Particles and Poles in Ising Field Theory

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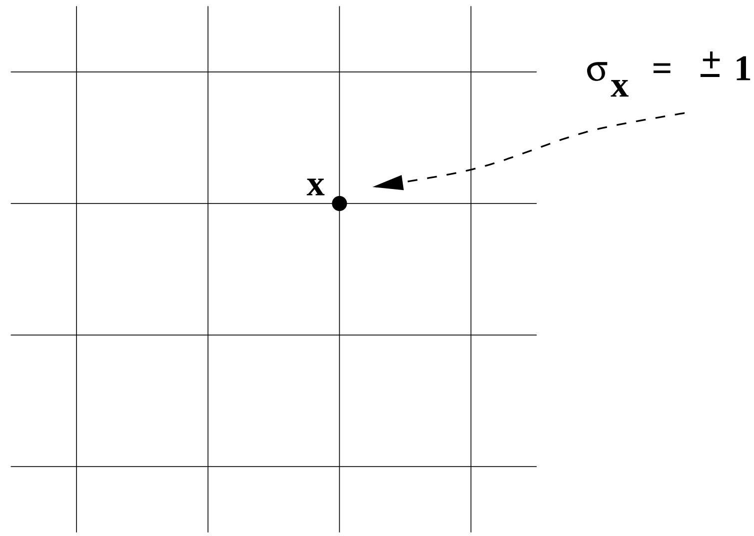
(2D) Ising Field Theory = scaling limit of the (2D) lattice Ising model **in a magnetic field** $H \neq 0$, near its ferromagnetic critical point, $T \to T_c, H \to 0$, as $R_c/a \to \infty$.

$$E\{\sigma_{\mathbf{x}}\} = -\sum_{<nn>} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - H\sum_{\mathbf{x}} \sigma_{\mathbf{x}}$$



• Defines 2D Euclidean quantum field theory

• At the critical point – Scale invariant $\to$ Conformal Field Theory.

Ising CFT := "Minimal Model" $\mathcal{M}_{3/4}$ ($c = 1/2$) = Free massless Majoranas (Onsager, 1944)

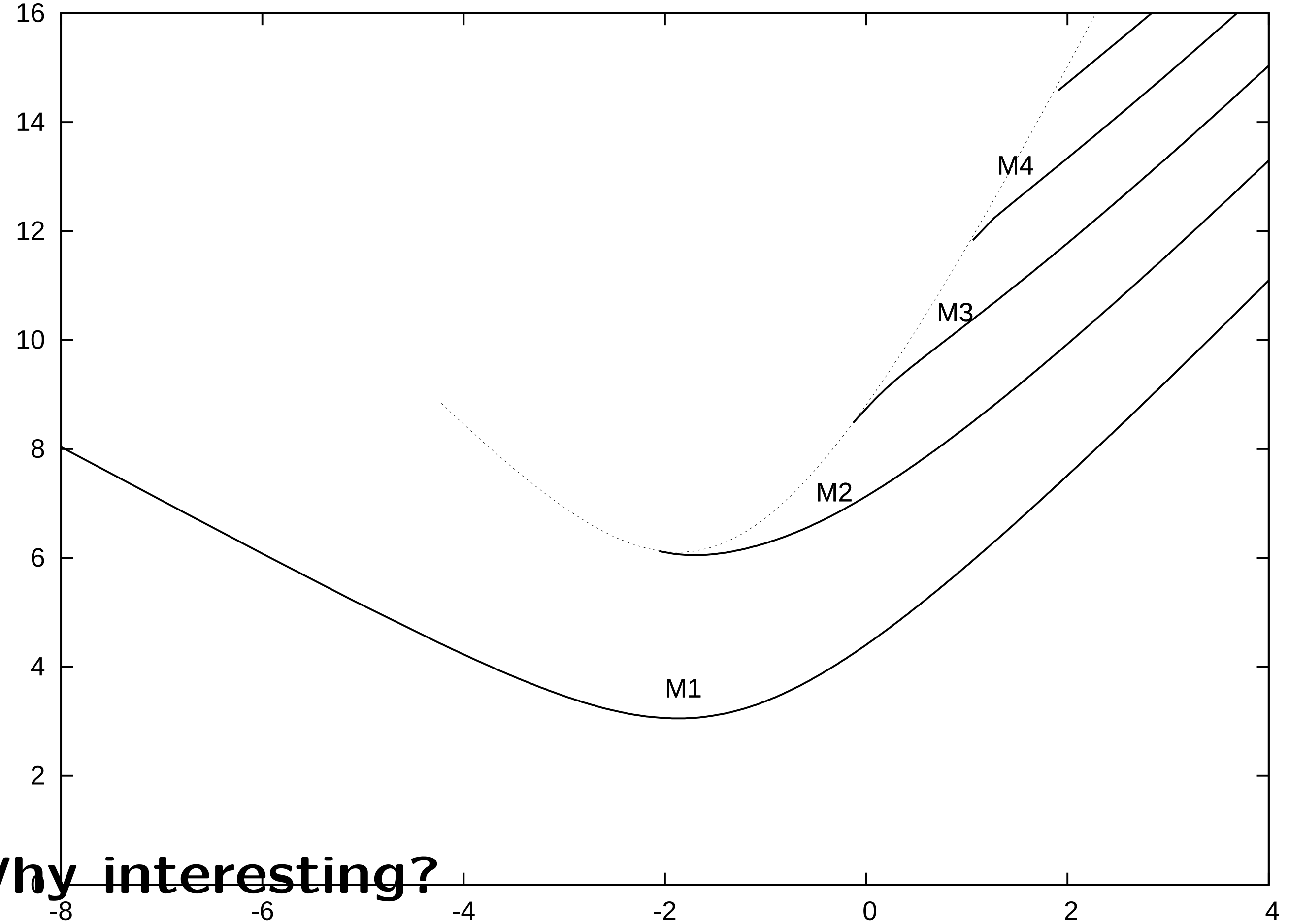
• Away from the critical point – massive QFT. Of interest in:

⋆ Stat-Mech: Universality class of 2D Curie transition, and liquid-vapor critical point (Determines universal scaling functions, correlation functions, ...)

⋆ Hep -Th: Massive QFT $\Rightarrow$ Particle theory (Mass spectrum of stable particles, resonance states, S-matrix)

IFT depends on one dimensionless parameter η (explained shortly) $\Rightarrow$ one-parameter family of particle theories. Toy model for a number of interesting phenomena, such as "quark confinement", first-order transition, false vacuum decay, resonance states, etc.

Masses $M_n(\eta)$, the functions of certain parameter of the theory: (from numerical analysis)



Why interesting?

1. Laboratory experiment

Quantum Criticality in an Ising Chain: Experimental Evidence for Emergent E_8 Symmetry

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Quantum phase transitions take place between distinct phases of matter at zero temperature. Near the transition point, exotic quantum symmetries can emerge that govern the excitation spectrum of the system. A symmetry described by the E_8 Lie group with a spectrum of eight particles was long predicted to appear near the critical point of an Ising chain. We realize this system experimentally by using strong transverse magnetic fields to tune the quasi–one-dimensional Ising ferromagnet $CoNb_2O_6$ (cobalt niobate) through its critical point. Spin excitations are observed to change character from pairs of kinks in the ordered phase to spin-flips in the paramagnetic phase. Just below the critical field, the spin dynamics shows a fine structure with two sharp modes at low energies, in a ratio that approaches the golden mean predicted for the first two meson particles of the E_8 spectrum. Our results demonstrate the power of symmetry to describe complex quantum behaviors.

Symmetry is present in many physical systems and helps uncover some of their fundamental properties. Continuous symmetries lead to conservation laws; for example, the invariance of physical laws under spatial rotation ensures the conservation of angular momentum. More exotic continuous symmetries have been predicted to emerge in the proximity of certain quantum phase transitions (QPTs) (*1*, *2*). Recent experiments on quantum magnets (*3* *5*) suggest that quantum critical resonances may expose the underlying symmetries most clearly. Remarkably, the simplest of systems, the Ising chain, promises a very complex symmetry, described mathematically by the E_8 Lie group (*2*, *6* *9*). Lie groups describe continuous symmetries and are important in many areas of physics. They range in complexity from the U(1) group, which appears in the low-energy description of superfluidity, superconductivity, and Bose-Einstein condensation (*10*, *11*), to E_8, the highest-order symmetry group discovered in mathematics (*12*), which has not yet been experimentally realized in physics.

The one-dimensional (1D) Ising chain in transverse field (*10*, *11*, *13*) is perhaps the most-studied theoretical paradigm for a quantum phase transition. It is described by the Hamiltonian

$$H = \Sigma_i - JS_i^z S_{i+1}^z - hS_i^x \qquad (1)$$

where a ferromagnetic exchange $J > 0$ between nearest-neighbor spin-½ magnetic moments $\mathbf{S}_i$ arranged on a 1D chain competes with an applied external transverse magnetic field h. The Ising exchange J favors spontaneous magnetic order along the z axis ($|\uparrow\uparrow\uparrow \cdots \uparrow\rangle$ or $|\downarrow\downarrow\downarrow \cdots \downarrow\rangle$), whereas the transverse field h forces the spins to point along the perpendicular $+x$ direction ($|\rightarrow\rightarrow\rightarrow \cdots \rightarrow\rangle$). This competition leads to two distinct phases, magnetically ordered and quantum paramagnetic, separated by a continuous transition at the critical field $h_C = J/2$ (Fig. 1A). Qualitatively, the magnetic field stimulates quantum tunneling processes between ↑ and ↓ spin states and these zero-point quantum fluctuations "melt" the magnetic order at h_C (*10*).

To explore the physics of Ising quantum criticality in real materials, several key ingredients are required: very good one-dimensionality of the magnetism to avoid mean-field effects of higher dimensions, a strong easy-axis (Ising) character, and a sufficiently low exchange energy J of a few meV that can be matched by experimentally attainable magnetic fields (10 T ~ 1 meV) to access the quantum critical point. An excellent model system to test this physics is the insulating quasi-1D Ising ferromagnet $CoNb_2O_6$ (*14* *16*), where magnetic Co^{2+} ions are arranged into near-isolated zigzag chains along the c axis with strong easy-

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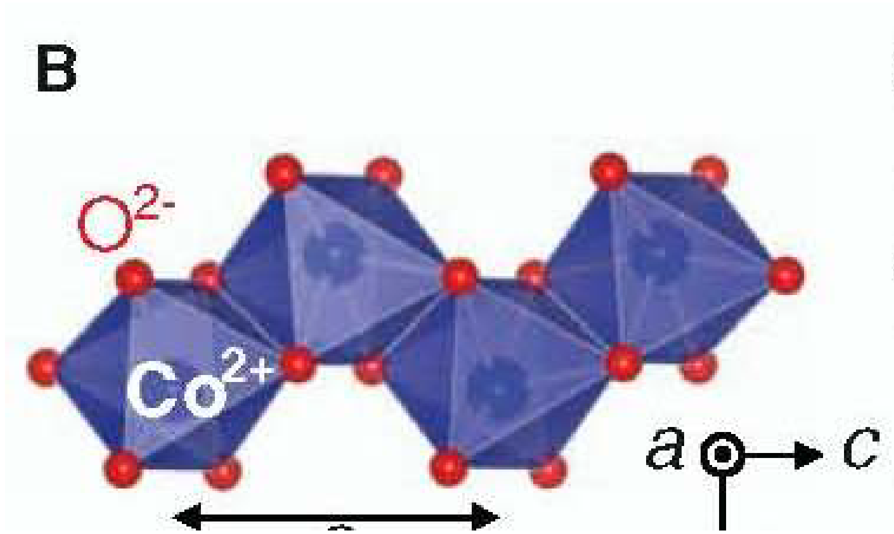

$$\hat{H} = -\sum_n \left[\sigma_n^z \sigma_{n+1}^z + \Delta\, \sigma_n^x\right] - H \sum_n \sigma_n^z$$

Quantum critical point at $\Delta = 1, H = 0$. IFT emerges in the scaling limit $\Delta \to 1$, $H \to 0$ with the ratio

$$\eta \ \sim \ (\Delta - 1)/H^{\frac{8}{15}}$$

kept fixed.

