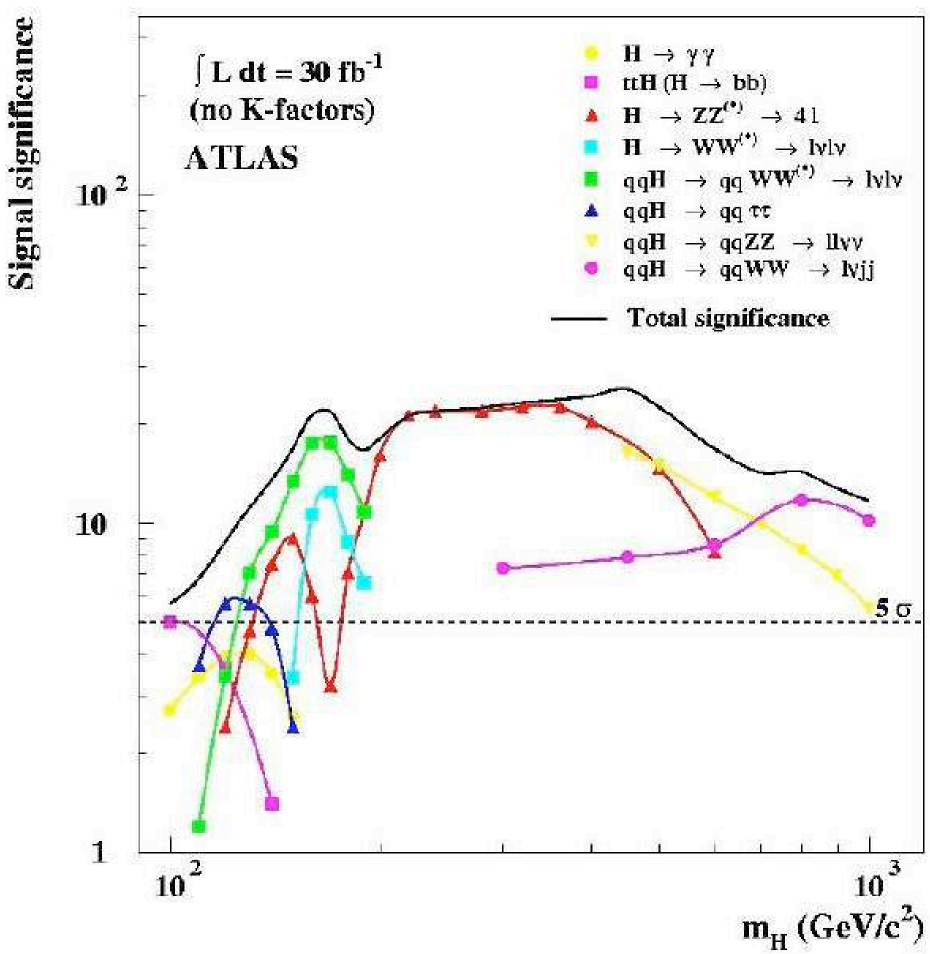
A good choice of jet-algorithm does matter!
Bad choice of algo $\Leftrightarrow$ lost in discrimination power!

Jet substructure: Z/W+ H (→bb)





⇒ Light Higgs hard: Higgs mainly produced in association with Z/W, decay H→bb is dominant, but overwhelmed by QCD backgrounds

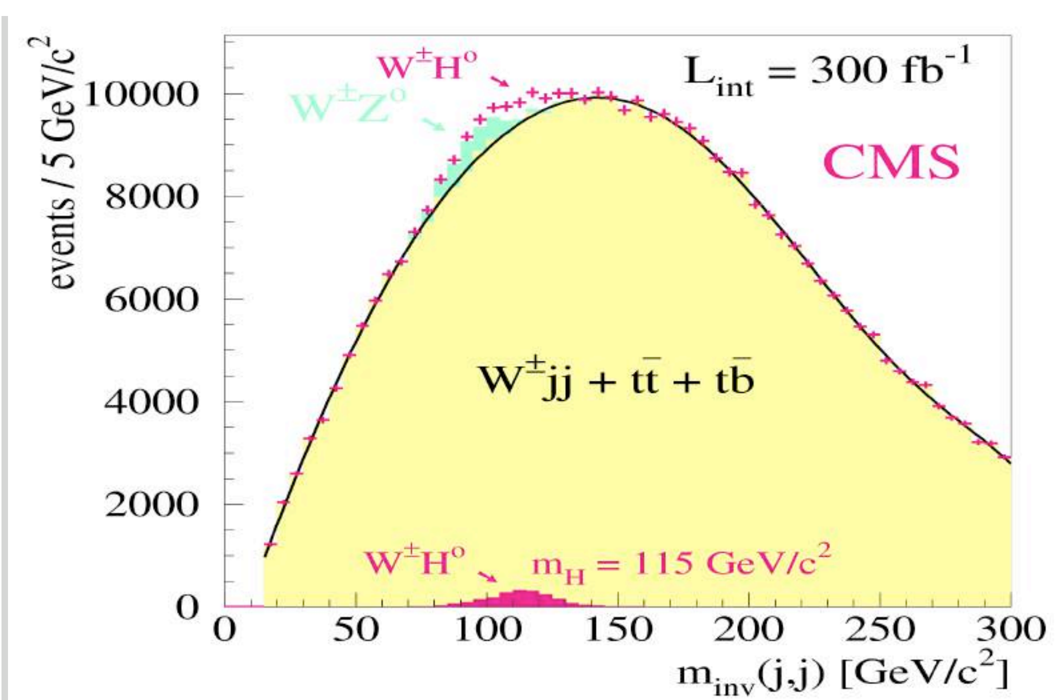
Z/W+ H (→bb)

Recall why searching for pp →WH(bb) is hard:

$\sigma(\mathrm{pp} \to WH(bb)) \sim \text{few pb}$ $\quad \sigma(\mathrm{pp} \to Wbb) \sim \text{few pb}$

$\sigma(\mathrm{pp} \to tt) \sim 800\mathrm{pb}$ $\sigma(\mathrm{pp} \to Wjj) \sim \text{few } 10^4\mathrm{pb}$ $\sigma(\mathrm{pp} \to bb) \sim 400\mathrm{pb}$

⇒ signal extraction very difficult



Conclusion [ATLAS TDR]:

