# Peak Estimator Design for Detector/Front-end Signals

# Summary

- Peak estimator design
- Cost function visualization
- Lagrange multiplier review
- Example based on simulation

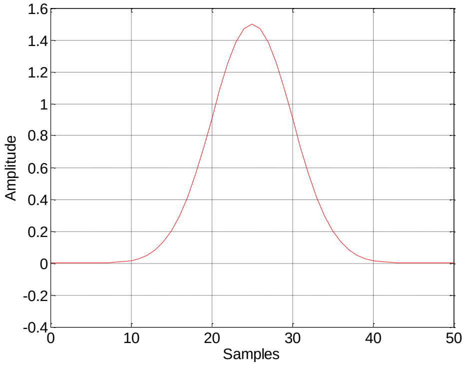
# Peak Estimator Design

# Signal Modeling

- Equation 1 describes the digitized signal $x[k]$ acquired at the front-end output. $h(t_k)$ is the signal and $n(t_k)$ is the additive noise.
- The signal $h(t_k)$ might be described in terms of its peak amplitude $A$ and the its shape normalized by the peak $g(t_k)$.

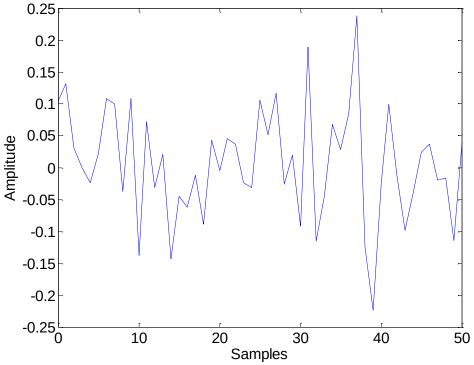
$$x[k] = h(t_k) + n(t_k) = Ag(t_k) + n(t_k)$$

$\boldsymbol{h(t_k) = Ag(t_k)\Big|_{A=1.5}}$



**+**

$\boldsymbol{n(t_k)}$



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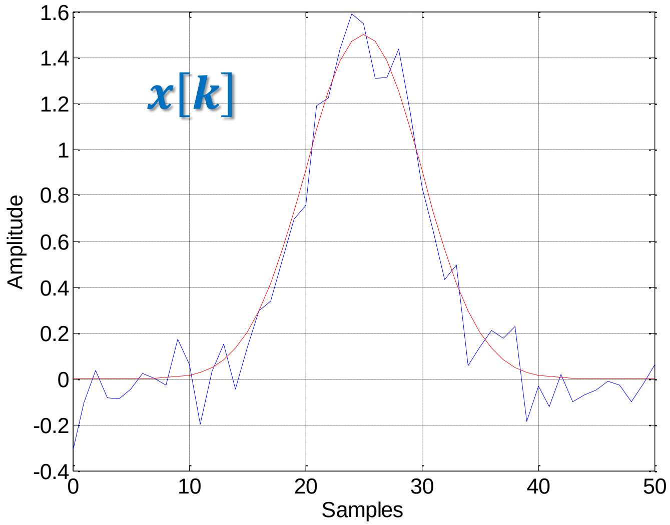

# Signal Modeling

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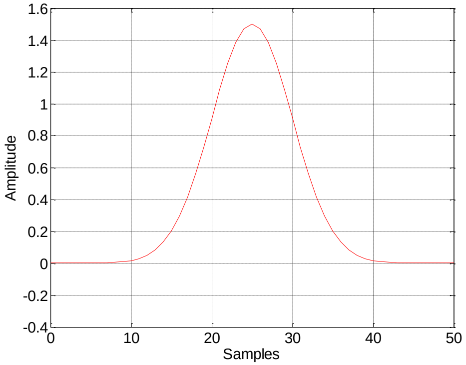
- The signal $h(t_k)$ might be described in terms of its peak amplitude $A$ and the its shape normalized by the peak $g(t_k)$.

