AdS / CFT Correspondence and Integrability

Large-N expansion

Yang-Mills theory:

$$S_{YM} = -\frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$A^i_{\mu j}$: $i = 1 \ldots N$

- use $1/N$ as expansion parameter ('t Hooft '74)

RG evolution:

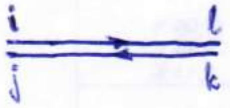
$$\frac{1}{g^2(\mu')} = \frac{1}{g^2(\mu)} + \frac{11N}{24\pi^2}\ln\frac{\mu'}{\mu}$$

$$g^2 \sim 1/N$$

$\boxed{\lambda = g^2 N}$ fixed at $N \to \infty$.

't Hooft coupling

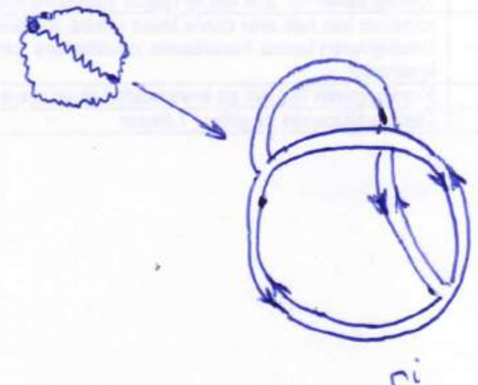
$$\langle A^i_{\mu j}(x)\, A^k_{\nu l}(y)\rangle = -\,\delta^i_l\, \delta^k_j\, \frac{g^2}{8\pi^2 (x-y)^2}$$



$\mathrm{tr}\, \partial_\mu A_\nu [A^\mu, A^\nu]$:

$\sim \frac{1}{g^2}$

"index conservation law"



$\delta^i_i\ \delta^j_j\ \delta^k_k$ $\qquad\qquad$ δ^i_i

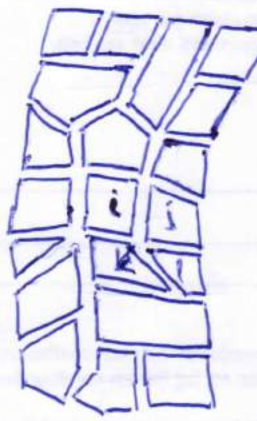
vertices: $\left(\frac{1}{g^2}\right)^2$

propagators: $(g^2)^3$

index loops: N^3 $\quad$ N

total: g^2N^3 $\quad$ $g^2 N$

planar $\quad$ non-planar



Double-line diagrams $\leftrightarrow$ Triangulations of 2d surfaces

P - propagators

V - vertices

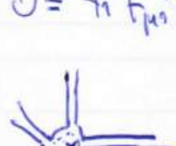
I - index loops

Th (Euler) $V - P + I = 2 - 2g$

g - genus of 2d surface

Single-trace operators:

$\mathcal{O} = \mathrm{tr}\, F_{\mu\nu}^2$ or $\mathrm{tr}\, D^3 F_{\mu\nu} D_\alpha F_{\mu\nu}$ etc.



missing index loop $\sim \frac{1}{N}$

$$\langle O_1 \dots O_n \rangle_{conn} = \sum_{g=0}^{\infty} F_g(\lambda)\, N^{2-2g-n}$$

- String theory with $g_s = \frac{1}{N}$

Large-N factorization:

$\langle O \rangle \sim N$

$\langle O_1 O_2 \rangle_{conn} \sim 1$

$\langle O_1 \dots O_n \rangle = \langle O_1 \rangle \dots \langle O_n \rangle + O(N^{n-2})$

If $\langle O_i \rangle = 0$:

$$\langle O_1 \dots O_n \rangle = \sum_{part.} \prod_{pairs} \langle O_i O_j \rangle + O(N^{-2})$$

"Wick th.": large-N $\Rightarrow$ free field built from collective variables (master field)

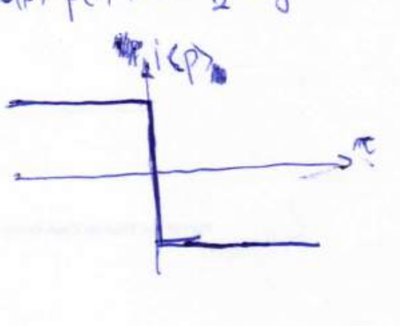
Bjorken-Johnson-Low formula

$$Z = \int Dx(\tau)\, e^{-\frac{1}{2}\int d\tau\, \dot{x}^2} \qquad \langle x(0)\, x(\tau) \rangle = \frac{1}{2}|\tau|$$

$\tau = it$

$p = \frac{dx}{dt} = +i \frac{dx}{d\tau}$ ← numeric fn $\forall$ trajectory $x(\tau)$. But $[x,p]=i$?