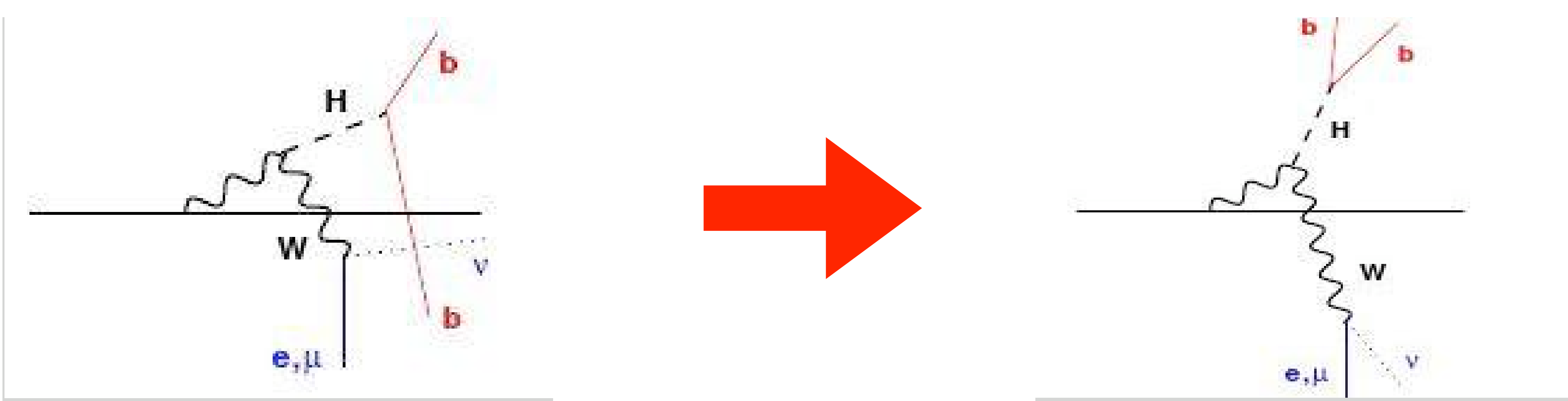
The extraction of a signal from H → bb decays in the WH channel will be very difficult at the LHC even under the most optimistic assumptions [...]

Z/W+ H (→bb) rescued ?

But ingenious suggestions open up to window of opportunity

Central idea: require high-p_T W and Higgs boson in the event

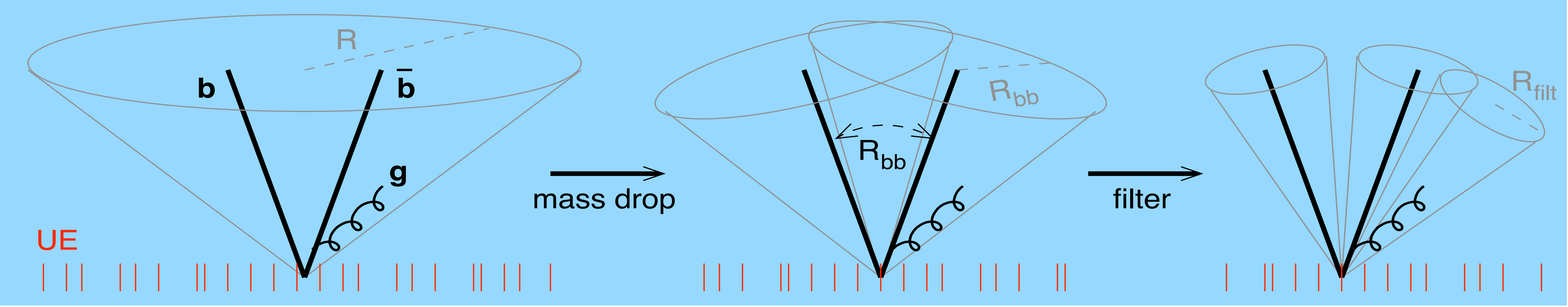
- leads to back-to-back events where two b-quarks are contained within the same jet
- high p_T reduces the signal but reduces the background much more
- improve acceptance and kinematic resolution



Z/W+ H (→bb) rescued ?

Then use a jet-algorithm geared to exploit the specific pattern of H → bb vs g → gg, q → gg

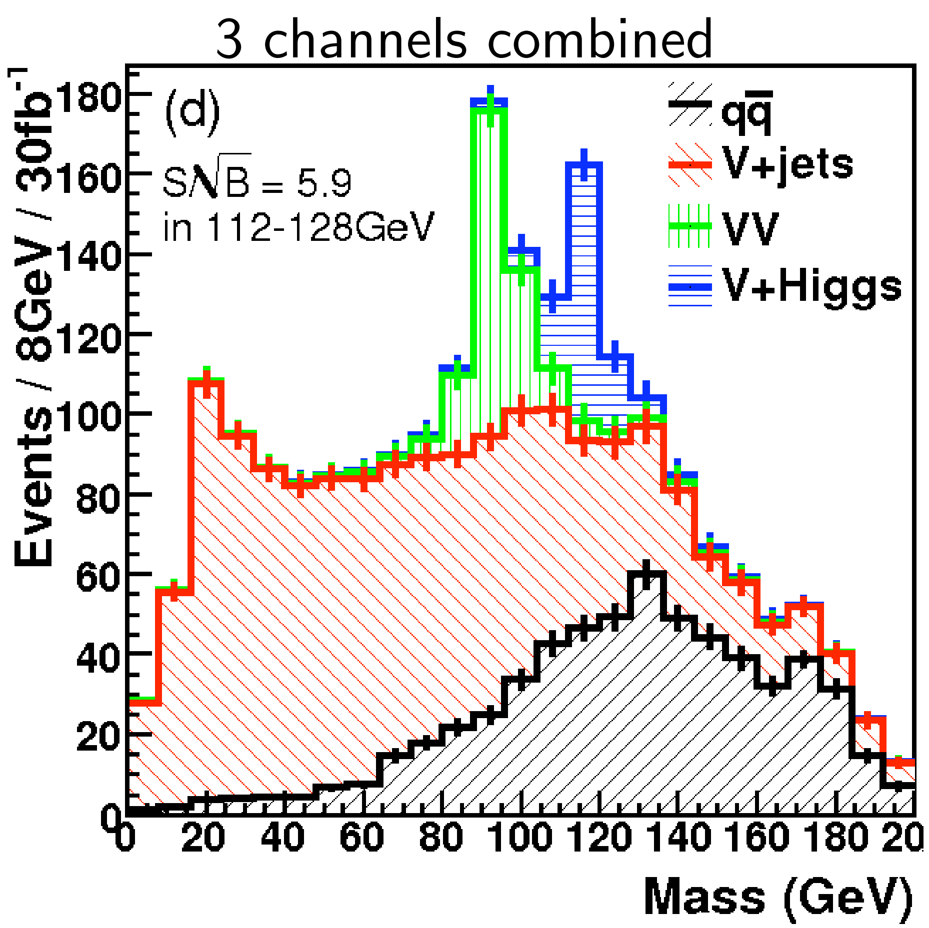
- QCD partons prefer soft emissions (hard → hard + soft)
- Higgs decay prefers symmetric splitting
- try to beat down contamination from underlying event
- try to capture most of the perturbative QCD radiation



1. cluster the event with e.g. CA algo and large-ish R
2. undo last recomb: large mass drop + symmetric + b tags
3. filter away the UE: take only the 3 hardest sub-jets

Z/W+ H (→bb) rescued ?