*Therefore we are supposing the signal shape is known*

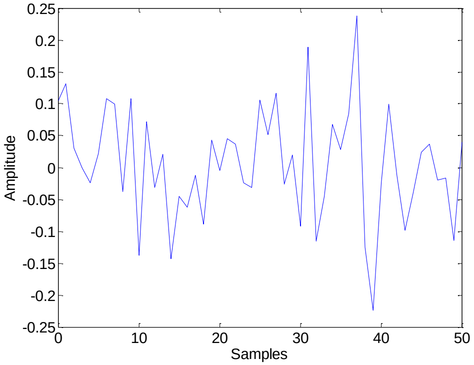
$$x[k] = h(t_k) + n(t_k) = Ag(t_k) + n(t_k)$$

$$\boldsymbol{h(t_k) = Ag(t_k)\Big|_{A=1.5}}$$



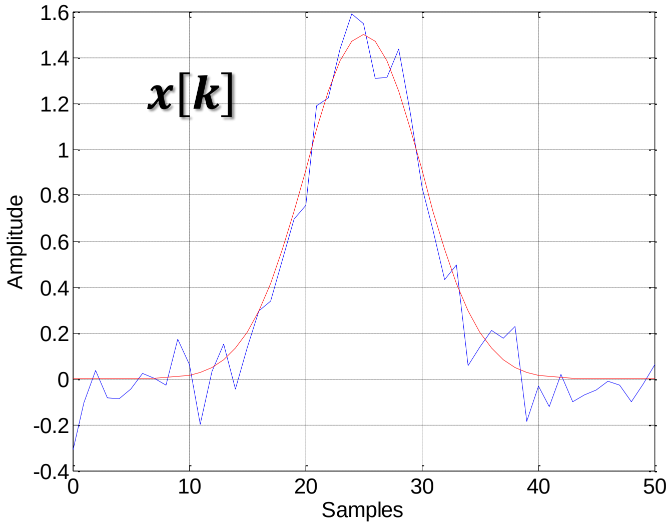
**+**

$$\boldsymbol{n(t_k)}$$



**=**

$$\boldsymbol{x[k]}$$



# Peak Estimator Design

- Now we can propose a simple linear equation to be used as a peak amplitude estimator

$$\hat{A} = \sum_{k=1}^{N} a[k]x[k]$$

$$\hat{A} = A\sum_{k=1}^{N} a[k]g[k] + \sum_{k=1}^{N} a[k]n[k]$$

- To make this equation an estimator of $A$ we should have the first summation equal to ONE and the second equal to ZERO
- In terms of its Expected Value, if we want an unbiased estimator → $E[\hat{A}] \ = A$

$$E[\hat{A}] \ = A\sum_{k=1}^{N} a[k]g[k] + \sum_{k=1}^{N} a[k]E\big[n[k]\big]$$

# Peak Estimator Design

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# Peak Estimator Design

- Therefore TWO restriction are necessary

$$E[\hat{A}] = A\underbrace{\sum_{k=1}^{N} a[k]g[k]}_{\sum_{k=1}^{N} a[k]g[k] = 1} + \underbrace{\sum_{k=1}^{N} a[k]E\big[n[k]\big]}_{\sum_{k=1}^{N} a[k] = 0}$$

# Peak Estimator Design

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$$E[\hat{A}] = A \underbrace{\sum_{k=1}^{N} a[k]g[k]}_{\displaystyle\sum_{k=1}^{N} a[k]g[k] = 1} + \underbrace{\sum_{k=1}^{N} a[k]E\big[n[k]\big]}_{\displaystyle\sum_{k=1}^{N} a[k] = 0}$$

