Third day:

1. Compute the partonic cross-section in DIS with a vector current

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2 \frac{\hat{s}}{Q^4} 2\pi\alpha_{em}(1 + (1-\hat{y})^2) \tag{13}$$

2. Compute the leading order δ-terms in P_{qq} and P_{gg} without performing any loop-integral. [Hint: use the fact that $\int_0^1 dz P_{ij}(z) = 0$.]

3. Show that convolutions in x-space become ordinary products in Mellin space, i.e.

$$(f \otimes g)(N) = f(N)g(N) \tag{14}$$

[Remember: $f(N) = \int_0^1 dx x^{N-1} f(x)$]

4. The plus-prescription is defined by

$$\int_0^1 dx f(x) g_+(x) \equiv \int_0^1 dx [f(x) - f(1)] g(x) \,. \tag{15}$$

Show that

$$\left(\frac{1+z^2}{1-z}\right)_+ = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \tag{16}$$

and

$$\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ = \frac{z}{(1-z)_+} + \frac{1}{2}z(1-z) + \frac{11}{12}\delta(1-z) \,. \tag{17}$$

The splitting functions are

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+ \tag{18}$$

$$P_{qg}(z) = T_R \left(z^2 + (1-z)^2\right) \tag{19}$$

$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z} \tag{20}$$

$$P_{gg}(z) = 2C_A \left[\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z} + \frac{1}{2}z(1-z)\right] - \frac{2}{3}n_f T_R \delta(1-z) \tag{21}$$

The anomalous dimensions are given by the moments of the splitting functions,

$$\gamma_{ij}(N,\alpha_S) = \sum_{n=0}^{\infty} \gamma_{ij}^{(n)}(N) \left(\frac{\alpha_S}{2\pi}\right)^{n+1}, \tag{22}$$

$$\gamma_{ij}^{(0)}(N) = \int_0^1 dz z^{N-1} P_{ij}(z)\,. \tag{23}$$

Show that

$$\gamma_{qq}^{(0)}(N) = C_F\left[-\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{k=2}^{N}\frac{1}{k}\right] \tag{24}$$

$$\gamma_{qg}^{(0)}(N) = T_R\left[\frac{2+N+N^2}{N(N+1)(N+2)}\right] \tag{25}$$

$$\gamma_{gq}^{(0)}(N) = C_F\left[\frac{2+N+N^2}{N(N+1)(N-1)}\right] \tag{26}$$

$$\gamma_{gg}^{(0)}(N) = 2C_A\left[-\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - 2\sum_{k=2}^{N}\frac{1}{k}\right] - \frac{2}{3}n_f T_R \tag{27}$$

Now consider the evolution of the singlet quark distribution

$$\Sigma(x) = \sum_i q_i(x) + \bar{q}_i(x) \tag{28}$$

which mixes with the gluon distribution via the evolution equations. In terms of moments with evolution variable $t = \log(Q^2/\Lambda^2)$ we have

$$\frac{d}{dt}\Sigma(N) = \frac{\alpha_S(t)}{2\pi}\left[\gamma_{qq}(N)\Sigma(N) + 2n_f\gamma_{qg}(N)g(N)\right] \tag{29}$$

$$\frac{d}{dt}g(N) = \frac{\alpha_S(t)}{2\pi}\left[\gamma_{gq}(N)\Sigma(N) + \gamma_{gg}(N)g(N)\right] \tag{30}$$

Verify that for $N = 2$ there are two eigenvalues to the above evolution equation and the corresponding anomalous dimensions are $\lambda_\pm = 0, -(16/9 + n_f/3)$ and find the corresponding eigenfunctions.