Mass of the three hardest sub-jets:



- with common & channel specific cuts: p_{tV}, p_{tH} > 200GeV , ...
- real/fake b-tag rate: 0.7/0.01
- NB: very neat peak for WZ (Z →bb) Important for calibration

Butterworth, Davison, Rubin, Salam '08

Suggested to have 5.9σ at 30 fb^{-1}. This and other works opened a new field of jet-substructure… (would be a whole new lecture)

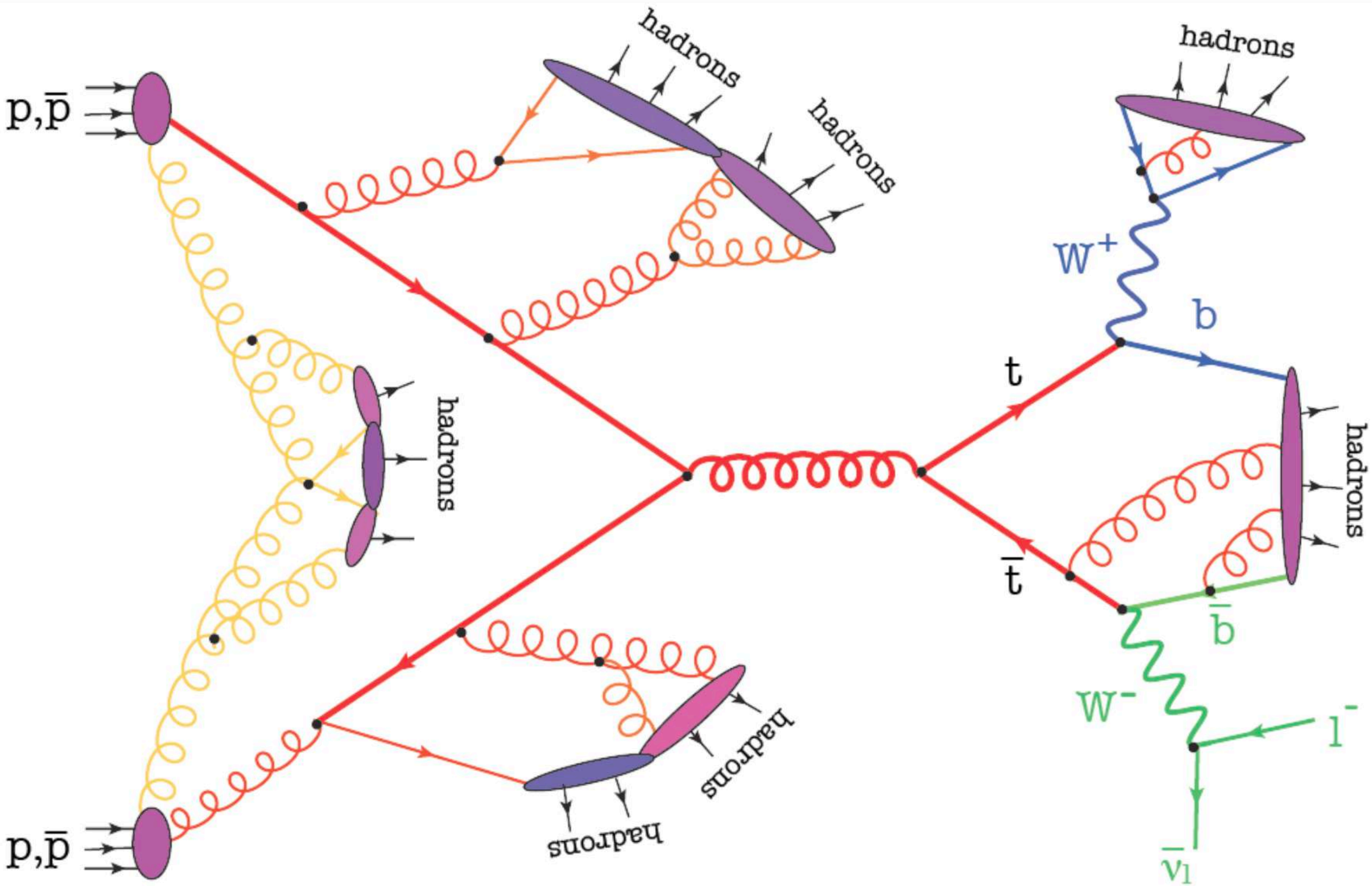
Recap on jets

- Two major jet classes: sequential (k_t, CA, ...) and cones (UA1, midpoint, ...)
- Jet algo is fully specified by: clustering + recombination + split merge or removal procedure + all parameters
- Standard cones based on seeds are IR unsafe
- SISCone is a infrared safe cone algorithm (no seeds)
- anti-kt a sequential algorithm used in most analyses now
- Using IR unsafe algorithms you can not use perturbative QCD calculations
- IR safe algorithm: sophisticated studies e.g. jet-area for pile-up subtraction
- Not all algorithms fare the same for BSM searches: quality measures
- Very active novel field of jet substructure [example of ZH(bb) with]

Parton shower & Monte Carlo methods

- today one can compute infrared-safe quantities at NLO, NNLO and very few ones at N^3LO. Progress is steady but somehow limited
- Fixed-order calculations involve few particles in the final state. This is quite different from "realistic" LHC events with hundreds of particles in the detectors
- we have also seen that sometimes large logs spoil the convergence of perturbative calculations, i.e. NLO (NNLO…) becomes unreliable
- now we adopt a different approach: we seek for an approximate result such that enhanced terms are taken into account to all orders
- this will lead to a 'parton shower' picture, which can be implemented in computer simulations, usually called Monte Carlo programs or event generators

Parton shower & Monte Carlo methods



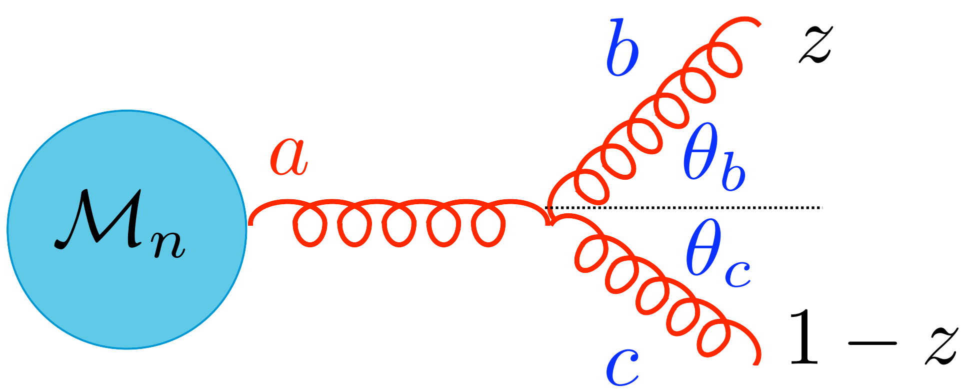
Credit: K. Hamilton

Parton branching: the time-like case

Assume: $p_b^2, p_c^2 \ll p_a^2 \equiv t$ (scale of the branching)

$$\begin{aligned} p_a &= (E_a, 0, 0, p_{az}) \\ p_b &= (E_b, 0, E_b \sin\theta_b, E_b \cos\theta_b) \\ p_c &= (E_c, 0, -E_c \sin\theta_c, E_c \cos\theta_c) \end{aligned}$$

Time-like branching: t > 0



Kinematics: $z = \frac{E_b}{E_a} = 1 - \frac{E_c}{E_a}$

small angle approx.