Our estimator is defined.... $\hat{A} = \sum_{k=1}^{N} a[k]x[k]$

Next step would be finding the coefficients a[k], k=1,2,3,…N

# Peak Estimator Design

- The coefficients a[k] will be found minimizing the estimator variance

$$Var[\hat{A}] = A\sum_{k=1}^{N} a[k]g[k] + \sum_{k=1}^{N} a[k]n[k]$$

$$Var[\hat{A}] = Var\left[\sum_{k=1}^{N} a[k]n[k]\right]$$

$$Var[\hat{A}] = \sum_{i=1}^{N}\sum_{j=1}^{N} a[i]a[j]C_{ij}$$

→ *Cost function to be minimized*

*Not forgetting both restrictions…*

$$\sum_{k=1}^{N} a[k]g[k] = 1 \qquad \sum_{k=1}^{N} a[k] = 0$$

# Visualizing the cost function

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- The model lead us to the following estimator variance

$$Var\left[\hat{A}\right] = Var\left[\mathbf{a}^T\mathbf{n}\right] = \mathbf{a}^T\mathbf{C}\mathbf{a}$$

  - **C** is the covariance matrix and **a** the filter weights

- Our goal is to minimize the estimator variance
  - Lets suppose there are two weights only...

$$Var\left[\hat{A}\right] = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$Var\left[\hat{A}\right] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22}$$

*Function to be minimized...*

*Find $a_i$ for min $Var\left[\hat{A}\right]$*

# Visualizing the cost function

- Lets suppose **C = I**

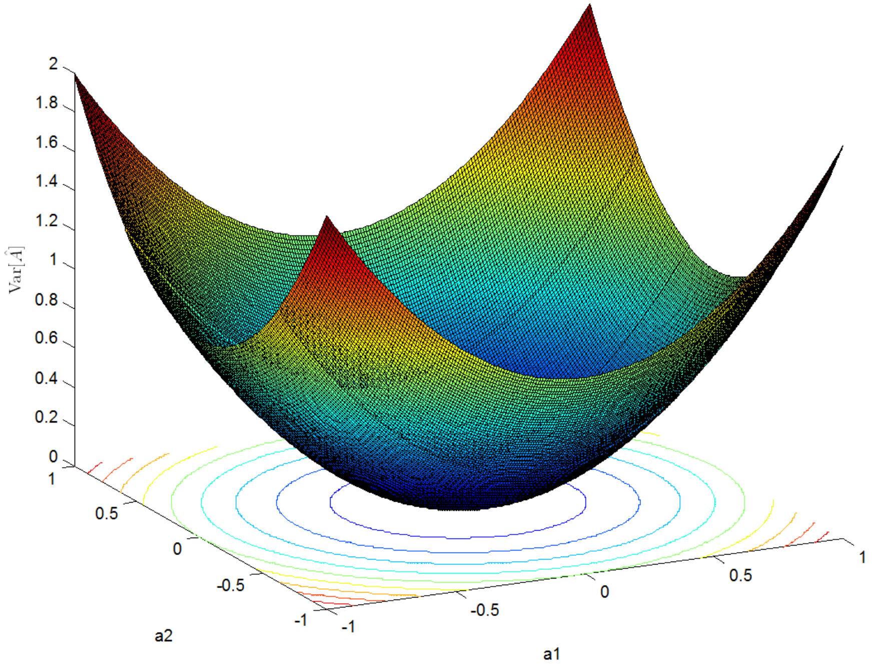
$$Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22} \Rightarrow Var[\hat{A}] = a_1^2 + a_2^2$$

*Function to be minimized...*

```
a1 = -1:0.01:1;
a2 = -1:0.01:1;
C = [1 0; 0 1];

for i=1:length(a1)
  for j=1:length(a2)
    a = [a1(i);a2(j)];
    Var(i,j) = a'*C*a;
  end
end

figure;
 hold on;
[X Y] = meshgrid(a1,a2);
surf(X, Y, Var');
contour(X, Y, Var');
axis([min(a1) max(a1) min(a2) max(a2)]);
xlabel('a1')
ylabel('a2')
zlabel('Var[$\hat{A}$]','interpreter','latex');
```