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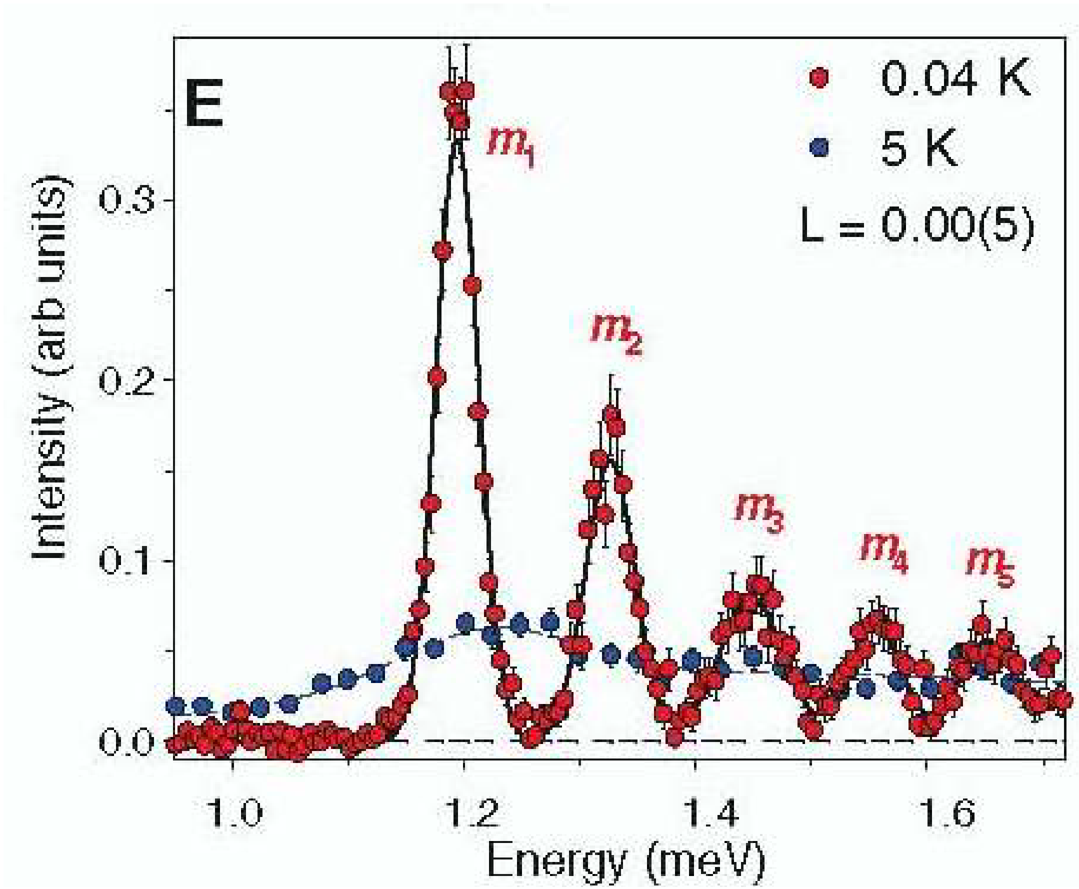
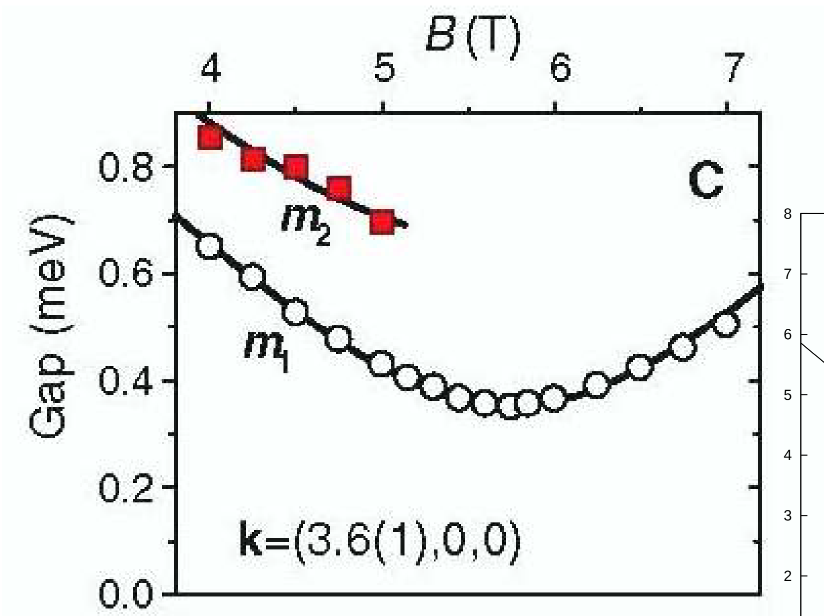
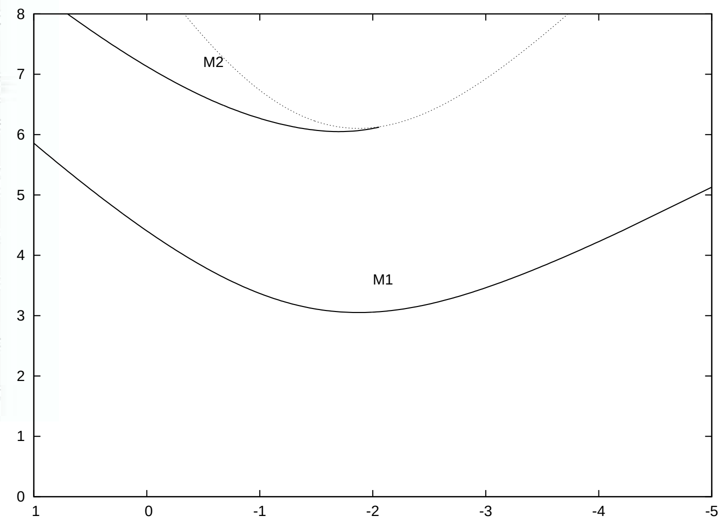

E
0.04 K
5 K
L = 0.00(5)
m_1
m_2
m_3
m_4
m_5
Intensity (arb units)
0.0
0.1
0.2
0.3
1.0
1.2
1.4
1.6
Energy (meV)


B (T)
4
5
6
7
0.8
0.6
0.4
0.2
0.0
Gap (meV)
C
m_2
m_1
k=(3.6(1),0,0)


8
7
6
5
4
3
2
1
0
1
0
-1
-2
-3
-4
-5
M2
M1

2. "Understanding" a QFT:

At generic η IFT is non-integrable - no analytic solution exists, nor is expected (except for few special **Integrable points** in the parameter space)

How much one can hope to "understand" a full-fledged, non-perturbative, non-integrable, (not even supersymmetric!) QFT?

I will discuss some features with data obtained by

- Numerical analysis (TCSA)

- Integrable QFT (integrable points)

- Interpolation between integrable points

- Special cases of resonance states accessible through

"integrability"

IFT: RG flow out of the fixed point $\mathcal{M}_{3/4}$

$$\mathcal{A}_{\text{IFT}} = \mathcal{A}_{\text{c}=1/2\ \text{CFT}} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x + h\int \sigma(x)\, d^2x\ ,$$

$\varepsilon(x)$ with $(\Delta, \bar{\Delta}) = (1/2, 1/2)$ ("energy density");

$$m \sim T_c - T \sim \Delta - 1$$

$\sigma(x)$ with $(\Delta, \bar{\Delta}) = (1/16, 1/16)$ ("spin density");

$$h \sim H$$

Apart from overall scale, the theory depends on a single dimensionless parameter

$$\eta = \frac{m}{|h|^{\frac{8}{15}}} \quad \sim \quad \frac{T_c - T}{H^{\frac{8}{15}}}\,.$$

Generally [i.e. except for $(m, h) = (0, 0)$] IFT is <u>massive</u>.

Qualitative picture ("McCoy-Wu scenario")

• $\underline{h=0}$. Onsager's theory: IFT = Free Majorana fermions

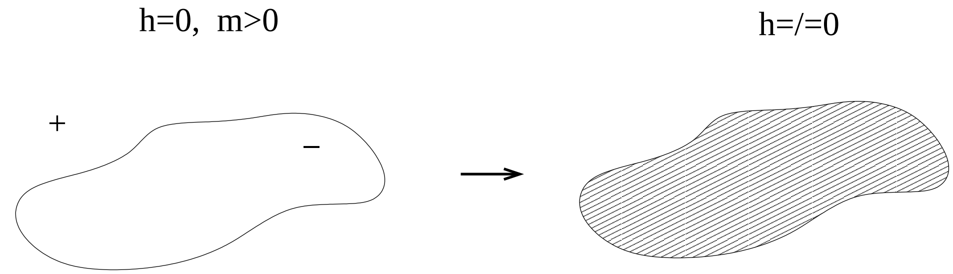
$\mathcal{A} = \int \left[\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi} + im\,\psi\bar{\psi}\right] d^2z \;\; \Rightarrow$ Free particles of mass $M_1 = |m|$

The spin field σ is "semilocal" w.r.t. the fermions. The $h = 0$ theory has two regimes:

$m < 0$ ("High-T" regime): Single vacuum $\langle\,\sigma\,\rangle = 0$, free particle of mass $|m|$ = excitation over the vacuum; $m \sim (T_c - T)/a$.

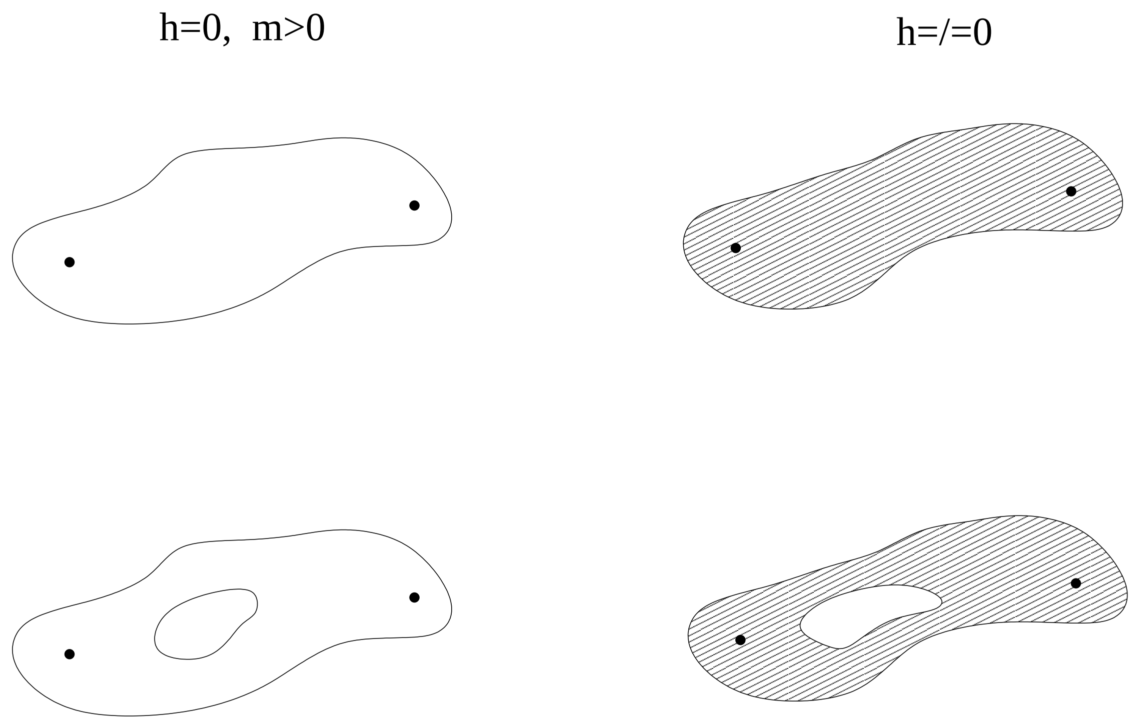
$m > 0$ ("Low-T" regime): Spontaneous breaking $\langle\,\sigma\,\rangle = \pm\,\bar{\sigma}$, free particle = kink interpolating between the two vacua.

• Low-T: Adding a weak magnetic field h $\Rightarrow$ confining attraction between the kinks (analogous to quarks) $\Rightarrow$ Tower of "mesons" (stable and resonances).



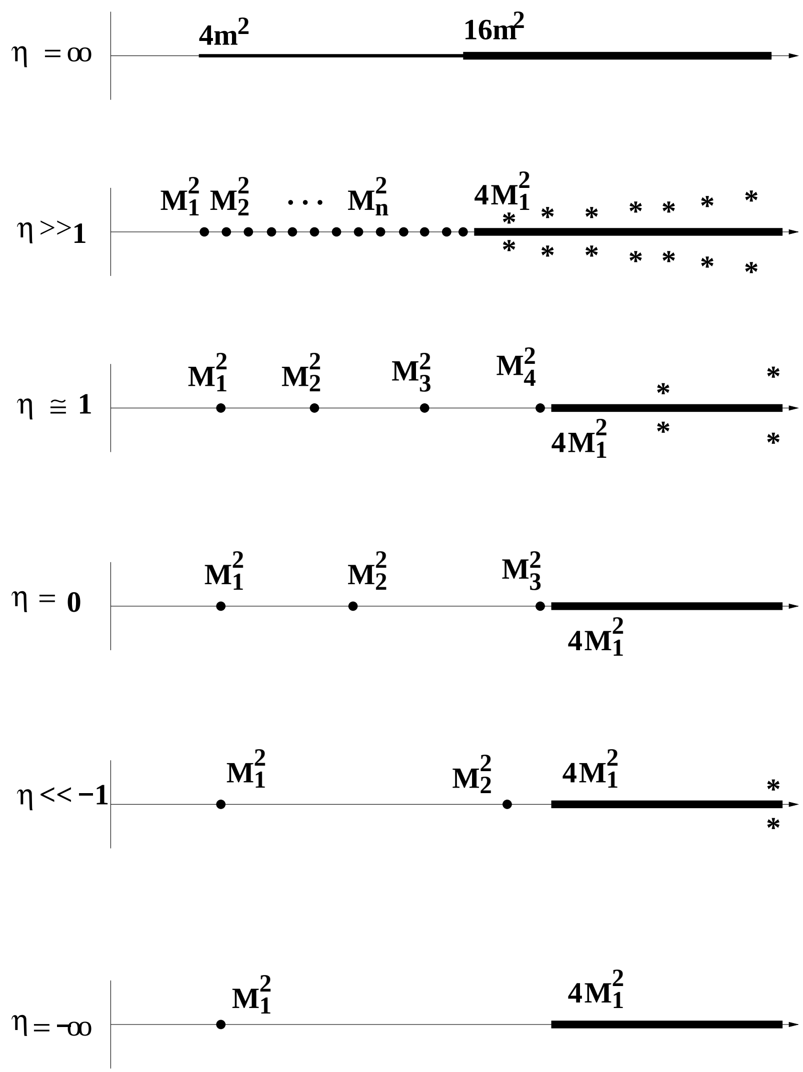
Adding h generates area-law interaction $e^{-h\bar{\sigma}\mathsf{Area}}$