$$t = (p_b + p_c)^2 = 2E_b E_c (1 - \cos\theta) \sim z(1-z) E_a^2 \theta^2$$

$$E_b \sin\theta_b = E_c \sin\theta_c \Rightarrow z\theta_b \sim (1-z)\theta_c$$

$$\theta = \theta_b + \theta_c = \frac{\theta_b}{1-z} = \frac{\theta_c}{z}$$

Parton branching: gluon case

Three-gluon vertex:

$$V_{ggg} = ig_s f_{ABC} \epsilon_a^\mu \epsilon_b^\nu \epsilon_c^\rho \left(g_{\mu\nu}(p_a - p_b)_\rho + g_{\nu\rho}(p_b - p_c)_\mu + g_{\rho\mu}(p_c - p_a)_\nu \right)$$

Parton branching: gluon case

Three-gluon vertex:

$$V_{ggg} = ig_s f_{ABC} \epsilon_a^\mu \epsilon_b^\nu \epsilon_c^\rho \left(g_{\mu\nu}(p_a - p_b)_\rho + g_{\nu\rho}(p_b - p_c)_\mu + g_{\rho\mu}(p_c - p_a)_\nu \right)$$

Use: $\epsilon_i \cdot p_i = 0$ and $p_a + p_b + p_c = 0$

$$V_{ggg} = -2ig_s f_{ABC} \left[(\epsilon_a \cdot \epsilon_b)(\epsilon_c \cdot p_b) - (\epsilon_b \cdot \epsilon_c)(\epsilon_a \cdot p_b) - (\epsilon_c \cdot \epsilon_a)(\epsilon_b \cdot p_c) \right]$$

Parton branching: gluon case

Three-gluon vertex:

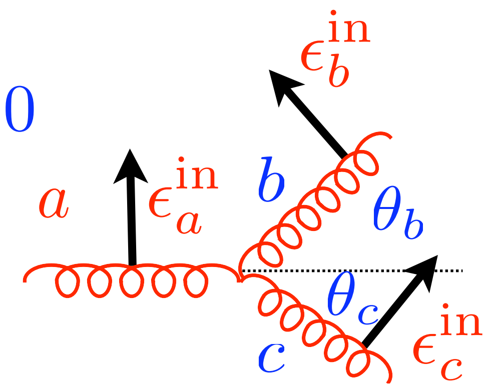
$$V_{ggg} = i g_s f_{ABC} \epsilon_a^\mu \epsilon_b^\nu \epsilon_c^\rho \left(g_{\mu\nu}(p_a - p_b)_\rho + g_{\nu\rho}(p_b - p_c)_\mu + g_{\rho\mu}(p_c - p_a)_\nu \right)$$

Use: $\epsilon_i \cdot p_i = 0$ and $p_a + p_b + p_c = 0$

$$V_{ggg} = -2 i g_s f_{ABC} \left[(\epsilon_a \cdot \epsilon_b)(\epsilon_c \cdot p_b) - (\epsilon_b \cdot \epsilon_c)(\epsilon_a \cdot p_b) - (\epsilon_c \cdot \epsilon_a)(\epsilon_b \cdot p_c) \right]$$

Branching: in a plane. Natural to split polarization vectors in $\epsilon_i^{\rm in}$ and $\epsilon_i^{\rm out}$

Properties: $\epsilon_i^{\rm in} \cdot \epsilon_j^{\rm in} = \epsilon_i^{\rm out} \cdot \epsilon_j^{\rm out} = -1 \qquad \epsilon_i^{\rm in} \cdot \epsilon_j^{\rm out} = \epsilon_i^{\rm out} \cdot p_j = 0$



Explicitly:

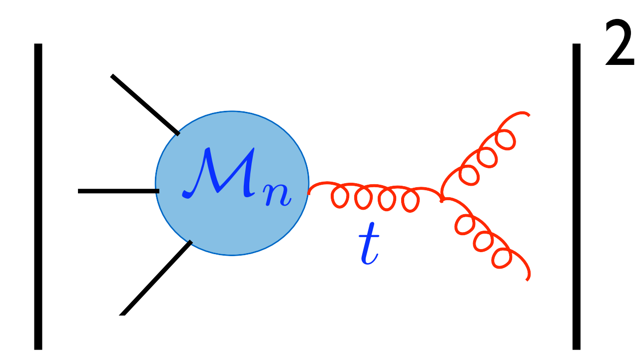
$$\begin{aligned} \epsilon_a^{\rm in} &= (0,0,1,0) \\ \epsilon_b^{\rm in} &= (0,0,\cos\theta_b, -\sin\theta_b) \\ \epsilon_c^{\rm in} &= (0,0,\cos\theta_c, \sin\theta_c) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \epsilon_a^{\rm in} \cdot p_b &= -E_b \theta_b = -z(1-z) E_a \theta \\ \epsilon_b^{\rm in} \cdot p_c &= E_c \theta = (1-z) E_a \theta \\ \epsilon_c^{\rm in} \cdot p_b &= -E_b \theta = -z E_a \theta \end{aligned}$$

Parton branching: the gluon case

Squared matrix element for n+1 partons becomes:



$$|\mathcal{M}_{n+1}|^2 = \frac{4g_s^2}{t} C_A F(z;\epsilon_a,\epsilon_b,\epsilon_c)|\mathcal{M}_n|^2$$

NB: one "t" cancels completely

a	b	c	F(z; ε$_a$, ε$_b$, ε$_c$)
in	in	in	(1-z)/z + z/(1-z) + z(1-z)
in	out	out	z(1-z)
out	in	out	(1-z)/z
out	out	in	z/(1-z)

Averaging over incoming and summing over outgoing pol. we get

$$C_A\langle F\rangle = \hat{P}_{gg} = C_a\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$$

The gluon case: remarks

Soft singularities ($z \to 0,1$) are associated to soft gluon in the plane of the branching

Correlation between plane of branching and polarization of incoming gluon: take polarization of gluon at an angle φ to the plane then

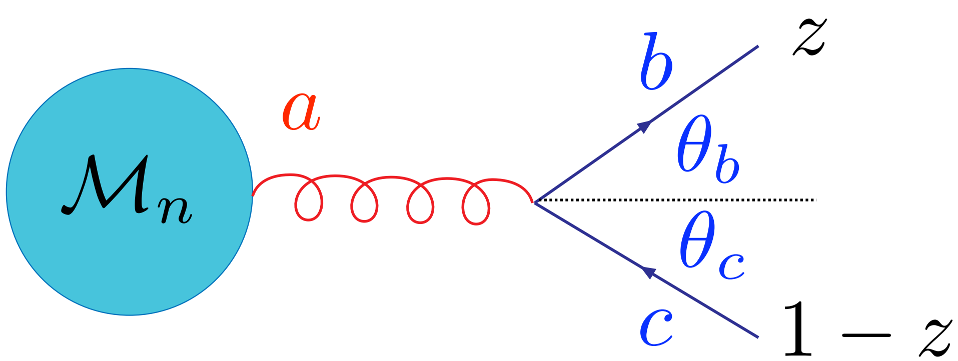
$$
\begin{aligned}
F_\phi &= \sum_{b,c} |\cos\phi\,\mathcal{M}(\epsilon_a^{\rm in},\epsilon_c,\epsilon_c) + \sin\phi\,\mathcal{M}(\epsilon_a^{\rm out},\epsilon_c,\epsilon_c)|^2 \\
&= \underbrace{\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)}_{\text{unpolarized result}} + \underbrace{z(1-z)\cos 2\phi}_{\text{correction}}
\end{aligned}
$$