# Visualizing the cost function

- Lets suppose **C = Diagonal**

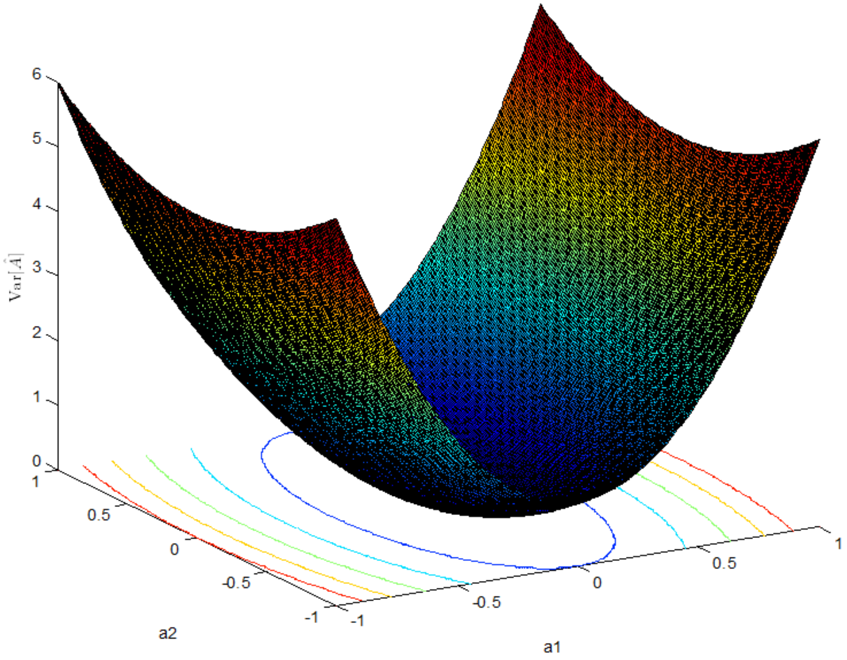
$$Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22} \quad \Rightarrow \quad \boxed{Var[\hat{A}] = 5a_1^2 + a_2^2}$$

*Function to be minimized…*

```
a1 = -1:0.01:1;
a2 = -1:0.01:1;
C = [5 0; 0 1];

for i=1:length(a1)
  for j=1:length(a2)
    a = [a1(i);a2(j)];
    Var(i,j) = a'*C*a;
  end
end

figure;
 hold on;
[X Y] = meshgrid(a1,a2);
surf(X, Y, Var');
contour(X, Y, Var');
axis([min(a1) max(a1) min(a2) max(a2)]);
xlabel('a1')
ylabel('a2')
zlabel('Var[$\hat{A}$]','interpreter','latex');
```



# Visualizing the cost function

- Lets suppose **C ≠ I**

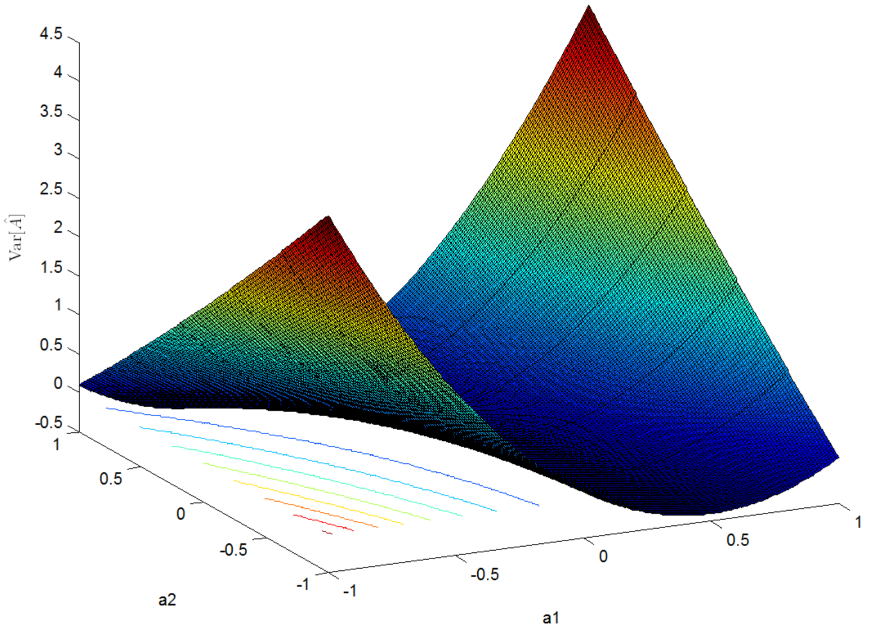
$$Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22} \quad \Rightarrow \quad \boxed{Var[\hat{A}] = 2a_1^2 a_2 + a_1 0.1 a_2^2}$$