Spin-spin correlation function $\langle \sigma(x_1)\sigma(x_2) \rangle$

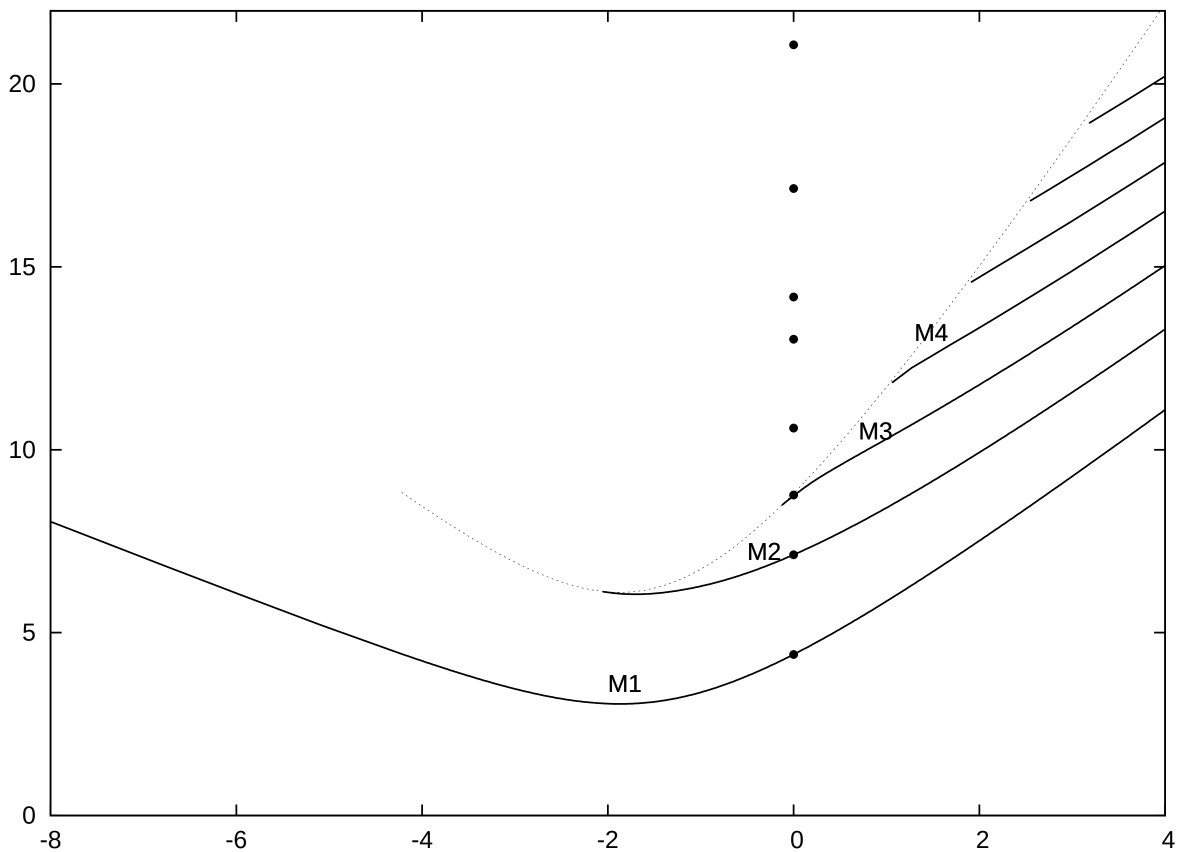


"McCoy-Wu scenario" (1978): The mass spectrum interpolates between the infinite tower of "mesons" at $\eta \to +\infty$ (Low-T regime) and one stable particle at $\eta \to -\infty$ (High-T regime).

E.g. for $G(k^2) = f.t.\langle\sigma(x)\sigma(0)\rangle \quad (k^2 = \omega^2 - p^2)$



Particle masses M_n (measured in the units of $|h|^{8/15}$), as the functions of η. Numerical results (via TCSA), and exact mass spectrum at integrable point $\eta = 0$.



I will refer to the stable particle as A_n, and their masses (measured in the units of $|h|^{8/15}$) as $M_n = M_n(\eta)$.

Questions one may ask:

What happens to the particle masses when they leave the spectrum of stable particles? ($\rightarrow$ resonances?)

The resonance states may also disappear. How this happens?

Analytic continuations of $M_n(\eta)$ as the functions of η?

It is useful to discuss in terms of the elastic $A_1 + A_1 \to A_1 + A_1$ scattering amplitude $S(\theta)$, defined as usual

$$\begin{aligned} \mid A_1(\theta_1)A_1(\theta_2)\rangle_{in} &= S(\theta_1 - \theta_2) \mid A_1(\theta_1)A_1(\theta_2)\rangle_{out} + \\ &+ \text{ inelastic terms} \end{aligned}$$

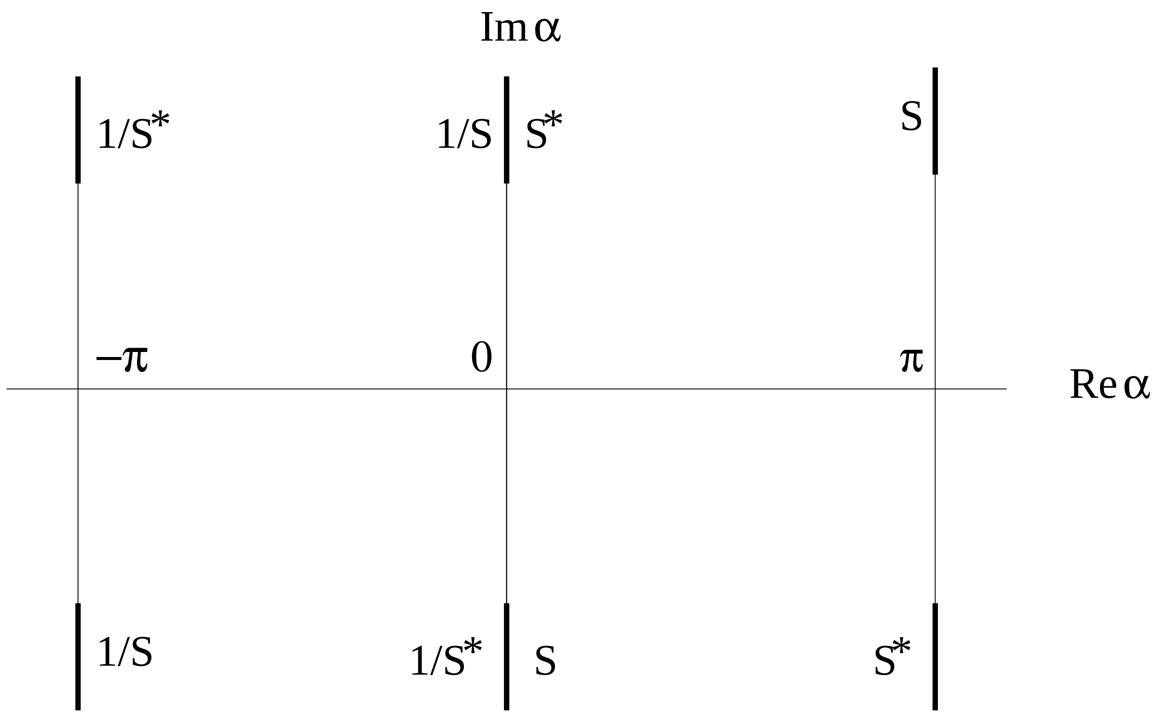
θ_1, θ_2 - rapidities: $(\omega_i, p_i) = (M_1 \cosh\theta_i, M_1 \sinh\theta_i)$;

$s = E^2_{\mathrm{CM}} = 4M_1 \cosh^2(\theta/2)$.

• $S(\theta)$ is analytic in the θ-plane with the branching singularities at $\theta = \pm\theta_X + i\pi\mathbb{Z}$, associated with the inelastic thresholds

$$A_1 + A_2 \to X$$

In complex α-plane ($\alpha = -i\theta$)



$S(\theta)$ satisfies

$$S(\theta)S(-\theta) = 1\,, \qquad S(\theta) = S(i\pi - \theta)$$

and hence periodicity,

$$S(\theta) = S(2\pi i + \theta)\,.$$

One can limit attention to the strip $-\pi < \Im m\,\theta < \pi$ ("Principal strip").

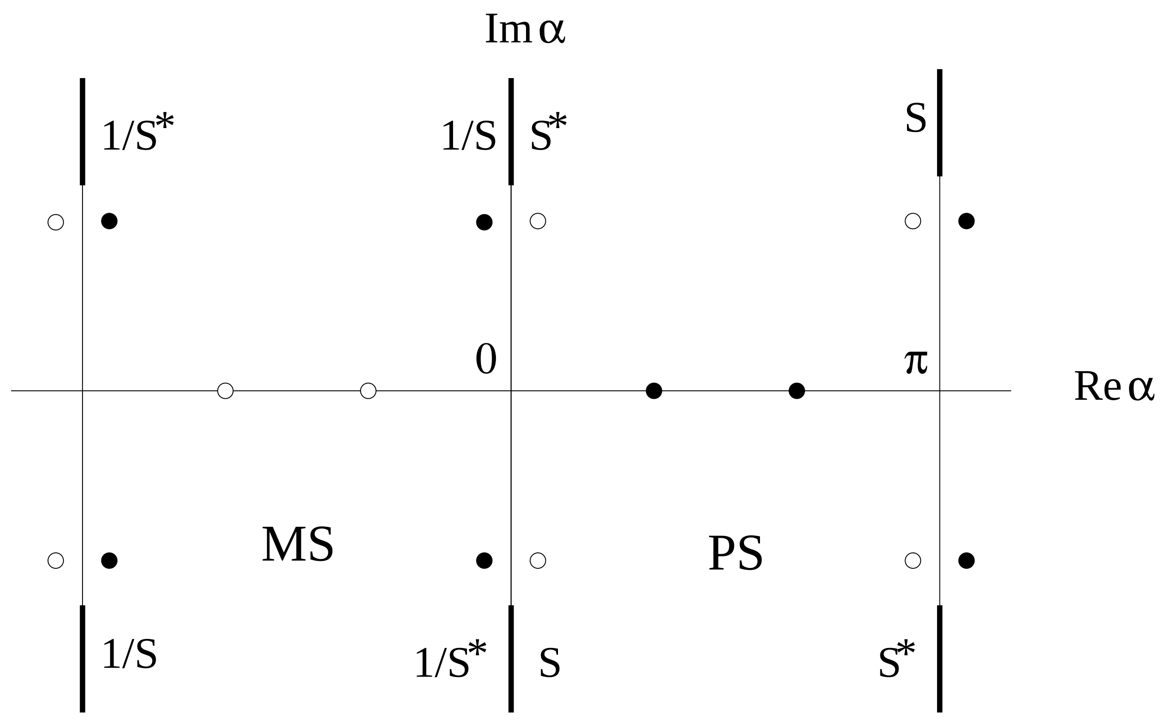
Poles

$S(\theta)$ may have poles in the Principal Strip. As $(\theta)S(-\theta) = 1$, any pole at $\alpha = \alpha_p$ has an associated "mirror" zero at $-\alpha_p$, and vice versa. I will call

<u>"Physical Strip"</u> (PS): $0 < \Im m\, \theta < \pi$. Poles in PS correspond to stable particles (s-channel, or u-channel). There can be no complex poles in PS.

<u>"Mirror Strip"</u> (MS): $-\pi < \Im m\, \theta < 0$. Poles in PS are manifested as "mirror" zeros in the MS. Real poles in MS are "anomalous thresholds", complex poles are resonance states.

Possible patters of poles $\bullet$ and zeros $\circ$ on the principal sheet



Residues

$$S(s) \simeq \frac{r_p}{s - M_p^2}\,, \qquad S(\theta) \simeq \frac{i r_p}{\theta - i\alpha_p}\,,$$

$$M_p = 2M_1 \cos\frac{\alpha_p}{2}\,.$$

Stable particle – real $\alpha_p \in PS$, and **positive** r_p. The cross-channel poles

$$S(s) \simeq \frac{r_p}{4M_1^2 - s - M_p^2}\,, \qquad S(\theta) \simeq -\frac{i r_p}{\theta - i\tilde{\alpha}_p}\,, \qquad \tilde{\alpha}_p = \pi - \alpha_p$$

have **negative** residues.

Many particle theories (including the one associated with IFT) have "φ^3 property": A_1 appears as a "bound state pole" in $A_1 A_1$ scattering $\Rightarrow$ Fixed-position poles at

$$\alpha_1 = 2\pi/3\,, \qquad \tilde{\alpha}_1 := \pi - \alpha_1 = \pi/3\,.$$

Resonances appear as complex poles in the MS.

Integrable points of IFT

(a) $\boldsymbol{\eta} = -\infty$ ($h = 0$, $m < 0$) Free particle A_1, of mass $M_1 = -m$

$$S(\theta) = -1\,.$$

Trivial pattern - no poles, no zeros.