Correction favors polarization of branching gluon in the branching plane, but is weak (no soft enhancements)

Gluon splitting to quarks

Similarly start from 3-particle vertex:

$$V_{q\bar{q}g} = -ig_s\, t^A_{bc}\, \bar{u}(p_b)\, \gamma_\mu \epsilon^\mu_a\, v(p_c)$$



Fix a representation of the Dirac algebra (called Dirac rep.):

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -1_{2\times2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma_i \\ -\sigma_i & 0_{2\times2} \end{pmatrix}$$

To first order in the small angles the spinors are

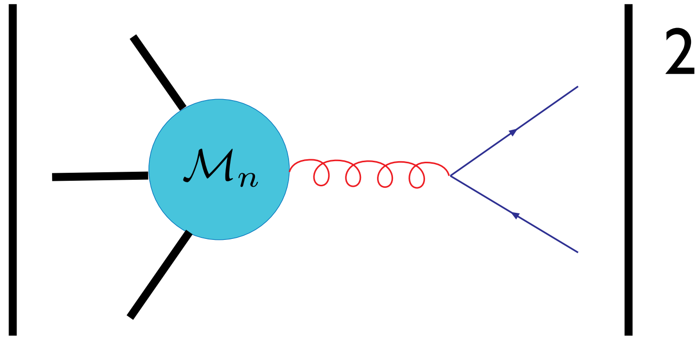
$$\frac{u_+(p_b)}{\sqrt{E_b}} = \begin{pmatrix} 1 \\ \theta_b/2 \\ 1 \\ \theta_b/2 \end{pmatrix} \quad \frac{u_-(p_b)}{\sqrt{E_b}} = \begin{pmatrix} \theta_b/2 \\ -1 \\ \theta_b/2 \\ -1 \end{pmatrix} \quad \frac{v_+(p_c)}{\sqrt{E_c}} = i\begin{pmatrix} -\theta_c/2 \\ -1 \\ \theta_c/2 \\ 1 \end{pmatrix} \quad \frac{v_-(p_c)}{\sqrt{E_c}} = i\begin{pmatrix} -1 \\ \theta_c/2 \\ -1 \\ \theta_c/2 \end{pmatrix}$$

Gluon splitting to quarks

Explicitly we find e.g.

$$-ig_s\bar{u}_+(p_b)\gamma_\mu\epsilon_a^{\text{in},\mu}v_-(p_c) = \sqrt{E_bE_c}(\theta_b-\theta_c) = \sqrt{z(1-z)}(1-2z)E_a\theta$$

Similarly to before define



a	b	c	F(z; ϵ_a, λ_b, λ_c)
in	$\pm$	$\mp$	$(1-2z)^2$
out	$\pm$	$\mp$	1

$$|\mathcal{M}_{n+1}|^2 = \frac{4g_s^2}{t}T_RF(z;\epsilon_a,\lambda_b,\lambda_c)|\mathcal{M}_n|^2$$

Averaged splitting function: $T_R\langle F\rangle \equiv \hat{P}_{qg}(z) = T_R\left[z^2+(1-z)^2\right]$

Angular correlation: $F_\phi = z^2+(1-z)^2-2z(1-z)\cos 2\phi$ (more important)

Last case: quark emitting gluon

Similarly to the two previous cases one obtains

$$|\mathcal{M}_{n+1}|^2 = \frac{4g_s^2}{t} C_F F(z;\lambda_a,\lambda_b,\epsilon_c)|\mathcal{M}_n|^2$$

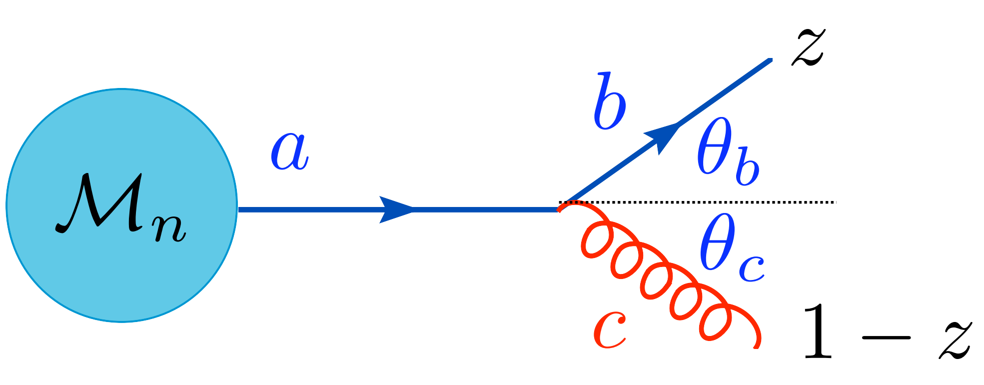
a	b	c	F(z; λ_a, λ_b, ε_c)
$\pm$	$\pm$	in	$(1+z)^2/(1-z)$
$\pm$	$\pm$	out	$1-z$

NB: helicity of the quark does not change during the branching

Averaged splitting function: $C_F\langle F\rangle \equiv \hat{P}_{qq}(z) = C_F\frac{1+z^2}{1-z}$

Angular correlation:

$$F_\phi = \frac{1+z^2}{1-z} + \frac{2z}{1-z}\cos 2\phi$$



Phase space

n-particle phase space (without branching): $d\Phi_n = d\Phi_{n-1}\frac{d^3p_a}{(2\pi)^3 2E_a}$

(n+1)-particle phase space (with branching): $d\Phi_{n+1} = d\Phi_{n-1}\frac{d^3p_b}{(2\pi)^3 2E_b}\frac{d^3p_c}{(2\pi)^3 2E_c}$

At fixed pb: $d^3p_a = d^3p_c \quad \Rightarrow \quad d\Phi_{n+1} = d\Phi_n \frac{d^3p_b}{(2\pi)^3 2E_b}\frac{E_a}{E_c}$

$$\begin{aligned} d^3p_b &= p_b^2 dp_b \ \sin\theta d\theta \ d\phi \\ &\sim E_b^2 dE_b \ \theta d\theta \ d\phi \\ &= E_a^3 z^2 dz \ \frac{dt}{2z(1-z)E_a^2} \ d\phi \end{aligned}$$

$\Rightarrow$

$$d\Phi_{n+1} = d\Phi_n \frac{1}{4(2\pi)^3} dt\, dz\, d\phi$$

N-particle cross-section: $d\sigma_n = \mathcal{F}\,|\mathcal{M}_n|^2\, d\Phi_n$ with $|\mathcal{M}_{n+1}|^2 = \frac{4g_s^2}{t} CF |\mathcal{M}_n|^2$

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz\, d\phi \frac{\alpha_s}{2\pi} C\,F$$

Azimuthal averaged result

Averaging over azimuthal angles:

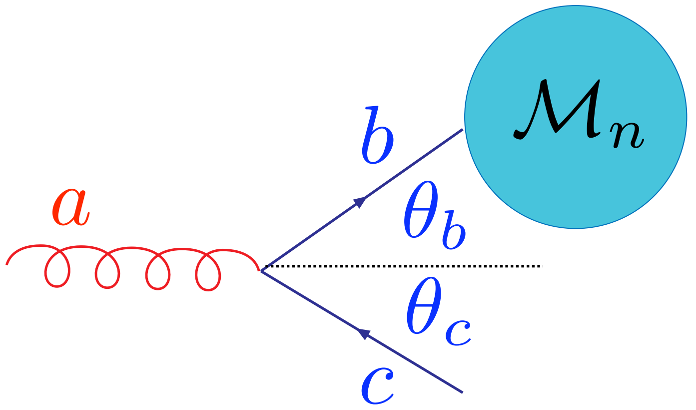
$$\int \frac{d\phi}{2\pi} C\, F = \hat{P}_{ba}(z)$$

The evolution equation becomes:

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz\, \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z)$$

Space-like branching

What are the modifications needed if an incoming parton splits?