*Function to be minimized...*

```
a1 = -1:0.01:1;
a2 = -1:0.01:1;
C = [2 1; 1 0.1];

for i=1:length(a1)
  for j=1:length(a2)
    a = [a1(i);a2(j)];
    Var(i,j) = a'*C*a;
  end
end

figure;
hold on;
[X Y] = meshgrid(a1,a2);
surf(X, Y, Var');
contour(X, Y, Var');
axis([min(a1) max(a1) min(a2) max(a2)]);
xlabel('a1')
ylabel('a2')
zlabel('Var[$\hat{A}$]','interpreter','latex');
```



# Visualizing the cost function

- The model lead us to the following estimator variance

$$Var\left[\hat{A}\right] = Var\left[\mathbf{a}^T\mathbf{n}\right] = \mathbf{a}^T\mathbf{C}\mathbf{a}$$

  - **C** is the covariance matrix and **a** the filter weights

- Our goal is to minimize the estimator variance

$$Var\left[\hat{A}\right] = [a_1 \quad a_2 \quad a_3 \quad \dots \quad a_n] \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Generalizing

$$Var\left[\hat{A}\right] = a_1^2 C_{11} a_2 C_{12} a_3 C_{13} \dots a_n C_{1n} + a_1 C_{21} a_2^2 C_{22} a_3 C_{23} \dots a_n C_{2n} + \cdots$$

*Function to be minimized...*

*Find $a_i$ for min $Var\left[\hat{A}\right]$*

# Visualizing the cost function

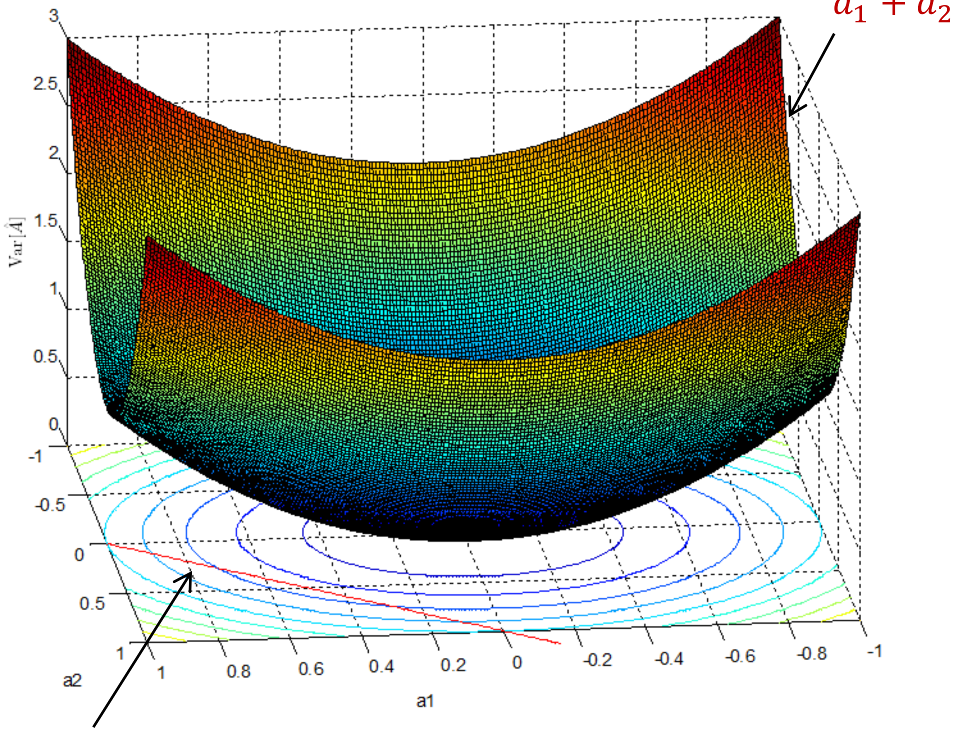
- Now applying restriction...
  - **C ≠ I** → $Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22}$ ➡ $Var[\hat{A}] = 2a_1^2 + a_2^2$
  - Lets suppose the following restriction: $a_1 + a_2 = 1$

```
clear all; close all
a1 = -1:0.01:1;
a2 = -1:0.01:1;
C = [2 0; 0 1];

for i=1:length(a1)
  for j=1:length(a2)
    a = [a1(i);a2(j)];
    Var(i,j) = a'*C*a;
  end
end

surf(a1, a2, Var);
hold on;
[X,Y] = meshgrid(a1,a2);
contour(a1, a2, X.^2+Y.^2);
plot(a1, 1-a1, '--r');
axis([min(a1) max(a1) min(a2) max(a2)]);

xlabel('a1')
ylabel('a2')
zlabel('Var[$\hat{A}$]','interpreter','latex');
```