$\langle x(0)\, p(\tau) \rangle = \frac{i}{2} \operatorname{sign} \tau$



But momentum is conserved...

Resolution: operators under path integral are automatically time-ordered: $\langle x(0)\, p(\tau)\rangle \to \langle T\, x(\cdot)\, p(\tau)\rangle$

BJL formula:

$$[\mathcal{O}_1, \mathcal{O}_2] = \lim_{\varepsilon\to 0} T\left(\mathcal{O}_1(0)\,\mathcal{O}_2(\varepsilon) - \mathcal{O}_1(0)\,\mathcal{O}_2(-\varepsilon)\right)$$

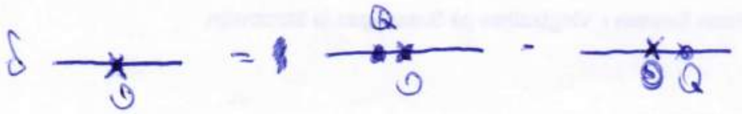
$$[x,p] = \lim_{\varepsilon\to 0} \left(\langle x(0)\,p(\varepsilon)\rangle - \langle x(0)\,p(-\varepsilon)\rangle\right) = \lim_{\varepsilon\to 0} \frac{i}{2}\left(\operatorname{sign}\varepsilon - \operatorname{sign}(-\varepsilon\ldots\right.$$

$= i$ ✓

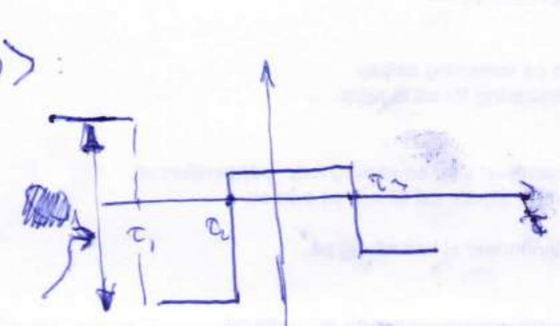
Symmetry transformations:

$$\delta\mathcal{O} = i\,[Q, \mathcal{O}]$$

generator of transformations (conserved charge)



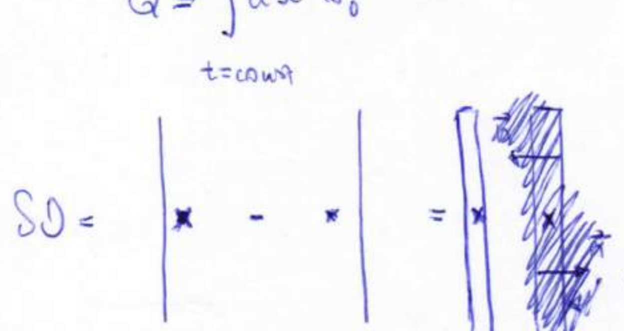
$\langle \mathcal{O}_1 \ldots \mathcal{O}_n\, Q(\tau)\rangle$:



$\langle \delta\mathcal{O}_1\, \mathcal{O}_2 \ldots \mathcal{O}_n\rangle$

In QFT:

$$Q = \int_{t=\mathrm{const}} d^3x\, J^0$$



$\oint d^3x\, n^\mu J_\mu$ inward unit normal

$$\oint_{C'} d^3x\, n^\mu J_\mu - \int_C d^3x\, n^\mu J_\mu = -\int_D d^4x\, \partial_\mu J^\mu = 0$$

$$\delta\mathcal{O}(x) = \lim_{\delta\to 0} \oint_{S_\delta} d^3y\, n^\mu(y)\, J_\mu(y)\, \mathcal{O}(x) \qquad \otimes \; S_\delta$$

Space-time transformations:

$$\delta_\xi x^\mu = \xi^\mu(x)$$

$$\delta_\xi \mathcal{O}(x) = \lim_{\delta\to 0} \int_{S_\delta} d^3y\, n^\mu(y)\, T_{\mu\nu}(y)\, \xi^\nu(y)\, \mathcal{O}(x)$$

↑ energy-momentum tensor.