(b) $\boldsymbol{\eta} = \mathbf{0}$ ($h \neq 0$, $m = 0$). There are eight particles A_1, A_2, ..., A_8, with purely elastic S-matrix ("E_8 structure").

$$(M_1, M_2, ..., M_8) \;\simeq\; \text{Perron-Frobenius vector of} \;\; C(E_8)$$

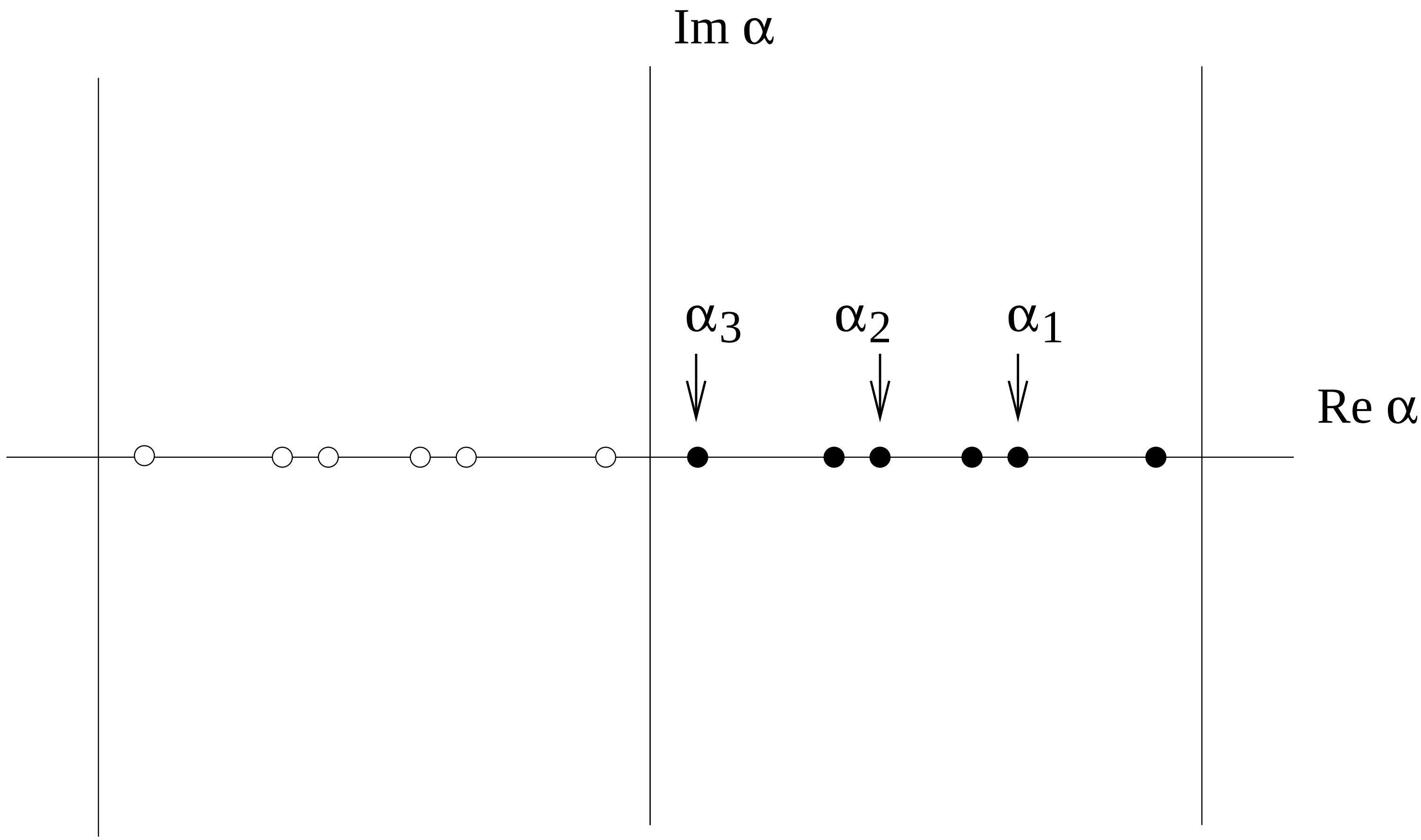
For $A_1 A_1 \to A_1 A_1$

$$S(\theta) = \frac{\sinh\theta + i\sin(2\pi/3)}{\sinh\theta - i\sin(2\pi/3)}\,\frac{\sinh\theta + i\sin(2\pi/5)}{\sinh\theta - i\sin(2\pi/5)}\,\frac{\sinh\theta + i\sin(\pi/15)}{\sinh\theta - i\sin(\pi/15)}\,.$$

Three pairs of poles α_p, $\tilde{\alpha}_p = \pi - \alpha_p$ in PS

$$\alpha_1 = \frac{2\pi}{3}\,, \qquad \alpha_2 = \frac{2\pi}{5}\,, \qquad \alpha_3 = \frac{\pi}{15}\,.$$

correspond to A_1, A_2, A_3 (A_4, ...,A_8 appear as poles in higher amplitudes)



$\eta = 0.00$

Suppose we start with $\eta = 0$, and then change this parameter to negative values, all the way down to $-\infty$. How the complicated pattern at $\eta = 0$ evolves into the trivial one corresponding to the free theory $\eta = -\infty$?

Small nonzero η: Perturbation theory in m

$$\mathcal{A}_{\text{IFT}} = \mathcal{A}_{\text{c=1/2 CFT}} + h \int \sigma(x)\, d^2x + \frac{m}{2\pi} \int \varepsilon(x)\, d^2x\,.$$

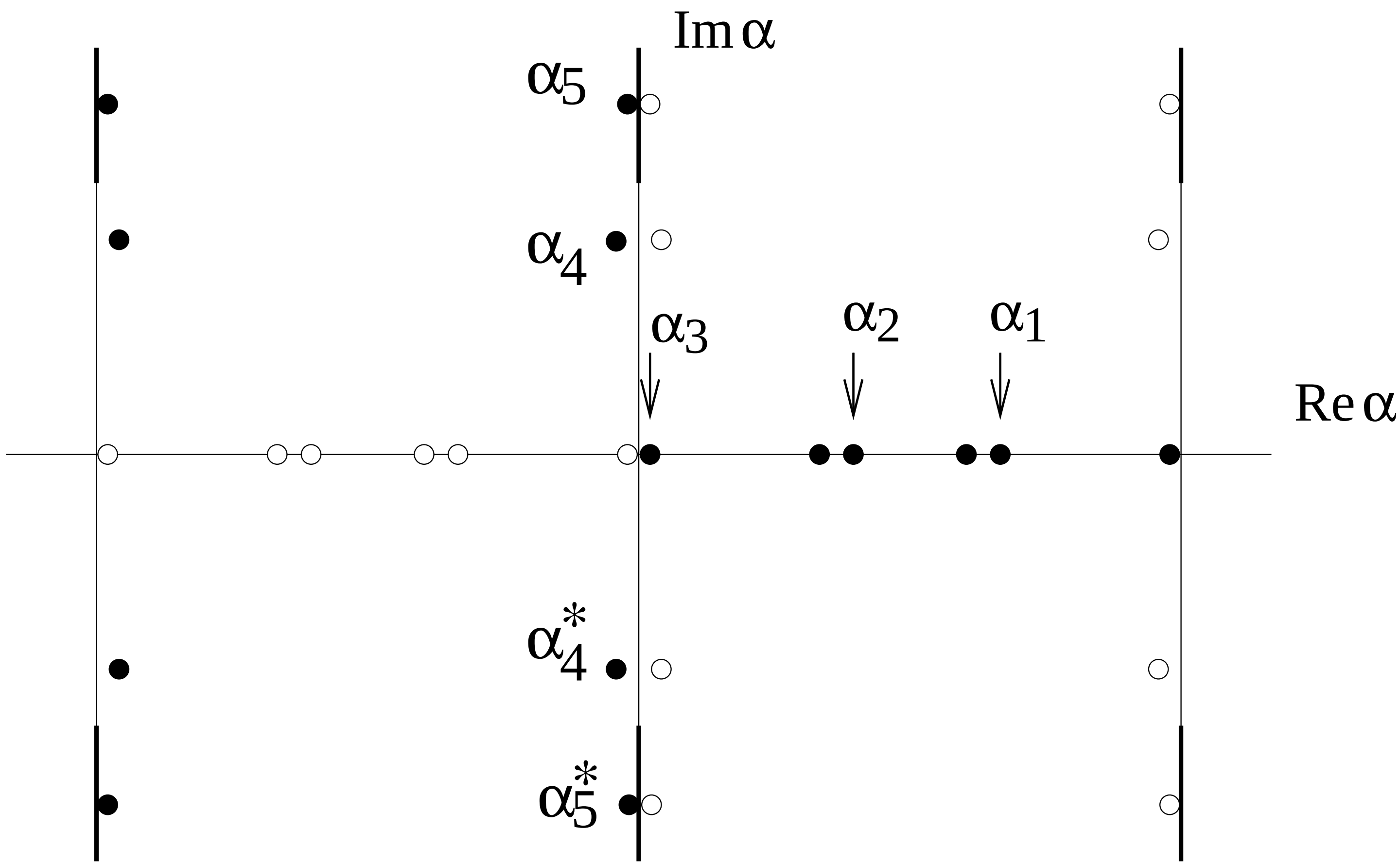
- $M_n(\eta) = M_n^{(0)} + M_n^{(1)}\,\eta + ...$

From $M_p/M_1 = 2\cos(\alpha_p/2)$

$$\alpha_2 = \frac{2\pi}{5} + \alpha_2^{(1)}\,\eta + ...\,, \qquad \alpha_3 = \frac{\pi}{15} + \alpha_3^{(1)}\,\eta + ...\,,$$

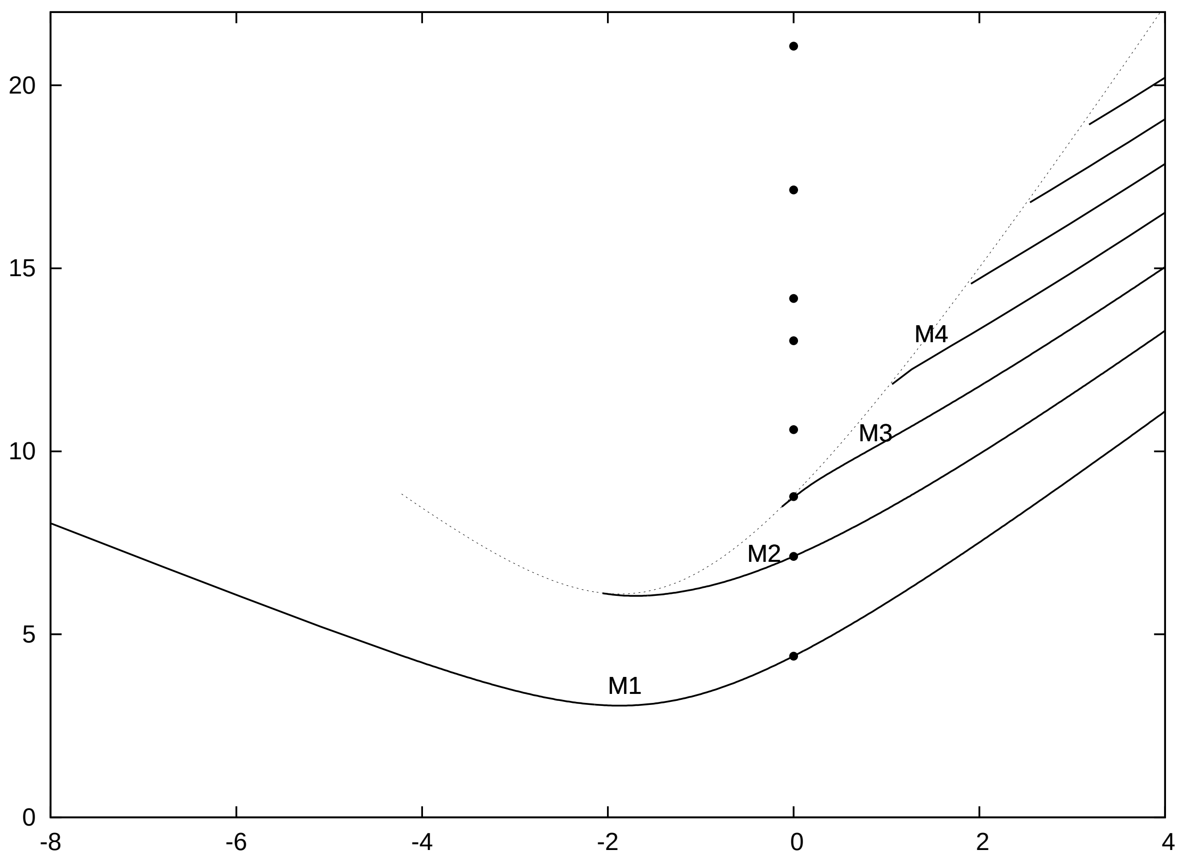
$$\alpha_2^{(1)} = 0.378325...\,, \qquad \alpha_3^{(1)} = 1.35226...\,,$$

When η becomes small negative both α_2 and α_3 move to the left ...

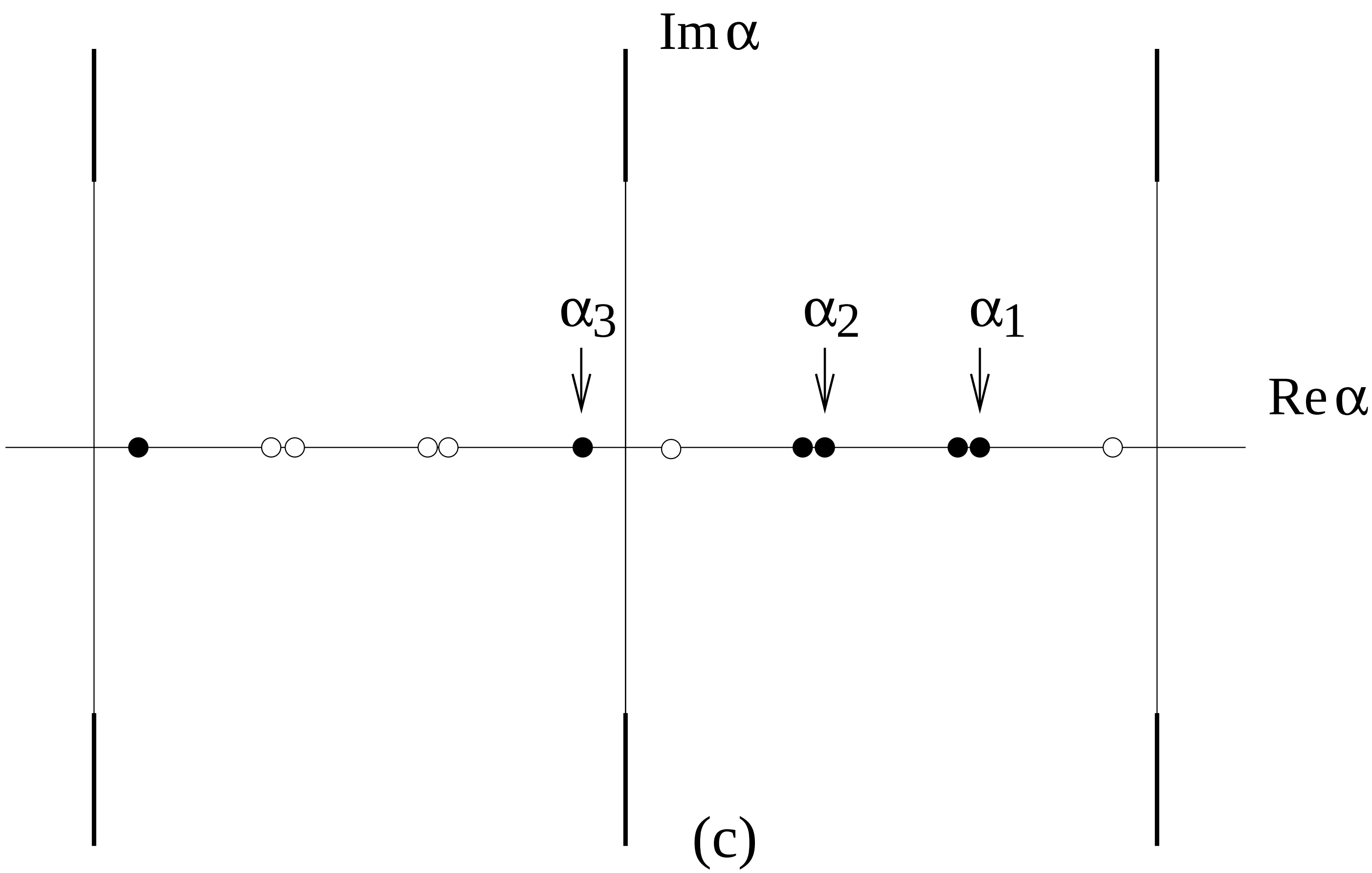


$\eta = -0.08$

Particle masses M_n (measured in the units of $|h|^{8/15}$), as the functions of η.