$a_1^2 + a_2^2$

$a_1 + a_2 - 1 = 0$

# Visualizing the cost function

- Now applying restriction...
  - **C ≠ I** → $Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22}$ ➡ $Var[\hat{A}] = 2a_1^2 + a_2^2$
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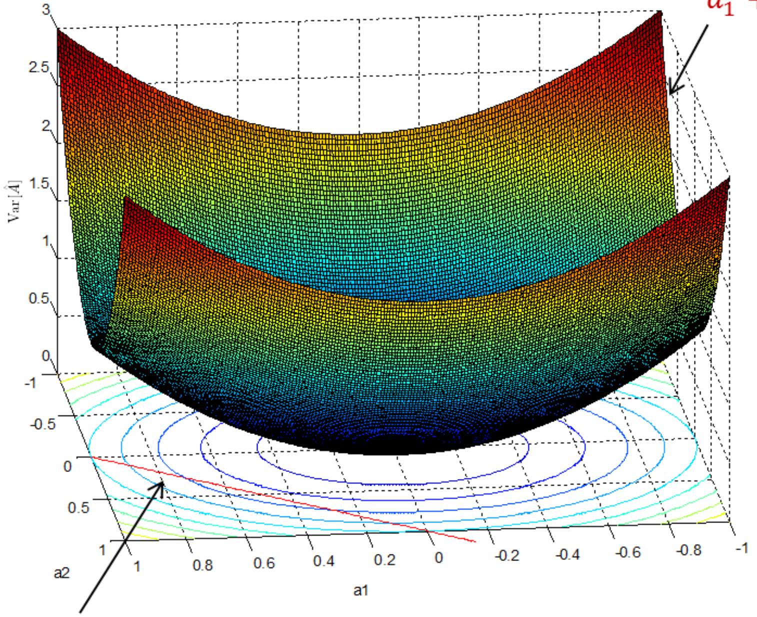
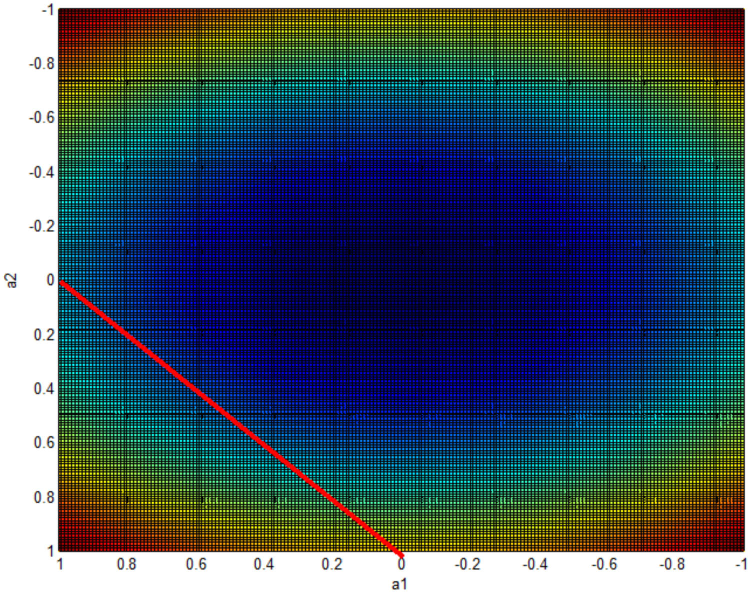