Conformal symmetry

Scale invariance:

$$\delta x^\mu = \varepsilon x^\mu$$

$$\Downarrow$$

$T_{\mu\nu}\xi^\nu$ is conserved for $\xi^\nu = x^\nu$

$$\partial_\mu (T^\mu{}_\nu x^\nu) = \underbrace{\partial_\mu T^\mu{}_\nu}_{0} x^\nu + T^\mu{}_\nu \delta^\nu_\mu = T^\mu{}_\mu$$

$\boxed{T^\mu{}_\mu = 0} \Rightarrow$ conformal symmetry

More conserved currents:

$$\partial_\mu (T^\mu{}_\nu \xi^\nu) = T^{\mu\nu} \partial_\mu \xi_\nu = \frac{1}{2} T^{\mu\nu}(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = 0$$

if $\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2}\partial_\lambda \xi^\lambda \eta_{\mu\nu}}$

conformal Killing equation

- solutions form Lie algebra w.r.t. Lie bracket:

$$[\rho, \xi]^\mu = \rho^\nu \partial_\nu \xi^\mu - \xi^\nu \partial_\nu \rho^\mu$$

Solutions:

- $\xi^\mu = c^\mu \qquad \Rightarrow$ translations (P^μ)

- $\xi^\mu = \omega^\mu{}_\nu x^\nu$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ $\Longrightarrow$ Lorentz transformations ($L_{\mu\nu}$)
- $\xi^\mu = x^\mu$ $\Longrightarrow$ Dilatations (D)
- $\xi^\mu = a^\mu x^2 - 2x^\mu a\cdot x$ $\Longrightarrow$ Special conformal transformations (K_μ)

15 generators: $\boxed{SO(4,2)} = \{L_{MN} \mid M,N=0\ldots5\}$

$$L_{\mu4} = \frac{1}{\sqrt{2}}(P_\mu + K_\mu)$$

$$L_{\mu5} = \frac{1}{\sqrt{2}}(P_\mu - K_\mu)$$

$$L_{45} = D$$

Finite conformal transformations (Liouville):

translation, rotations, dilatations + inversion: $x^\mu \to \frac{x^\mu}{x^2}$

Conformal Field Theory

Primary operators:

$$\delta_\xi O^A = \xi^\mu \partial_\mu O^A + \frac{\Delta}{4}\partial_\mu \xi^\mu O^A + \frac{1}{2}\partial_\mu \xi_\nu \Sigma^{\mu\nu A}{}_B O^B$$

Lorentz generators in spin (s_1, s_2) irrep of $SO(3,1)$

Δ - dimension of O^A

- Irrep. of $SO(4,2)$ is characterized by (Δ, s_1, s_2)

At $x=0$:

raising $\to$ $P_\mu \cdot O = \partial_\mu O$

$L_{\mu\nu} \cdot O = \Sigma_{\mu\nu} O$

"Hamiltonian" $\to$ $D \cdot O = \Delta O$

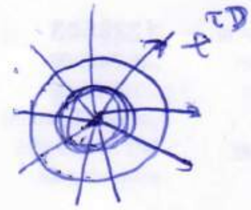
$K_\mu \cdot O = 0$

lowering

Descendants: $\partial_{\mu_1}\dots\partial_{\mu_n}\mathcal{O}$.

Correspond to radial quantization:

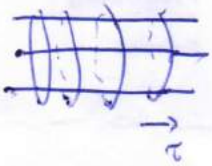
$\mathbb{R}^4$



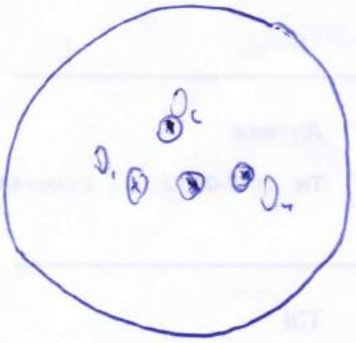
$\tau = \ln|x|$

$\downarrow \; |x| = e^{\tau}$

$\mathbb{R}\times S^3$



Conformal Ward identities:



$$0 \overset{!}{=} \oint dy^\mu n^\mu \xi^\nu \langle T_{\mu\nu}\mathcal{O}_1 \dots \mathcal{O}_n\rangle = \langle \delta_\xi \mathcal{O}_1 \dots \mathcal{O}_n\rangle + \dots + \langle \mathcal{O}_1 \dots \delta\mathcal{O}_n\rangle$$

by cluster decomposition

Ex $\xi^\mu = x^\mu$,

$$\delta\mathcal{O}(x) = x^\mu\partial_\mu\mathcal{O} + \Delta\mathcal{O}$$

$$(x^\mu\partial_\mu + 2\Delta)\langle\mathcal{O}(x)\,\mathcal{O}(0)\rangle = 0.$$

$$\Downarrow$$

$$\langle\mathcal{O}(x)\,\mathcal{O}(0)\rangle = \frac{\text{const}}{|x|^{2\Delta}}$$