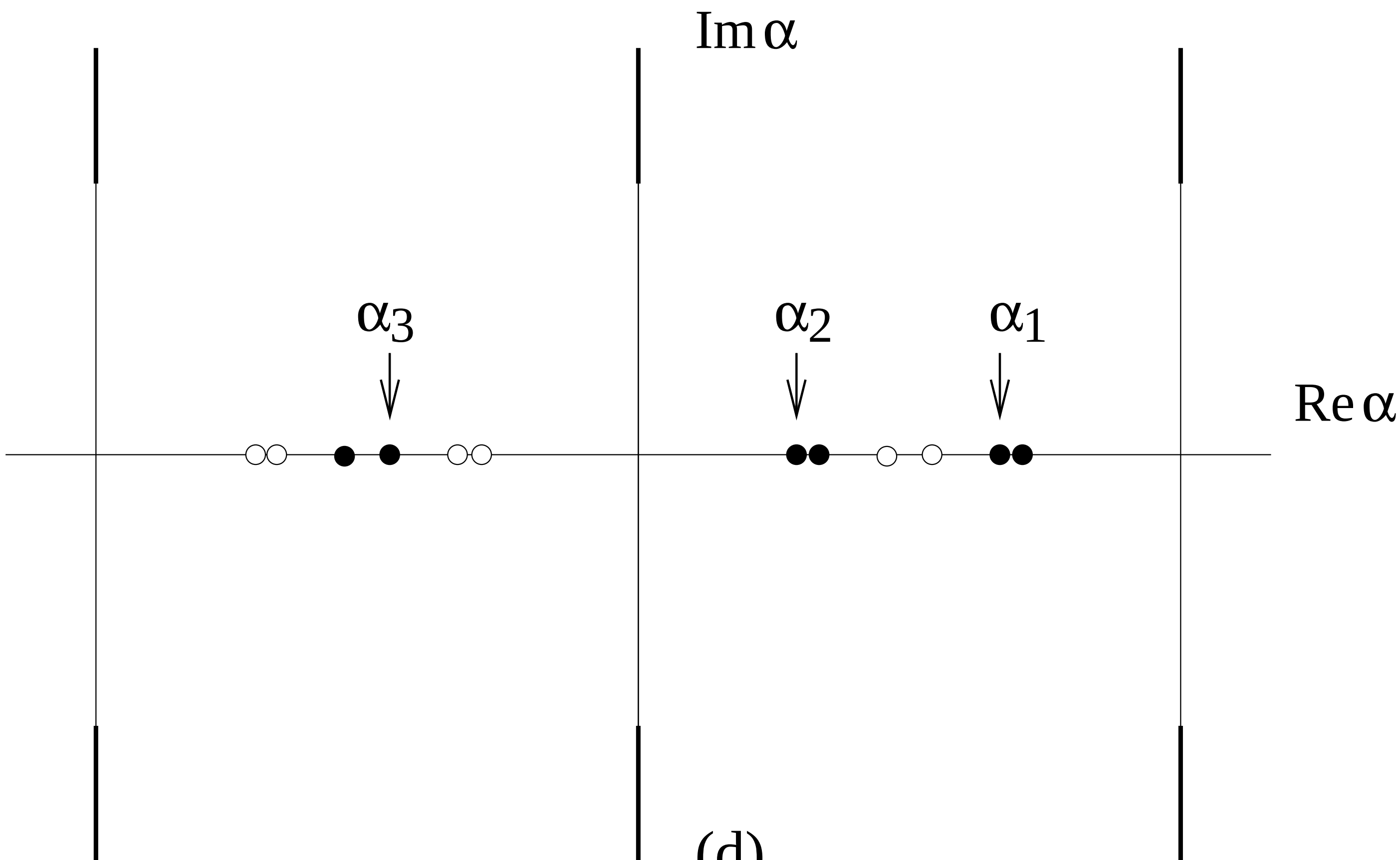
At certain $\eta_3 \approx -0.138$ α_3 leaves PS, and enters MS. A_3 disappears as a stable particle



$\eta = -0.27$

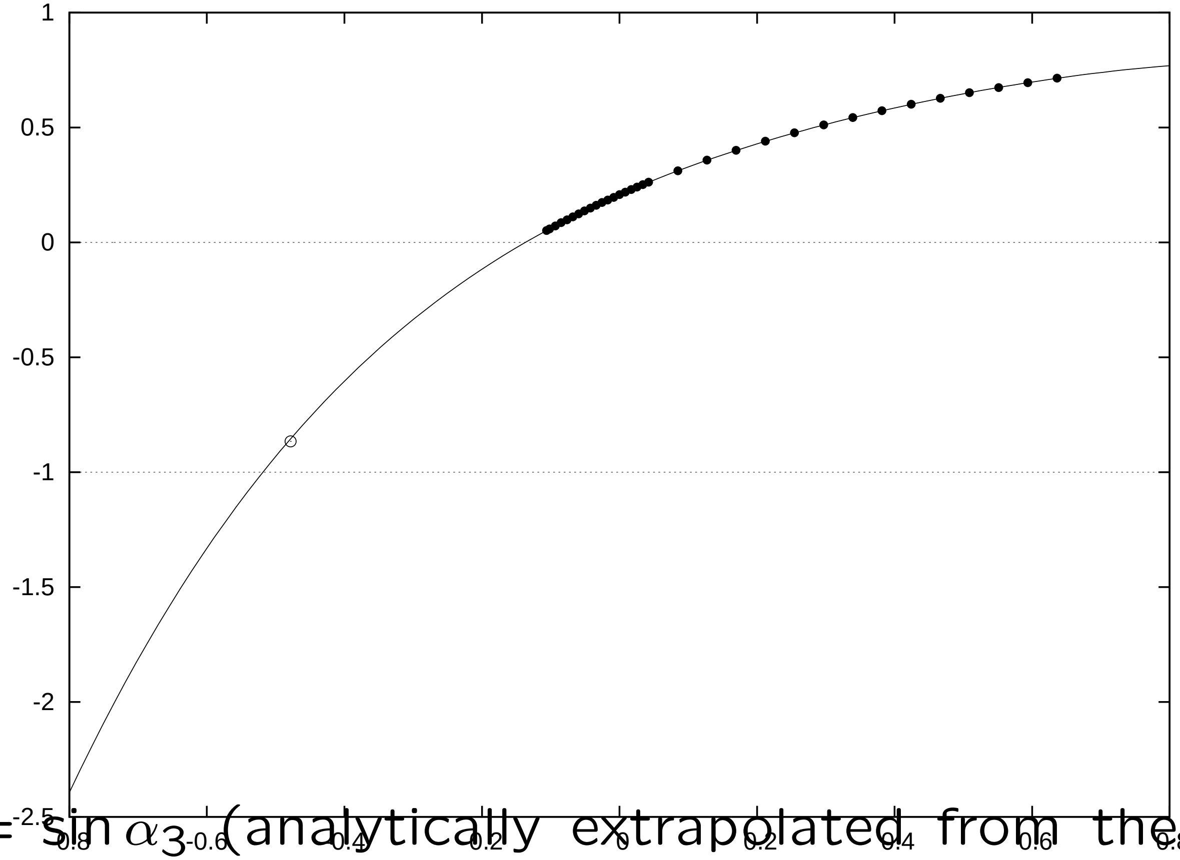
At $\eta_{12} \approx -0.477$ α_2 crosses $\pi - \alpha_1 = \pi/3$. Simultaneously, α_3 must cross $-\pi/3$, which happens when $M_2/M_1 = \sqrt{3}$. I.e.

$$\eta_{12}: \quad M_3/M_1 = M_2/M_1 = \sqrt{3}$$



(d)

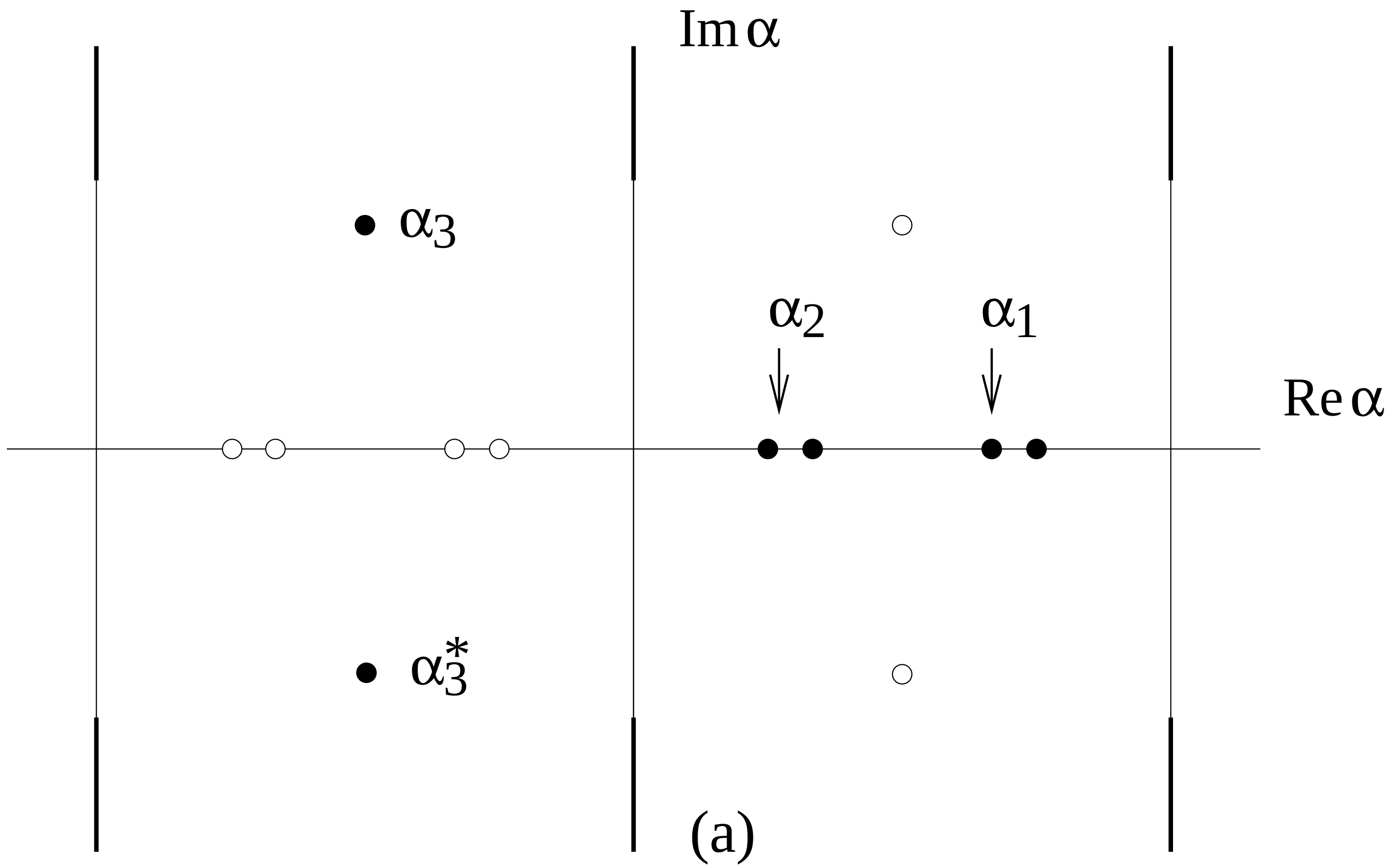
$\eta = -0.49$



$B_3 = \sin \alpha_3$ (analytically extrapolated from the domain $\eta > \eta_3$, where M_3/M_1 is directly available through TCSA