The kinematics changes: $p_a^2, p_c^2 \ll |p_b^2| \equiv t$

Space-like branching: t < 0

Small angle approximation: $t = E_a E_c \theta_c^2$ (verify)

(n+1) particle phase space becomes: $d\Phi_{n+1} = d\Phi_n \frac{1}{4(2\pi)^3} dt \, \frac{dz}{z} \, d\phi$

The additional "z" is compensated by the different flux-factor, we find

Space-like or time-like braching: $d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \, \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z)$

Perturbative evolution

In exact analogy with what done for parton densities inside hadrons we want to write an evolution equation for the probability to have partons at the momentum scale Q^2 with momentum fraction z during PT branching

Start from DGLAP equation

$$Q^2 \frac{\partial f(x,Q^2)}{\partial Q^2} = \int_0^1 dz \frac{\alpha_s}{2\pi} \hat{P}(z) \left(\frac{1}{z} f\left(\frac{x}{z}, Q^2\right) - f(x,Q^2) \right)$$

Introduce a cut-off to regulate divergences

$$Q^2 \frac{\partial f(x,Q^2)}{\partial Q^2} = \int_0^{1-\epsilon} \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, Q^2\right) - f(x,Q^2) \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} \hat{P}(z)$$

Introduce a Sudakov form factor

$$\Delta(Q^2) = exp\left\{ - \int_{Q_0}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} \hat{P}(z) \right\}$$

Perturbative evolution

The DGLAP equation becomes

$$Q^2 \frac{\partial}{\partial Q^2}\left(\frac{f(x,Q^2)}{\Delta(Q^2)}\right) = \frac{1}{\Delta(Q^2)}\int_0^{1-\epsilon}\frac{dz}{z}\frac{\alpha_s}{2\pi}\hat{P}(z) f\left(\frac{x}{z},Q^2\right)$$

Integrating the above equation one gets

$$f(x,Q^2) = f(x,Q_0^2)\frac{\Delta(Q^2)}{\Delta(Q_0^2)} + \int_{Q^0}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\frac{\Delta(Q^2)}{\Delta(k_\perp^2)}\int_0^{1-\epsilon}\frac{dz}{z}\frac{\alpha_s}{2\pi}\hat{P}(z) f\left(\frac{x}{z},k_\perp^2\right)$$

This equation has a probabilistic interpretation

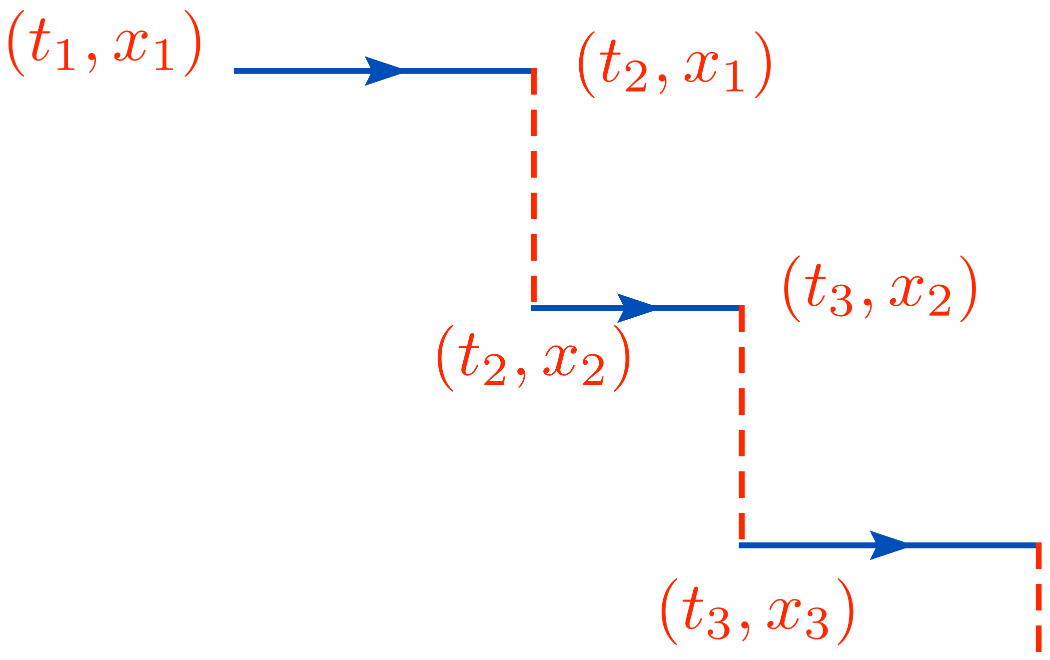
- First term: probability of evolving from Q_0^2 to Q^2 without emissions (ratio of Sudakovs $\Delta(Q^2)/\Delta(Q_0^2)$)
- Second term: emission at scale $k_\perp^2$ and evolution from $k_\perp^2$ to Q^2 without further emissions

Multiple branchings

Multiple branching can now be described using the above probabilistic equation

Denote by t the evolution variable (e.g $t = Q^2$)
Start from one parton at scale t_1 and momentum fraction x_1

The question is how to generate the values of t_2, x_2 and φ_2



Multiple branchings

1. t_2 generated with the correct probability by solving the equation (r = random number in [0,1])

$$\Delta(t_1)/\Delta(t_2) = r$$

If t_2 smaller than cut-off evolution stops (no further branching)

Multiple branchings

1. t_2 generated with the correct probability by solving the equation (r = random number in [0,1])

$$\Delta(t_1)/\Delta(t_2) = r$$

If t_2 smaller than cut-off evolution stops (no further branching)

2. Else, generate momentum fraction z = x_2/x_1 with $\text{Prob.} \sim \frac{\alpha_s}{2\pi} P(z)$

$$\int_\epsilon^{x_2/x_1} dz \frac{\alpha_s}{2\pi} P(z) = r' \int_\epsilon^{1-\epsilon} dz \frac{\alpha_s}{2\pi} P(z)$$