General case:

$$\sum_i \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + \Delta_i\right)\langle\mathcal{O}_1\dots\mathcal{O}_n\rangle = 0$$

For scalar operators:

$$\sum_i \left(2x_i^\lambda x_i^\mu \frac{\partial}{\partial x_i^\mu} - x_i^2 \frac{\partial}{\partial x_{i\lambda}} + 2\Delta_i x_i^\lambda\right)\langle O_1 \dots O_n\rangle = 0$$

Solution:

$$\langle O_1 \dots O_n \rangle = \frac{F(u_1 \dots u_{n(n-3)/2})}{\prod_{i<j}(x_i - x_j)^{2\alpha_{ij}}}$$

$$\alpha_{ij} = \frac{1}{n-2}\left(\Delta_i + \Delta_j - \frac{1}{n-1}\sum_k \Delta_k\right)$$

$$u_a = \prod_{i<j}(x_i - x_j)^{2\beta_{ij}}$$

$$\beta_{ij} = \beta_{ji}, \quad \beta_{ii} = 0, \quad \sum_j \beta_{ij} = 0.$$

3pt:

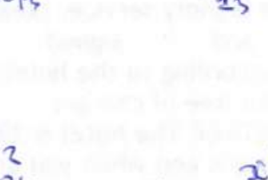
- no cross-ratios

Structure constants of operator algebra.

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\, x_{13}^{\Delta_1+\Delta_3-\Delta_2}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1}}$$

4pt:

- 2 cross-ratios: $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ $\quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}$



Anti-de-Sitter space

$$ds^2 = \frac{dx_\mu dx^\mu + dz^2}{z^2}$$

translations, rotations, dilatations $(x^\mu \to \lambda x^\mu,\ z \to \lambda z)$

+ inversion: $x^\mu \to \frac{x^\mu}{x^2+z^2}$ $\quad z \to \frac{z}{x^2+z^2}$ are isometries of AdS_{d+1}

- Geometric realization of $SO(4,2)$: $AdS_5 = SO(4,2)/SO(4,1)$

Fields in AdS_5:

$$S_{bulk} = -\frac{1}{2}\int d^5x \sqrt{g}\left(g^{MN}\partial_M\phi\,\partial_N\phi + m^2\phi\right)$$

$$\left(\frac{1}{\sqrt{g}}\partial_M\sqrt{g}\,g^{MN}\partial_N - m^2\right)\phi = 0$$

$$\left(z^5\frac{\partial}{\partial z}\frac{1}{z^3}\frac{\partial}{\partial z} + z^2\partial_\mu^2 - m^2\right)\phi = 0$$

$$\phi(z,x) \sim z^\Delta e^{ipx} \quad \text{at } z\to 0$$

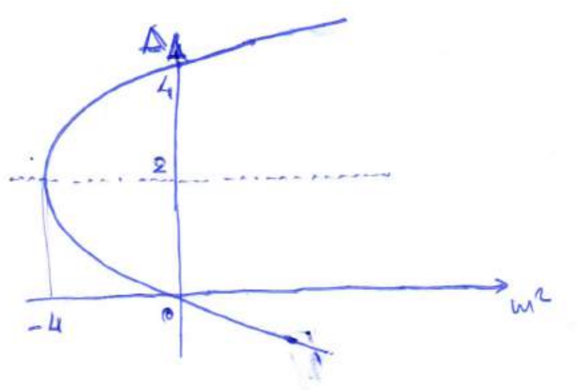
$$\Delta(\Delta-4) - m^2 = 0$$

$$\Delta_\pm = 2 \pm \sqrt{m^2+4}$$

Measure $= d^4x\,\frac{dz}{z^5}$

$\Delta > 2$: normalizable

$\Delta \leq 2$: non-normalizable



$m^2 \geq -4$ Breitenlohner – Freedman bound

Natural b.c.'s: $\phi(z,x) \to \eta(x)\,z^{4-\Delta}$

Solution:

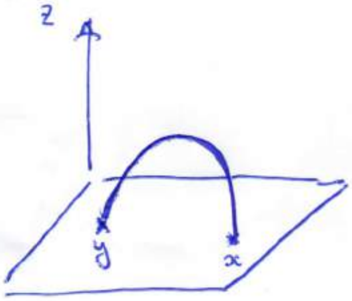
$$\phi(z,x) = \frac{(\Delta-1)(\Delta-2)}{\pi^2}\int d^4y\left[\frac{z}{z^2+(x-y)^2}\right]^\Delta \eta(y)$$

bulk-to-boundary propagator

- $\eta(y)$ has finite support D
- For x outside D: $\phi(z,x) \overset{z\to 0}{\simeq} z^\Delta\,\frac{(\Delta-1)(\Delta-2)}{\pi^2}\int d^4y\,\frac{1}{(x-y)^{2\Delta}}\,\eta(y)$

boundary-to-boundary propagator

2pt function of primary op's of dim. Δ in a CFT_4



Holographic duality:

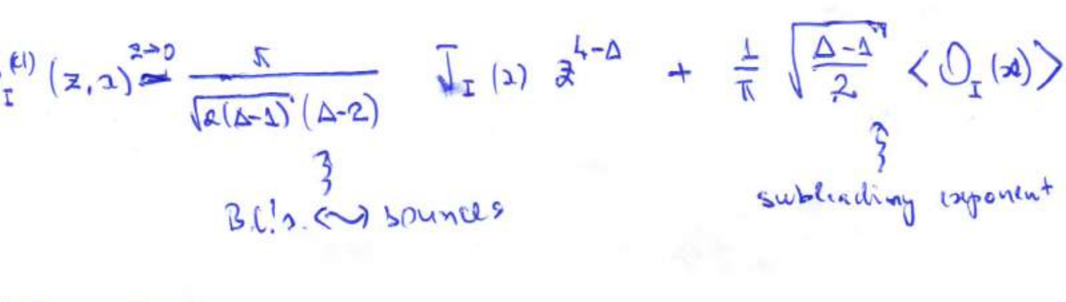
$$\phi_I(z,x) \longleftrightarrow \mathcal{O}_I(x)$$
$$m^2 \longleftrightarrow \Delta$$

$$\langle \mathcal{O}_k(x) \rangle_J = \left\langle \mathcal{O}_k(x)\, e^{\int d^4x \sum_I J_I \mathcal{O}_I} \right\rangle_{CFT}$$

$\hookrightarrow$ to find <u>quantum</u> $\langle \ldots \rangle$ in CFT, need to solve <u>classical</u> EOM in AdS:

$$\phi_I^{(1)}(z,x) \overset{z\to 0}{\simeq} \underbrace{\frac{\pi}{\sqrt{2(\Delta-1)}\,(\Delta-2)}\, J_I(x)\, z^{4-\Delta}}_{\text{B.C.'s} \sim \text{sources}} + \underbrace{\frac{1}{\pi}\sqrt{\frac{\Delta-1}{2}}\, \langle \mathcal{O}_I(x) \rangle\, z^{\Delta}}_{\text{subleading exponent} \sim \text{exp. values}}$$

Witten diagrams:



AdS/CFT correspondence

$$\boxed{\begin{array}{c} \mathcal{N}=4 \quad D=4 \\ \text{Super-Yang-Mills} \end{array}} \equiv \boxed{\begin{array}{c} \text{IIB strings} \\ \text{on } AdS_5 \times S^5 \end{array}}$$

$\mathcal{N}=4$ SYM : A_μ, Φ_I, $\Psi_{A\alpha}$

$I=1\ldots6 \quad A=1\ldots4$

$$\mathcal{L}=\frac{1}{g^2}\,\mathrm{tr}\left\{-\frac{1}{2}F_{\mu\nu}^2+\left(D_\mu\Phi_I\right)^2+\frac{1}{2}\left[\Phi_I,\Phi_J\right]^2+\text{fermions}\right\}$$

- g^2 does not run $\quad \lambda=g^2N$
- no mass scale $\Rightarrow$ conformal symmetry

Global symmetry : $SO(4,2)\times SO(6)$

$$\begin{matrix} SO(4,2) & \times & SO(6) \\ \wr & & \wr \\ AdS_5 & \times & S_5 \end{matrix}$$

String tension : $T=\dfrac{\sqrt{\lambda}}{2\pi}$

String coupling: $g_s=\dfrac{\lambda}{4\pi N}$

λ - fixed $\quad N\to\infty$ ('t Hooft limit)	T - fixed $\quad g_s\to 0$ (free strings)
$\lambda \gg 1$ (strong coupling)	$T\gg 1$ (classical gravity)
$\lambda\ll 1$ (weak coupling)	$T\ll 1$ ("tensionless strings")
$\{O_I(x),\ \Delta_I\}$	String states, m_I^2