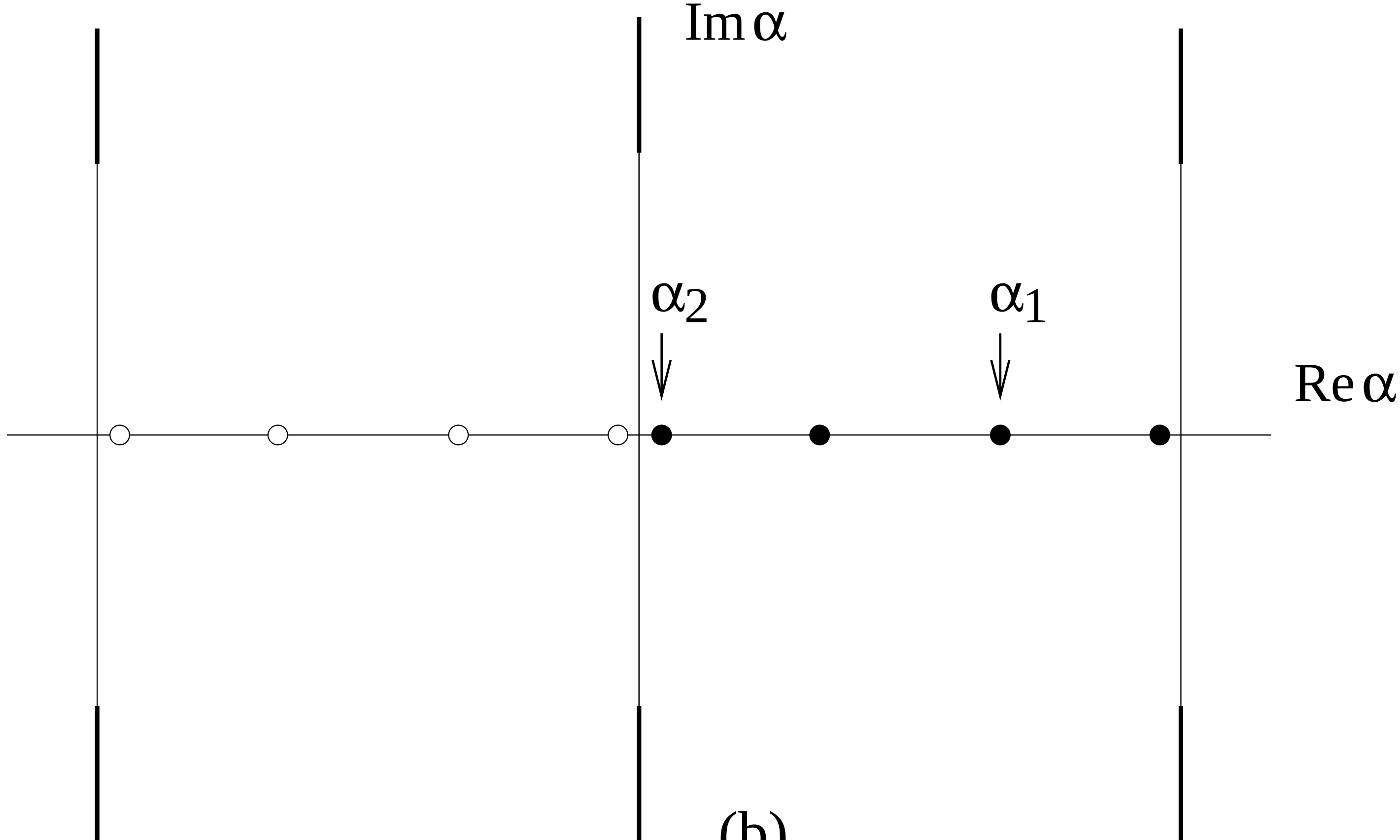
$B_3 < -1$ (α_3 is complex) at $\eta < \eta_{33} \approx -0.55$

At $\eta_{33} \approx -0.51$ the poles α_3 and $-\pi - \alpha_3$ collide at $-\pi/2$, and become complex poles



$\eta = -0.94$

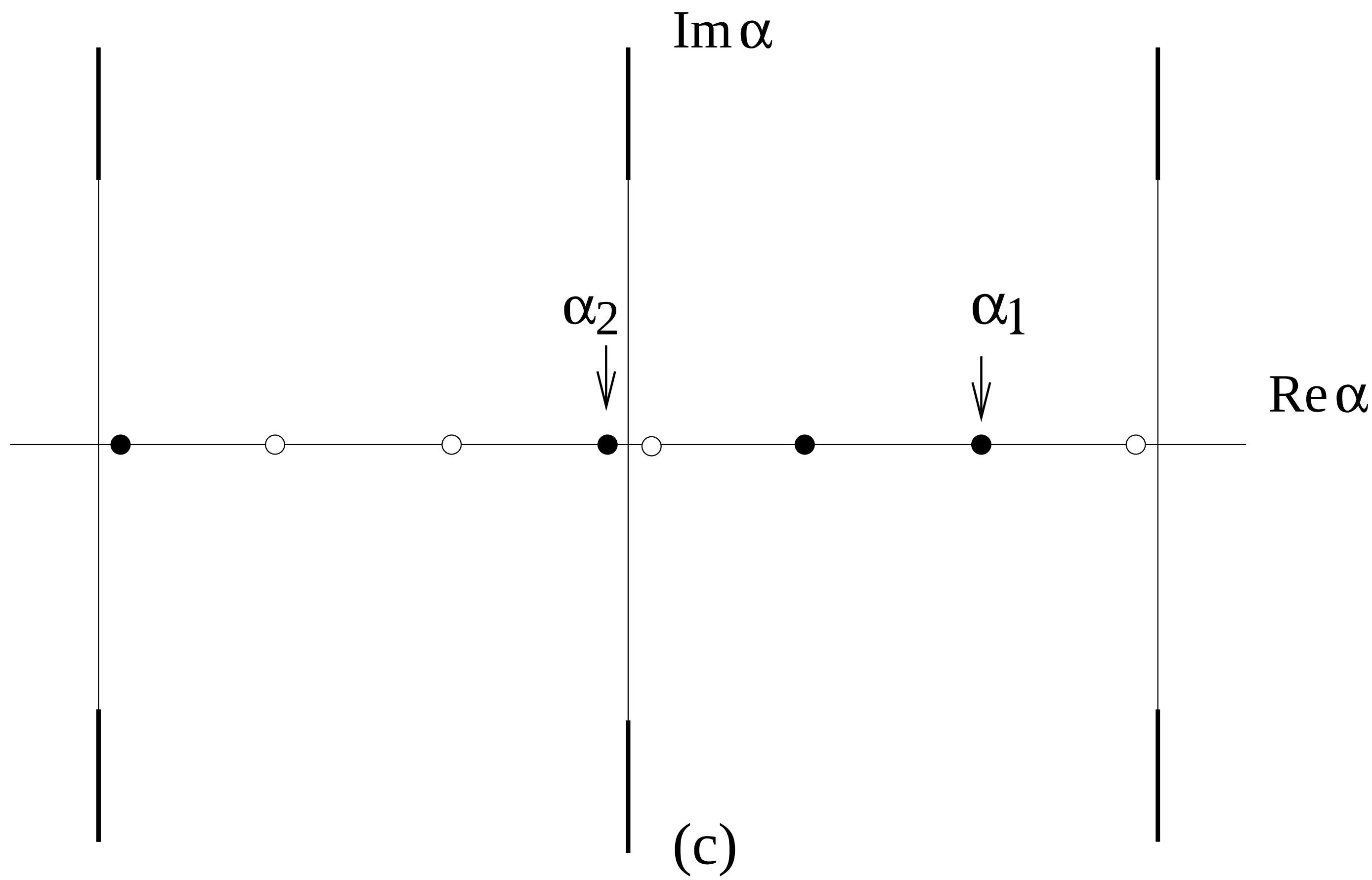
At $\eta \to \eta_2 + 0$, $\eta_2 \approx -2.08$ the pole α_2 approaches zero



(b)

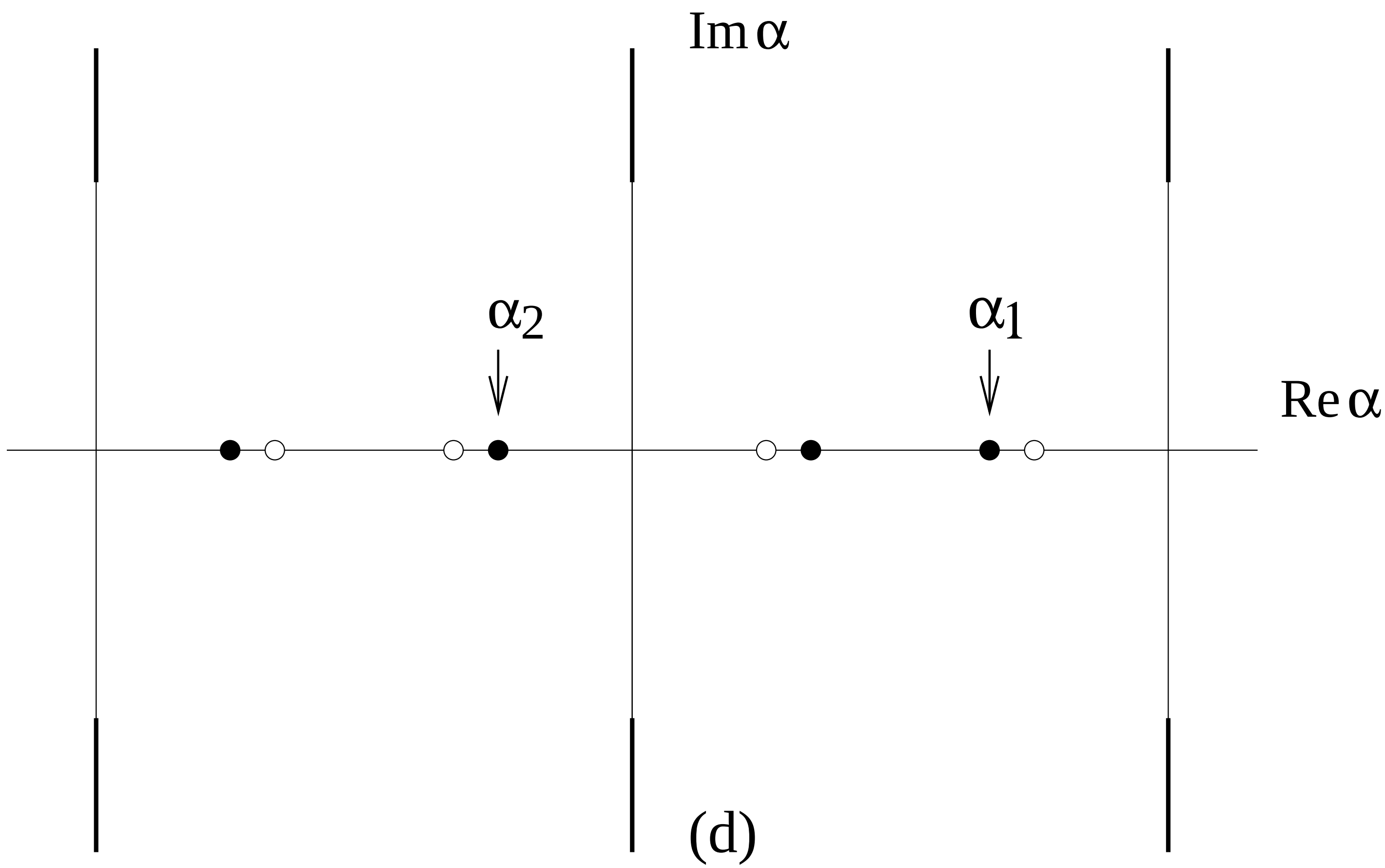
$\eta = -1.87$

And at $\eta < \eta_2$ it crosses into MS. A_2 disappears from the spectrum, so that below η_2 only A_1 is left.



$\eta = -2.29$

At $\eta \to -\infty$

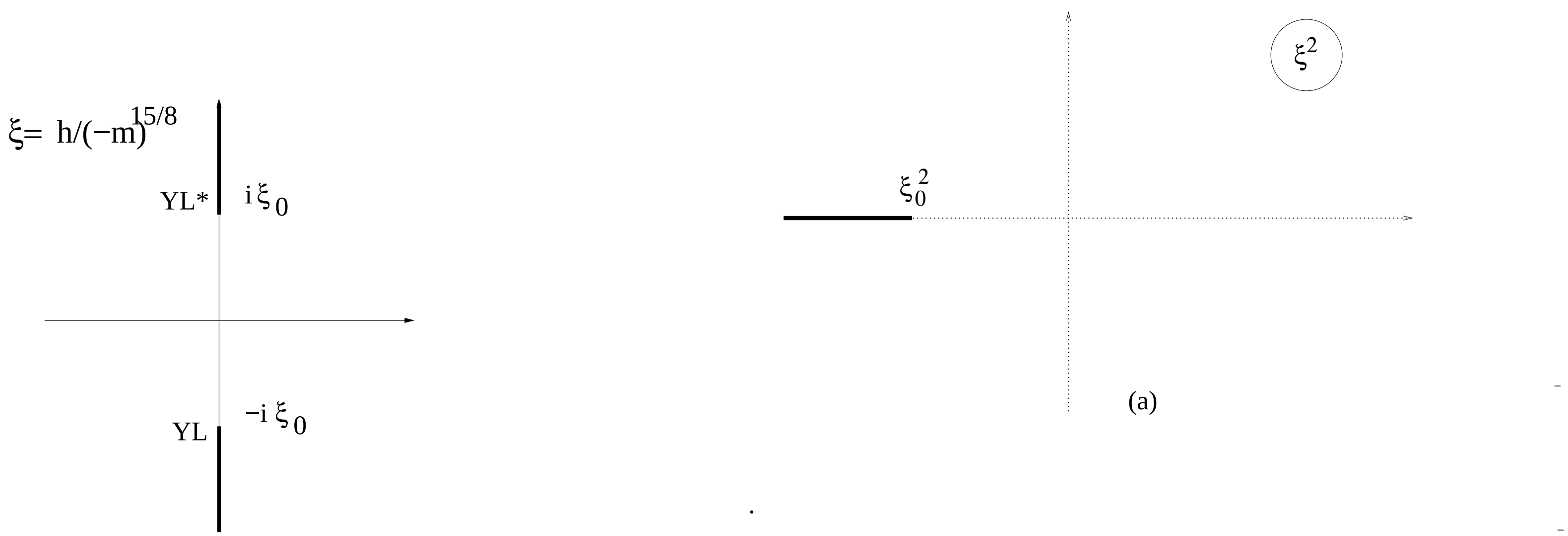


$\eta = -4.35$

Pure imaginary h at $m < 0$.