ε: IR cut-off for resolvable branching

Multiple branchings

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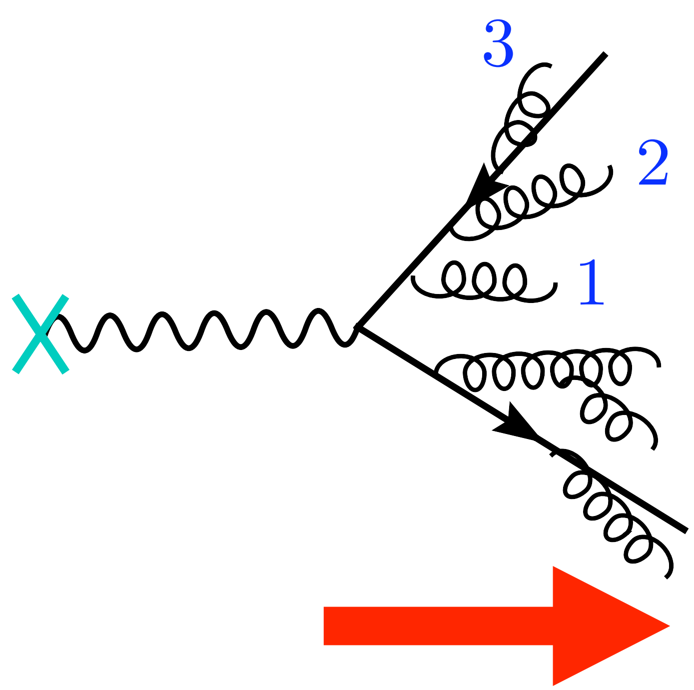
ε: IR cut-off for resolvable branching

3. Azimuthal angles: generated uniformly in $(0,2\pi)$ (or taking into account polarization correlations)

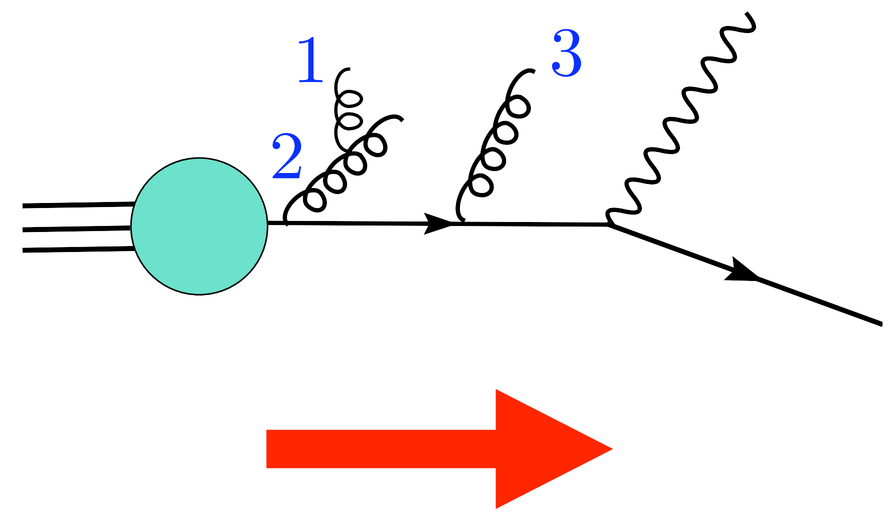
Space-like vs time-like evolution

Time-like: t evolves from a hard-scale downwards to an IR cut-off



$$Q > t_1 > t_2 > \cdots > Q_0$$

Space-like: t increases in the evolution up to the hard scale Q^2



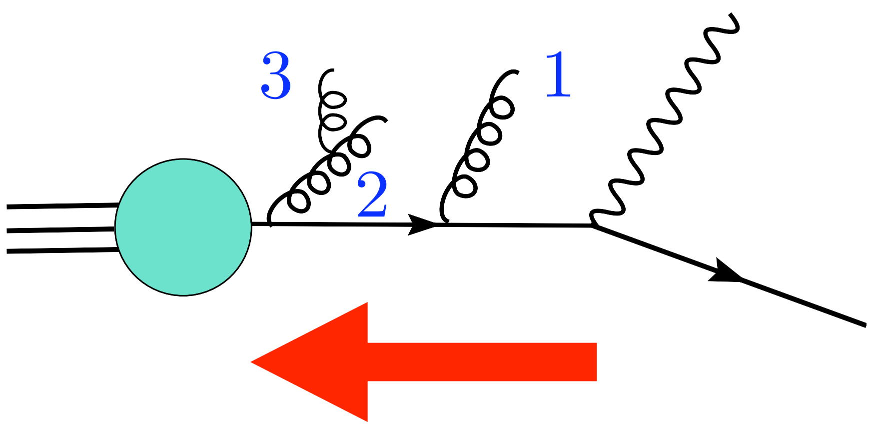
$$Q_0 < t_1 < t_2 < \ldots, Q$$

Each outgoing parton becomes a source of the new branching until the "no-branching" step is met (cut-off essential in parton shower)

⇒ a parton cascade develops, when all branchings are done partons are converted into hadrons via a hadronization model

Backward evolution

In space-like cases it is more convenient to start from the momentum fraction of the outgoing parton x_n and generate x_{n-1}, .. x_0 by backward evolution



$$Q > t_1 > t_2 > \cdots > Q_0$$

Essentially, the evolution proceeds as before but with a modified form factor which take the local parton density into account

We will not discuss backward evolution, despite its wide-spread use

Angular ordering

In the branching formalism discussed now we considered collinear enhancements to all orders in PT. But there are also soft enhancements.

When a soft gluon is radiated from a ($p_i p_j$) dipole one gets a universal eikonal factor

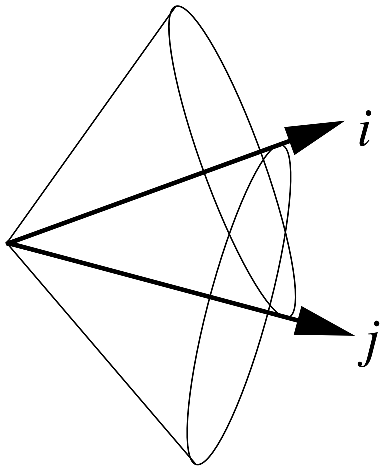
$$\omega_{ij} = \frac{p_i p_j}{p_i k \, p_j k} = \frac{1 - v_i v_j \cos\theta_{ij}}{\omega_k^2 (1 - v_i \cos\theta_{ik})(1 - v_j \cos\theta_{jk})}$$

Massless emitting lines $v_i = v_j = 1$, then

$$\omega_{ij} = \omega_{ij}^{[i]} + \omega_{ij}^{[j]} \qquad \omega_{ij}^{[i]} = \frac{1}{2}\left(\omega_{ij} + \frac{1}{1-\cos\theta_{ik}} - \frac{1}{1-\cos\theta_{jk}}\right)$$

Angular ordering

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \omega_{ij}^{[i]} = \begin{cases} \frac{1}{\omega_k^2 (1-\cos\theta_{ik})} & \theta_{ik} < \theta_{ij} \\ 0 & \theta_{ik} > \theta_{ij} \end{cases}$$