$m^2=0 \longleftrightarrow \Delta=4$

Graviton $h_{\mu\nu}(x,z) \longleftrightarrow$ EM tensor $T_\mu^\nu=\mathrm{tr}\left(F_{\mu\lambda}F^{\lambda\nu}-\frac{1}{4}\delta_\mu^\nu F_{\lambda\rho}F^{\lambda\rho}+\ldots\right)$

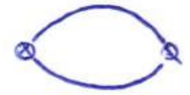
Dilaton $\phi(x,z) \longleftrightarrow$ Lagrangian density $\mathcal{L}=\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}+\ldots\right)$

Axion $a(x,z) \longleftrightarrow$ Topological density $\chi=\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{tr}\,F_{\mu\nu}F_{\lambda\rho}$

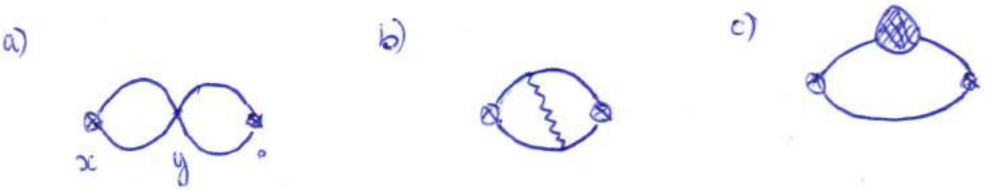
Anomalous dimensions

Konishi operator: $K = \mathrm{tr}\, \Phi_I \Phi_I$

$\Delta_K = 2$

$\langle K(x)\, K(0)\rangle =$  $= \left(\frac{g^2}{8\pi^2 x^2}\right)^2 \cdot N^2 \cdot 6 \cdot 2 = \frac{3\lambda^2}{16\pi^4 x^4}$

One-loop correction:



(a): $$-\frac{15\lambda^3}{128\pi^8}\int \frac{d^4y}{(x-y)^4\, y^4} = -\frac{15\lambda^3}{32\pi^6}\,\frac{1}{x^4}\,\ln(\Lambda|x|)$$

log div. @ $y \to x$ or 0

$$2\times \frac{1}{x^4}\times 2\pi^2 \times \int_{1/\Lambda} \frac{dy}{y}$$

(a) + (b) + (c):

$$\langle K(x)\, K(0)\rangle = \frac{3\lambda^2}{16\pi^4 x^4}\left(1-\frac{3\lambda}{2\pi^2}\ln \Lambda x\right) \approx \frac{3\lambda^2}{16\pi^4}\,\frac{1}{x^4}\,(\Lambda x)^{-\frac{3\lambda}{2\pi^2}} = \text{const}\times\frac{\Lambda^{-\frac{3\lambda}{2\pi^2}}}{x^{4+\frac{3\lambda}{2\pi^2}}}$$

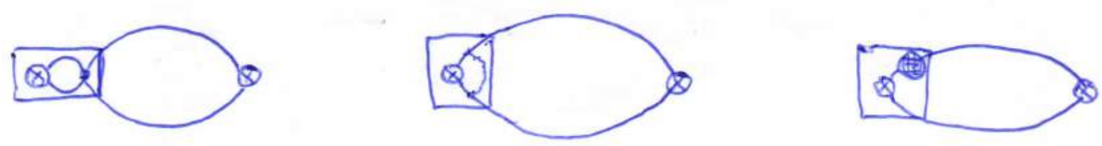
- Multiplicative renormalization of the operator:

$$K_R = \Lambda^{\frac{3\lambda}{4\pi^2}}\, K$$

- Connection to scaling dimension:

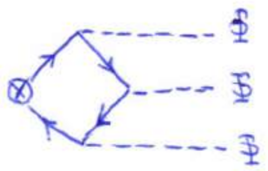
$$\Delta_K = 2 + \frac{3\lambda}{4\pi^2} + O(\lambda^2)$$