IFT remains "real" at pure imaginary h, below the Yang-Lee singularity

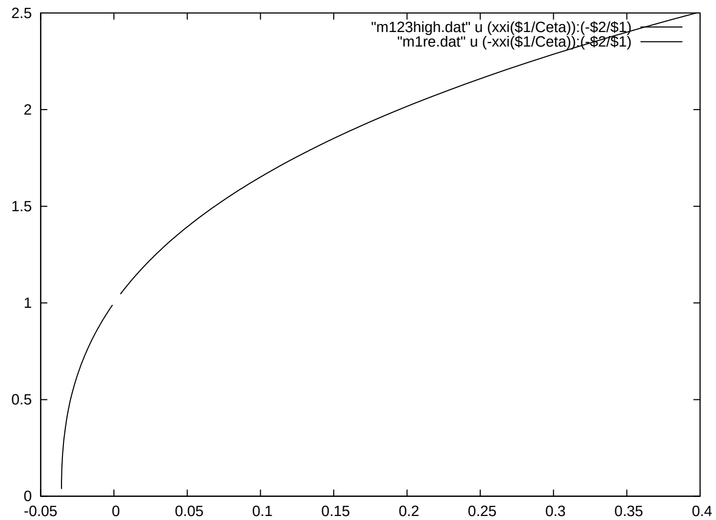
$$\xi^2 \equiv h^2/(-m)^{15/4} > -\xi_0^2 \ \approx -0.035846$$



$$\xi^2 = 1/(-\eta)^{15/4}$$

• YL = critical point. CFT = $\mathcal{M}_{2/5}$, with $c = -22/5$ (**J.Cardy, 1985**),

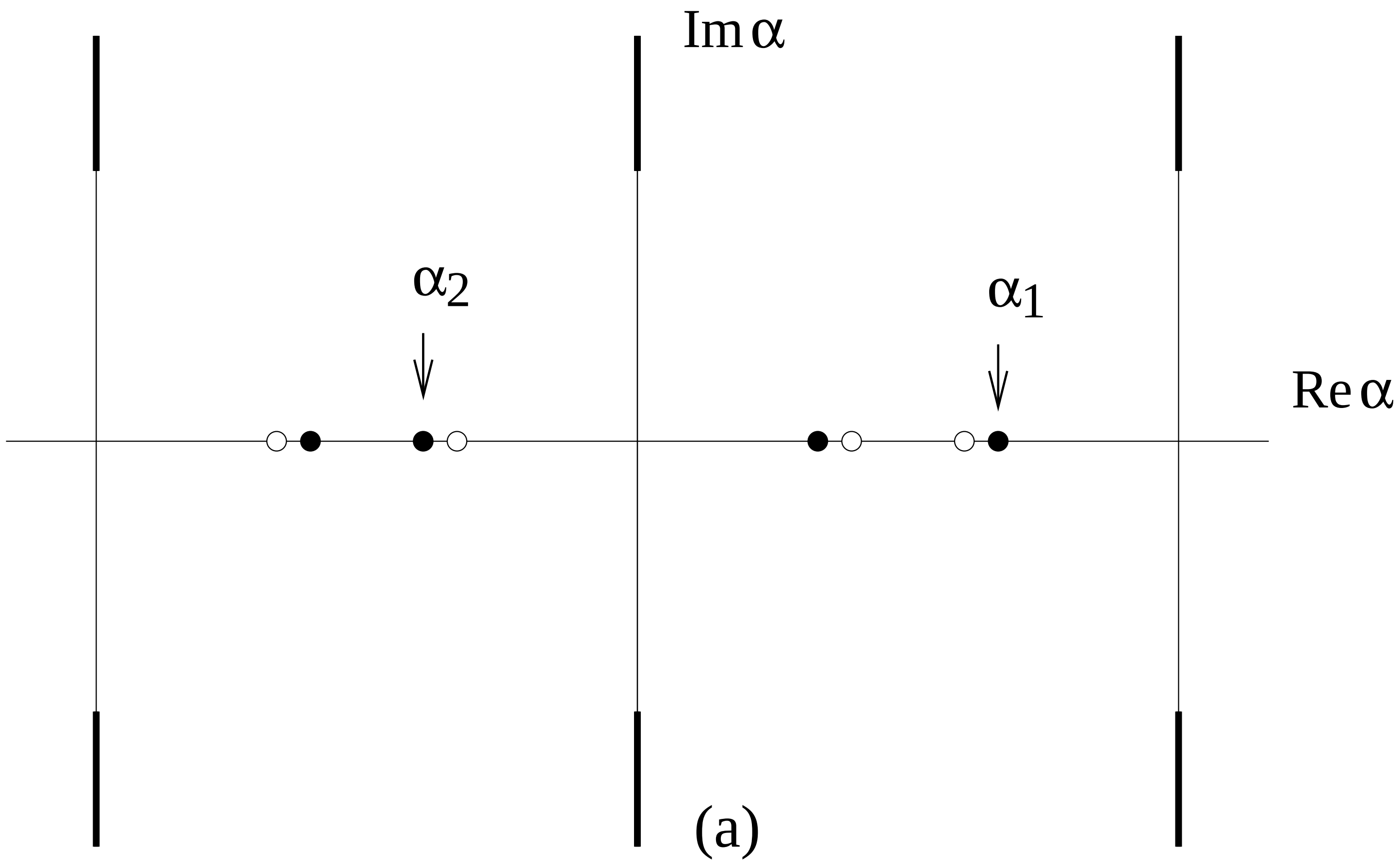
$$M_1 \sim (\xi^2 + \xi_0^2)^{5/12}$$



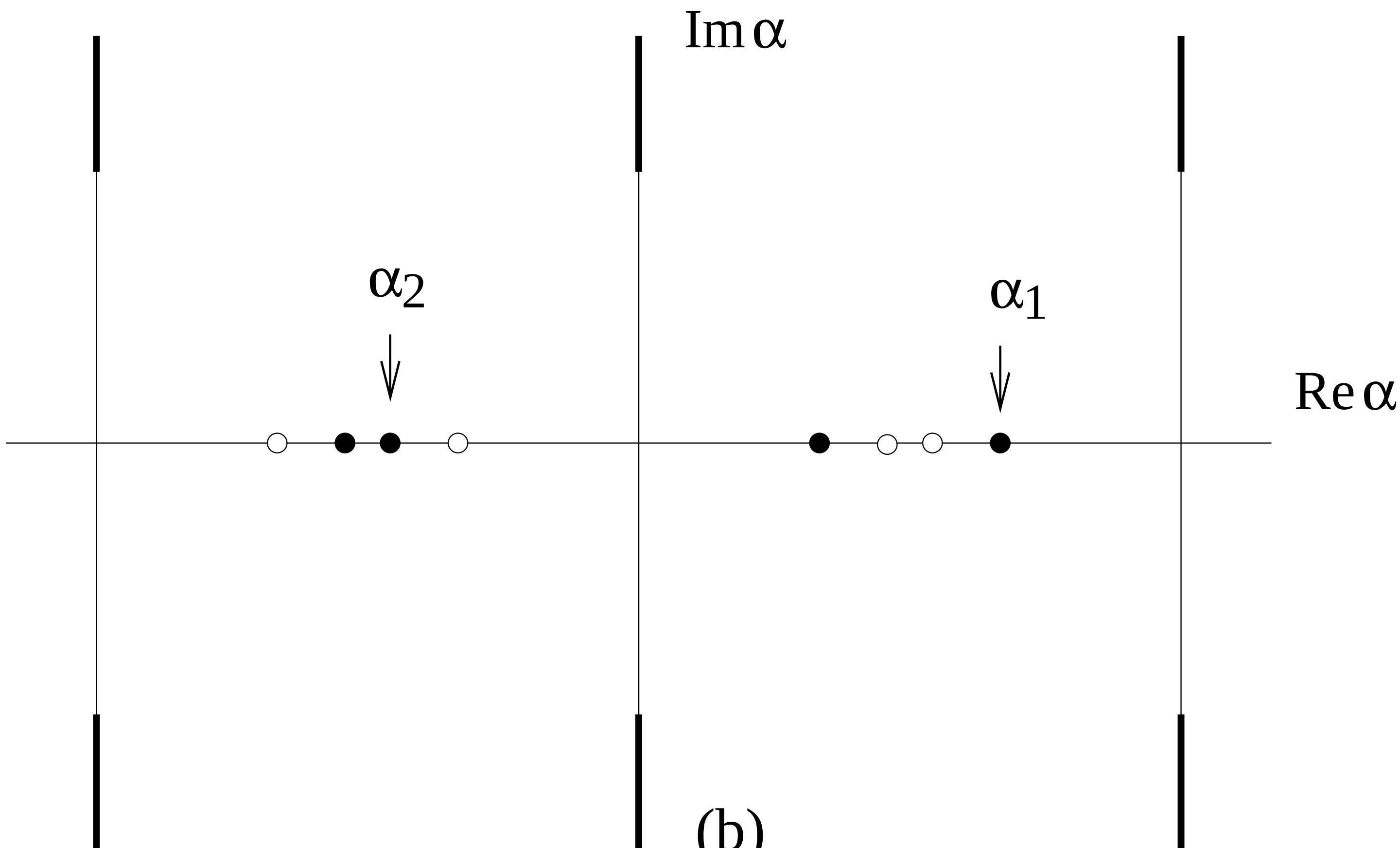
• Integrable at $\xi^2 \to -\xi_0^2$, with one particle, and

$$S(\theta) = \frac{\sinh\theta + i \sin\alpha_1}{\sinh\theta - i\sin\alpha_1}\,, \qquad \alpha_1 = \frac{2\pi}{3}\,.$$

(**J. Cardy, G. Mussardo, 1989**)



(a)