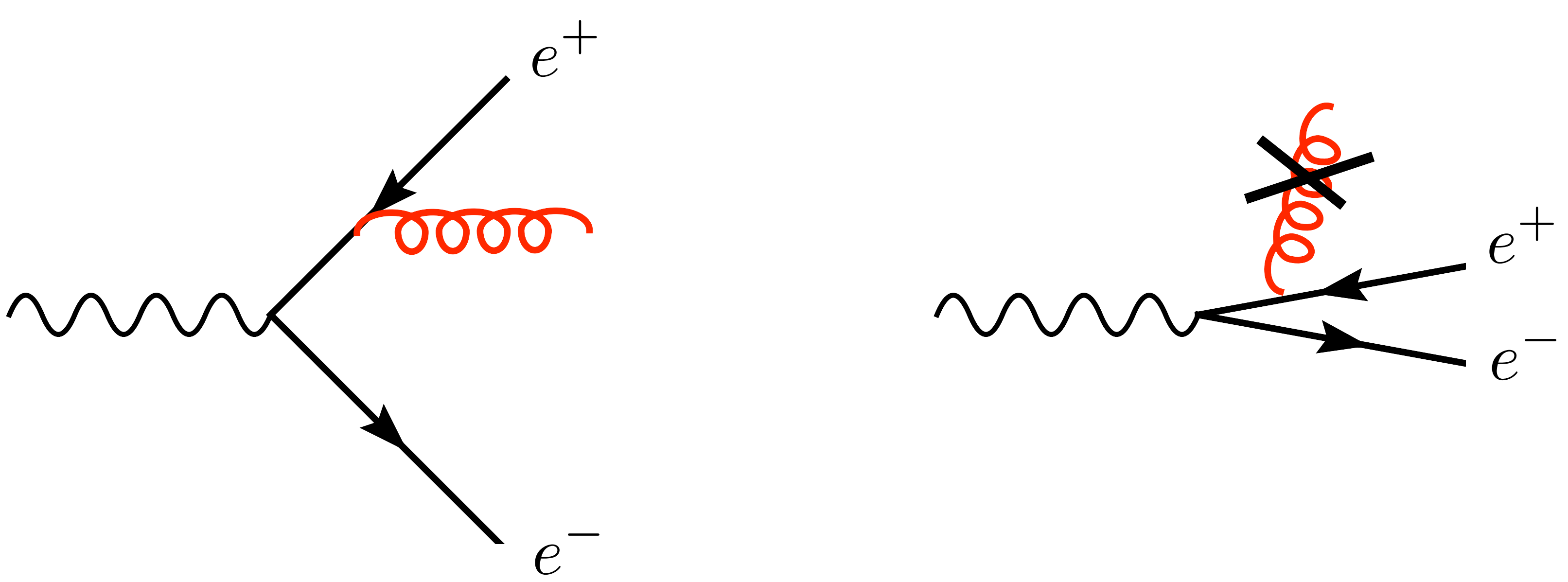


Proof: see e.g. QCD and collider physics, Ellis, Stirling, Webber

Angular ordering & coherence

A. O. is a manifestation of coherence of radiation in gauge theories

<u>In QED</u>
suppression of soft bremsstrahlung from an e+e- pair (Chudakov effect)
At large angles the e^+e^- pair is seen coherently as a system without total charge ⇒ radiation is suppressed



Angular ordering & coherence

Coherent a → b + c branching: replace the ordering variable $t = p_a^2$ with

$$\zeta = \frac{p_b p_c}{E_b E_c} \sim 1 - \cos\theta_{bc}$$

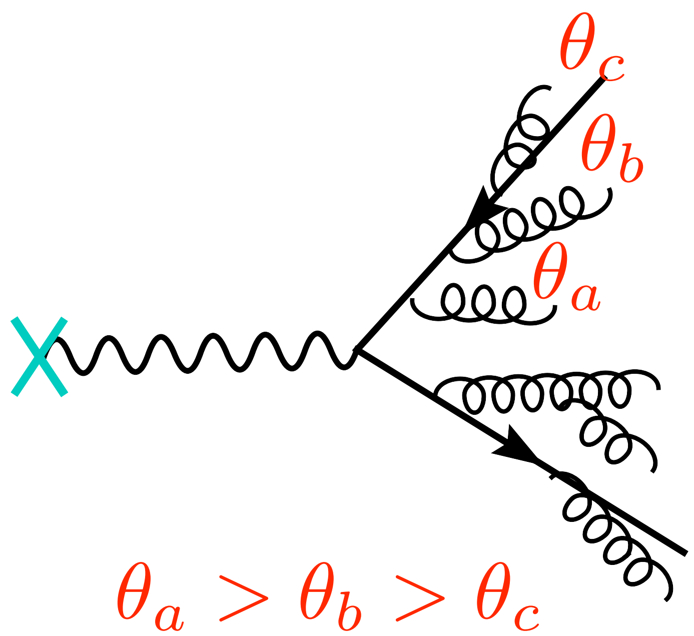
and require $\zeta' < \zeta$ at successive branchings

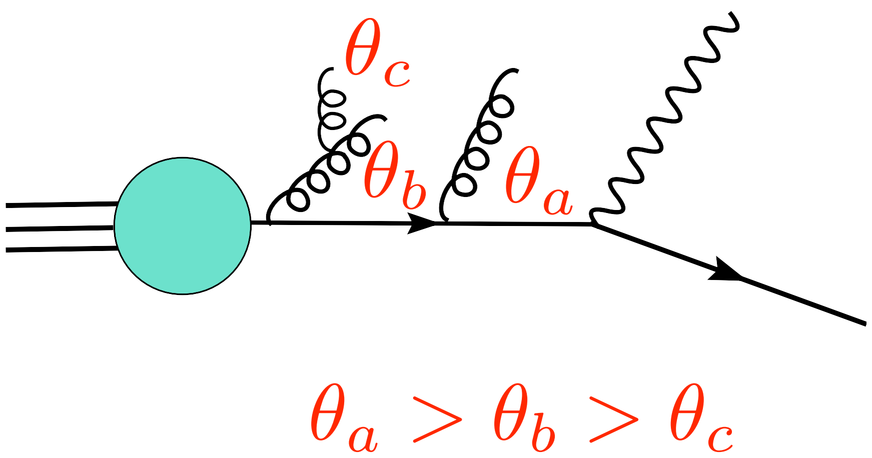
The basic formula for coherent branching

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \, \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z)$$

NB: need collinear cut-off. Simplest choice: $\zeta_0 = \frac{t_0}{E^2}$

AO: time like vs space-like case





NB: angles decrease when moving away from the hard vertex, i.e. in the space-like case angles increase during the evolution

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So, despite not guaranteeing NLL accuracy, they fare usually better than Leading Log analytic calculations

The real issue is that it is very difficult to estimate the uncertainty

Warning

The above discussion is a simplification

- many details/subtleties not discussed enough, some not at all
- various MC differ in the choice of the ordering variable and in many details, but the basic idea remains the same
- purpose was to give an overall idea of how Monte Carlos and what they can/can't do

Recap on Monte Carlos

- parton evolution as branching process from higher to lower x
- parton shower based on Sudakov form factor (Prob. of evolving without branching) with corresponding evolution equation
- branching described by picking randomly 3 numbers (t, x, φ) with the right prob. distributions
- virtuality ordered shower: collinear approximation
- angular ordering needed to describe also soft effects
- parton shower supplemented by hadronization + U.E. (various models $\Rightarrow$ MC tuning) $\Rightarrow$ full event generator
- by construction PS fail to describe multiple hard radiation
- Lots of work on merging/matching parton shower and fixed order calculations (POWHEG, MC@NLO, NNLOPS …)