1 loop anomalous dimension



Operator mixing

$$(\bar{\Psi}\Psi)_R = Z_1\, \bar{\Psi}\Psi + Z_2\, \Phi^3$$



In general:

$$O^a_R = Z^a{}_b\, O^b$$

Mixing matrix:

$$\Gamma = Z^{-1}\frac{dZ}{d\ln\Lambda}$$

$$\Gamma \circ O_n = \gamma_n O_n \qquad \Delta_n = \Delta_n^{(0)} + \gamma_n$$

Spin chains

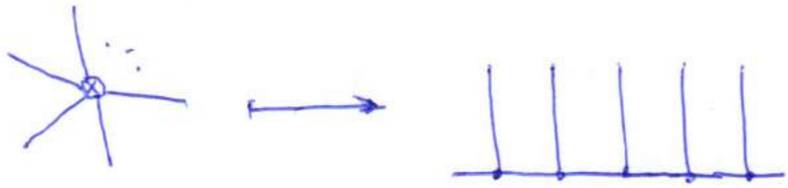
$$O = \Psi^{I_1 \ldots I_L}\, \mathrm{tr}\, \Phi_{I_1} \ldots \Phi_{I_L}$$

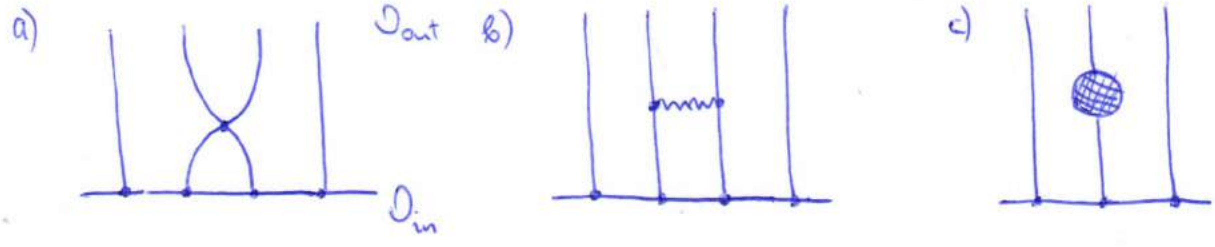
↑

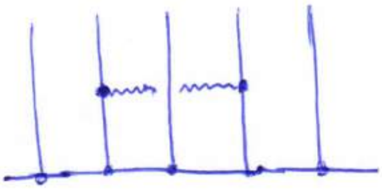
"wave function"

$$\mathcal{H} = \underbrace{\mathbb{R}^6 \otimes \ldots \otimes \mathbb{R}^6}_{L} \,/\, \text{cyclic perm.}$$

Renormalization of $\mathrm{tr}\, \Phi_{I_1} \cdots \Phi_{I_L}$:







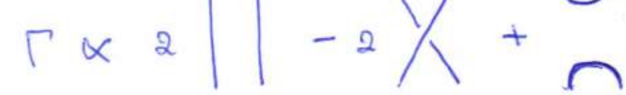
$1/N$ suppressed.

One-loop mixing matrix:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(2 - 2P_{l,l+1} + K_{l,l+1} \right)$$

$$P \, a \otimes b = b \otimes a$$

$$K \, a \otimes b = (a \cdot b)\, 1$$



- Unique integrable spin chain with $SO(6)$ symmetry.

Ex (1) Konishi operator

$$K = \mathrm{tr}\, \Phi_I \Phi_I$$

$$\Psi^{I_1 I_2} = \delta^{I_1 I_2}$$

$$P_{12} |K\rangle = |K\rangle$$

$$K_{12} |K\rangle = 6 |K\rangle$$

$$\gamma_K = \frac{\lambda}{16\pi^2} \cdot 2 \cdot \left(2 - 2 + 6 \right) = \frac{3\lambda}{4\pi^2} \quad \checkmark$$

Ex (2) Chiral Primary Operators

$$CPO = \underset{\substack{\uparrow \\ \text{symmetric traceless}}}{C^{I_1 \dots I_L}} \operatorname{tr} \Phi_{I_1} \dots \Phi_{I_L}$$

$\gamma_{CPO} = 0$ (true to all orders in λ)

Complex fields:

$$Z = \Phi_1 + i\,\Phi_2 \qquad \bar{Z}$$

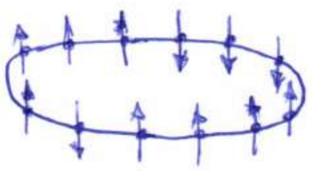
$$W = \Phi_3 + i\,\Phi_4 \qquad \bar{W}$$

$$Y = \Phi_5 + i\,\Phi_6 \qquad \bar{Y}$$

BMN vacuum: $\operatorname{tr} Z^L$

su(2) subsector:

$$\mathcal{O} = \operatorname{tr}\left(Z^{L-M} W^M + \text{perm.}\right)$$



$\operatorname{tr} ZWZZZZWWWZZZ$

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} \left(1 - P_{l,l+1}\right) = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}\right)$$

Heisenberg Hamiltonian.