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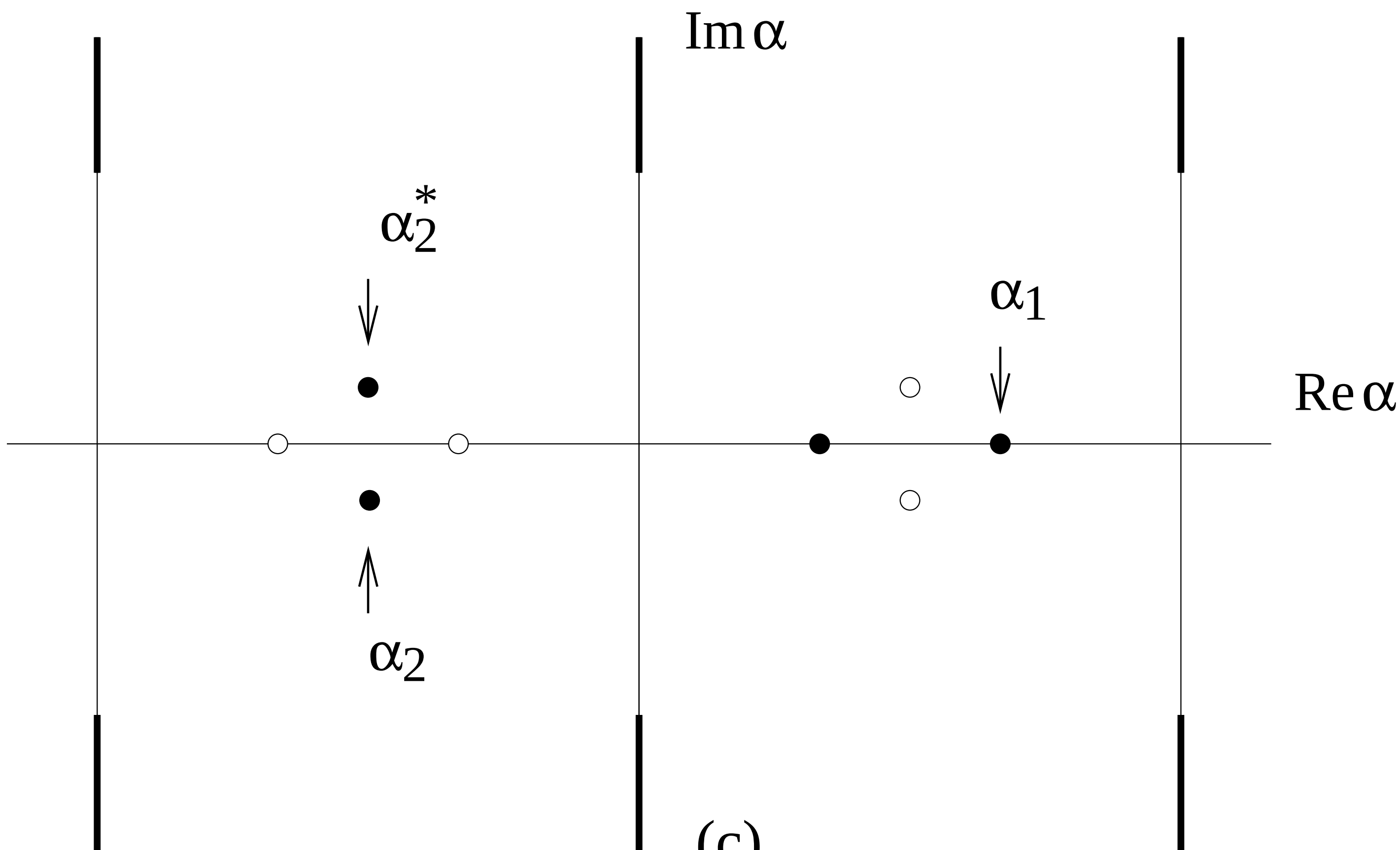
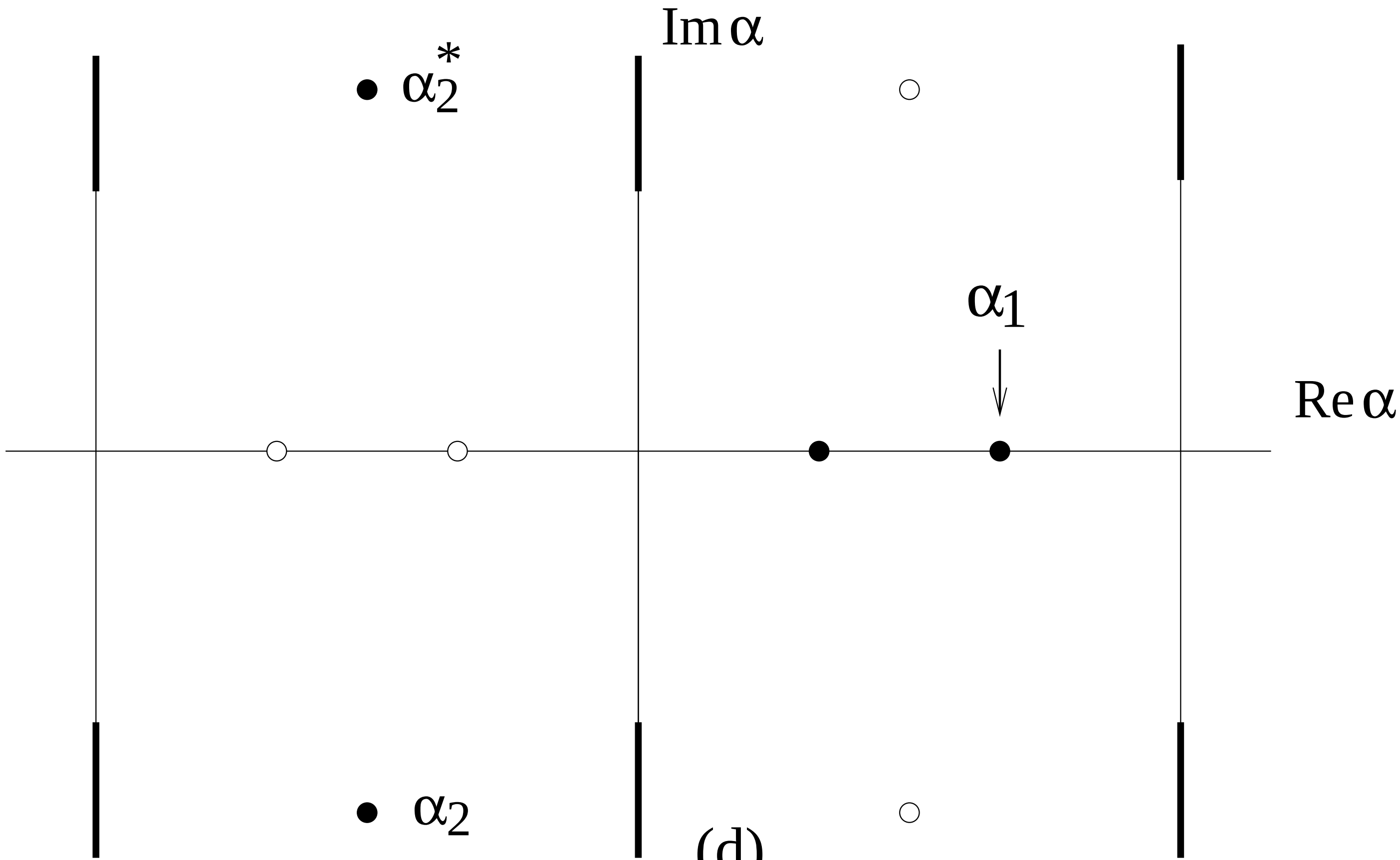


(c)



(d)

$$S_{\mathsf{YL}}(\theta) = \frac{\sinh\theta + i\sin 2\pi/3}{\sinh\theta - i\sin 2\pi/3}\,.$$

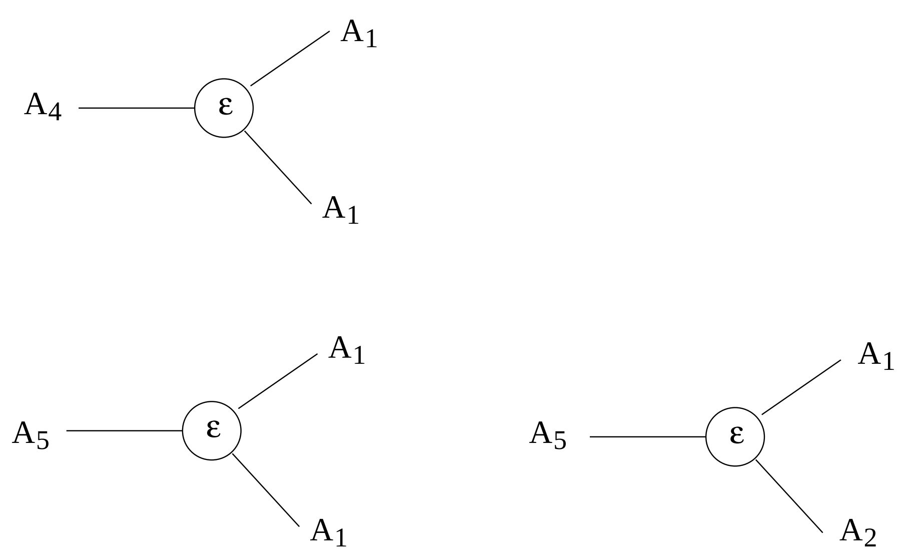
Resonances

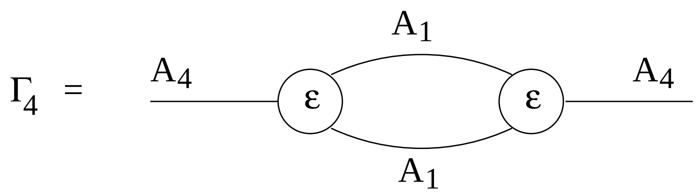
As η decreases from 0 to $-\infty$, first A_3, and then A_2, become resonances. But there are many more resonances ...

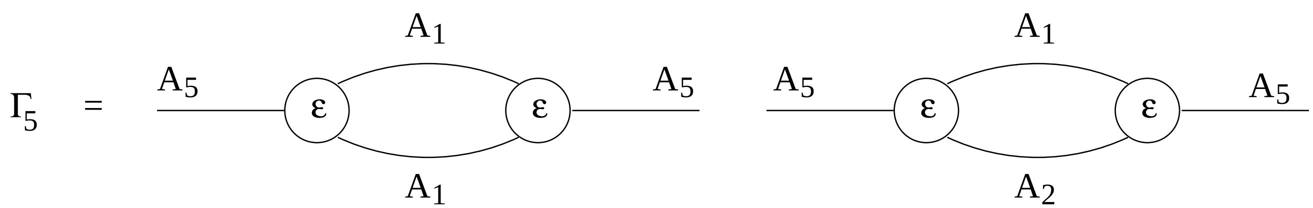
- $\eta = 0$: Eight stable particles, with $M_4, M_5, ..., M_8 > 2M_2$. At $\eta \neq 0$ integrability is violated $\Rightarrow$ decay channels open

Perturbation theory in η: $A_n \to A_m A_k$ decay amplitude $\sim$ the form-factor

$$\langle A_n(0) \mid \varepsilon(0) \mid A_m(\theta_1) A_k(\theta_2) \rangle \neq 0$$







$$\Gamma_4 = (0.047587\ \eta^2)\, M_1\,, \qquad \Gamma_5 = (0.011000\ \eta^2)\, M_1\,,$$

(G. Delfino, P. Grinza, G. Mussardo, 2005).

Away from $\eta = 0$ all five "heavy" particles A_4, A_5, ..., A_8 become resonances.

What happens to A_4, A_5, ..., A_8 when η decreases from zero?

• "Experimental" observation: Some of the resonances, e.g. A_5 (but not A_4), remain very narrow when η is not too far from zero (As seen from numerics). Why?

• At $\eta = 0$ M_n are given by components of Perron-Frobenius vector of the cartan matrix of E_8, e.g.

$$A_2: \qquad M_2 = 2M_1 \cos\frac{\pi}{5} = 1.61803\ M_1$$

$$A_3: \qquad M_3 = 2M_1 \cos\frac{\pi}{30} = 1.98904\ M_1$$

A_3 can be understood as weakly coupled $A_1 A_1$ bound state

$$\varepsilon_2 \equiv 2M_1 - M_3 = 4M_1 \sin^2\frac{\pi}{60} \approx 0.0109562\ M_1$$

Then one expects to have three- and four- and multi-particle bound states:

In 1+1, particles in weakly bound states are well approximated by non-relativistic QM with δ-function attraction:

$$\hat{H} = -\sum_{i=1}^{k} \frac{d^2}{dx_i^2} - u \sum_{i<j} \delta(x_i - x_j)\,, \qquad u > 0$$

$\exists$ k-particle bound states with the binding energies

$$\varepsilon_k \simeq \frac{k^3 - k}{3!}\,\varepsilon_2\,.$$

At small η, one then expects to have particles with the masses close to $3M_1$, $4M_1$, etc - weakly bound states of 3, 4, etc particles A_1.

Indeed, we have

$$M_5 = 4M_1 \cos\frac{\pi}{5} \cos\frac{2\pi}{15} = 2.95629\ M_1 = 3M_1 - \text{small}$$
$$M_7 = 8M_1 \cos^2\frac{\pi}{5} \cos\frac{7\pi}{30} = 3.89115\ M_1 = 4M_1 - \text{small}$$
$$M_8 = 8M_1 \cos^2\frac{\pi}{5} \cos\frac{2\pi}{15} = 4.78338\ M_1 = 5M_1 - \text{small}$$

$$3M_1 - M_5 = 0.043704\ M_1\,, \qquad \varepsilon_3 = 0.043824\ M_1\,,$$
$$4M_1 - M_7 = 0.108843\ M_1\,, \qquad \varepsilon_4 = 0.109562\ M_1\,,$$
$$5M_1 - M_8 = 0.216613\ M_1\,, \qquad \varepsilon_5 = 0.219124\ M_1\,.$$

Here M_5, M_7, M_8 are exact, but ε_k are given by the approximation

$$\varepsilon_k = \frac{k^3 - k}{3!}\,\varepsilon_2\,.$$

Remarkably, the PF vector of $C(E_8)$ "knows' about weakly interacting particles!

- At $\eta = 0$ there are weakly coupled multi-particle bound states

$$A_3 = (A_1 A_1)\,, \qquad A_5 = (A_1 A_1 A_1)\,,$$
$$A_7 = (A_1 A_1 A_1 A_1)\,, \quad A_8 = (A_1 A_1 A_1 A_1 A_1)$$

Predictions:

• When η is small negative, the "binding energy" of A_3

$$\varepsilon_2 \equiv 3M_1 - M_3 = 4M_1 \sin^2 \frac{\alpha_3}{2}$$

becomes even smaller $\Rightarrow$ A_5, A_7, A_8 (now resonances) are even better approximated as the weakly coupled 3, 4, 5 particle bound states.

The approximation

$$M_5 = 3M_1 - \varepsilon_3 \,, \quad M_7 = 4M_1 - \varepsilon_4 \,, \quad M_8 = 5M_1 - \varepsilon_5$$

is expected to work even better. Also, the imaginary parts ($\Gamma_n = -\Im m \, M_n$) are expected to be small

$$\Gamma_5 \,,\ \Gamma_7 \,,\ \Gamma_8 \ \sim\ \varepsilon_2^2$$

(Analog of tetra-quark and higher exotic states in QCD?)

• At $\eta < \eta_3$ the resonances A_5, A_7, A_8 disappear (poles leave the principal sheet)

These predictions are well consistent with numerical data.

Q:

• What about six-, seven-, and higher multi-particle bound states at small η? $\varepsilon_5 \approx 0.219\ M_1 \sim M_1$

Interference with another particle channels? At $\eta = 0$

$$6M_1 - \varepsilon_6 \ \approx 5.617\ M_1\,, \qquad M_4 + M_6 \approx 5.623\ M_1$$

• What about "missing" resonances A_4 and A_6?

At $\eta = 0$

$$M_4 = 4M_1 \cos\frac{\pi}{5}\cos\frac{7\pi}{30} = 2.404867\ M_1\,,$$
$$M_6 = 4M_1 \cos\frac{\pi}{5}\cos\frac{\pi}{30} = 3.218340\ M_1$$

As η approaches η_2

$$2M_1 - M_2 = 4M_1 \sin^2 \frac{\alpha_2}{2}$$

becomes small. Now A_2 becomes weakly coupled $(A_1 A_1)$, and again, one expects to see weakly-coupled multi-particle bound states. It is plausible that as $\eta \to \eta_2 + 0$

$$A_4 \approx (A_1 A_1 A_1)\,, \quad A_6 \approx (A_1 A_1 A_1 A_1)$$

There may be many more resonances

(a) Do not stem from any stable particles at integrable points

(b) Wide ($\Gamma > M_1$)

(c) High energy ($\bar{M} >> M_1$)

Because of (b) and (c) - difficult/impossible to extract from existing numerics.

Requires data about high energy scattering (hep-th)...

Open questions and directions:

• Evolution of poles at $\eta > 0$. How E_8 spectrum evolves into infinite tower of "mesons" at $\eta \to +\infty$?

• Fate of resonances

• Analytic continuation towards Yang-Lee critical point (at pure imaginary h).

- **Who is gonna to win today?**