BMN vacuum $\longleftrightarrow$ Ferromagnetic ground state

$$\operatorname{tr} Z^L \qquad\qquad |0\rangle = |\uparrow \dots \uparrow\rangle$$

Magnon:

$$\varepsilon(p) = \frac{\lambda}{2\pi^2} \sin^2 \frac{p}{2}$$

Spectrum:

$$e^{ip_j L} \prod_{k\neq j} S(p_j, p_k) = 1 \qquad \text{(Bethe eqs)}$$

$$\sum_j p_j = 0 \quad \text{(trace cyclicity)}$$

$$\gamma = \sum_j \varepsilon(p_j)$$

Rapidity variable:

$$e^{ip} = \frac{u + \frac{i}{2}}{u - \frac{i}{2}}$$

$$\varepsilon = \frac{\lambda}{8\pi^2} \frac{1}{u^2 + \frac{1}{4}}$$

$$S(u, u') = \frac{u - u' - i}{u - u' + i}$$

String theory in $AdS_5 \times S^5$

Geometry of AdS_{d+1}:

$$X^\mu = \frac{x^\mu}{z}$$

$$X^{-1} = \frac{x^2 + z^2 + 1}{2z}$$

$$X^d = \frac{x^2 + z^2 - 1}{2z}$$

$$\eta_{MN} X^M X^N + 1 = 0$$

$$\eta_{MN} = \underset{-1}{-}\ \underbrace{- + \dots +}_{0 \dots d-1}\ \underset{d}{+}$$

• AdS_5 & S^5 are symmetric homogeneous spaces $\Rightarrow$ σ-model on $AdS_5 \times S^5$ is integrable

$AdS_2 \times S^2$:

$ds^2 = d\theta^2 + \cos^2\theta \, d\varphi^2$

$ds^2 = d\rho^2 - \cosh^2\rho \, dt^2$

$$S_{str} = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \left[-\cosh^2\rho\,(\partial t)^2 + (\partial\rho)^2 + \cos^2\theta\,(\partial\varphi)^2 + (\partial\theta)^2 \right]$$

Energy $(t \to t + \delta t) \longleftrightarrow \Delta$

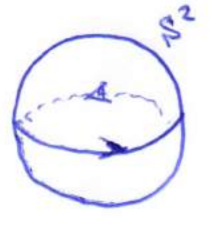
Angular momentum $(\varphi \to \varphi + \delta\varphi) \longleftrightarrow L$

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\int_0^{\ell} d\sigma\, \cosh^2\rho\; \partial_\tau t$$

$$L = \frac{\sqrt{\lambda}}{2\pi}\int_0^{\ell} d\sigma\, \cos^2\theta\; \partial_\tau\varphi$$

Simple solution (BMN string):

$$t = \tau = \varphi \quad \text{at} \quad \rho = 0 = \theta$$



$$L = \frac{\sqrt{\lambda}\,\ell}{2\pi} \quad \text{and} \quad \Delta = L$$

- Dual to $\mathrm{tr}\, Z^L$

Expand in small ρ and θ:

$$S_{str} \simeq \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \left[(\partial\rho)^2 - \rho^2 + (\partial\theta)^2 - \theta^2 \right]$$

$$0 < \sigma < \ell_{str} = \frac{2\pi L}{\sqrt{\lambda}}$$

$$\sigma \to \frac{2\pi}{\sqrt{\lambda}}\sigma$$

$$\mathcal{S} = \frac{1}{2}\int d\tau \int_0^L d\sigma \left[(\partial_\tau\varphi_i)^2 - \frac{\lambda}{4\pi^2}(\partial_\sigma\varphi_i)^2 - \dots \right.$$

where $\varphi_1 = \rho,\ \varphi_2 = \theta$

- String modes are massive particles:

$$\varepsilon(p) = \sqrt{1 + p_{str}^2} - 1$$

$$\ell_{str} = \frac{2\pi L}{\sqrt{\lambda}} \qquad \ell_{spin\text{-}chain} = L$$

$$\cancel{p_{str} = \frac{\sqrt{\lambda}}{2\pi}p}$$

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda p^2}{4\pi^2}} - 1 \quad \text{vs.} \quad \varepsilon(p) = \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}$$

(string) (spin chain)

- Agree at $p \to 0$.

non-relativistic limit (string)

continuum limit (spin chain)

Exact magnon dispersion relation:

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} - 1$$