$a_1^2 + a_2^2$

$a_1 + a_2 - 1 = 0$



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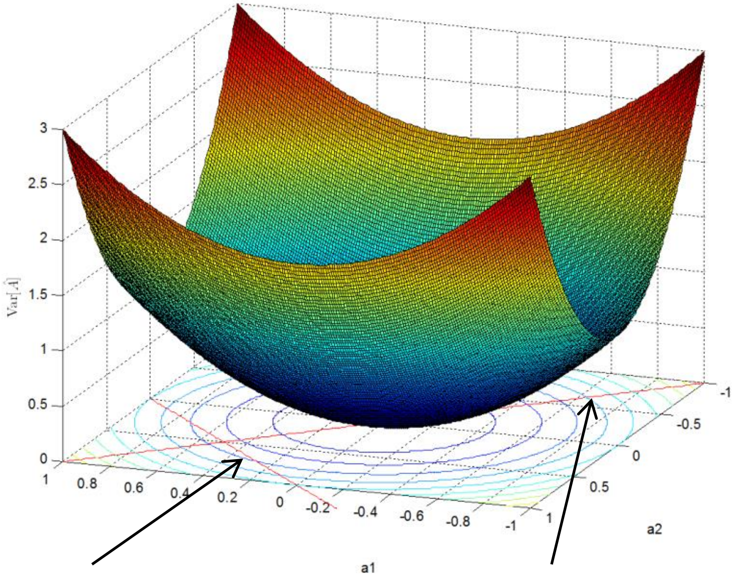
  - **C ≠ I** → $Var[\hat{A}] = a_1^2C_{11}a_2C_{12} + a_1C_{21}a_2^2C_{22}$ ➡ $Var[\hat{A}] = 2a_1^2 + a_2^2$
  - Lets suppose the following restriction: $a_1 + a_2 = 1$
  - Now a second restriction: $a_1 - a_2 = 0$

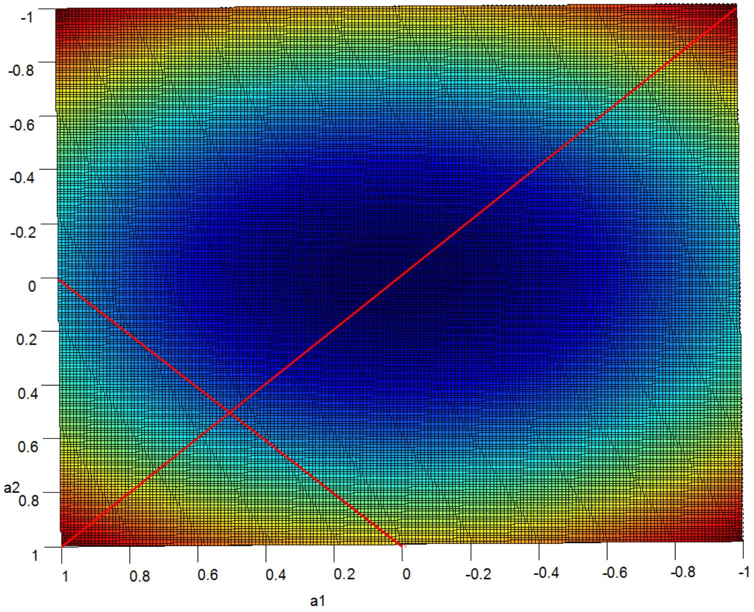


$a_1 + a_2 = 1$

plot(a1, 1-a1, '--r');

$a_1 - a_2 = 0$

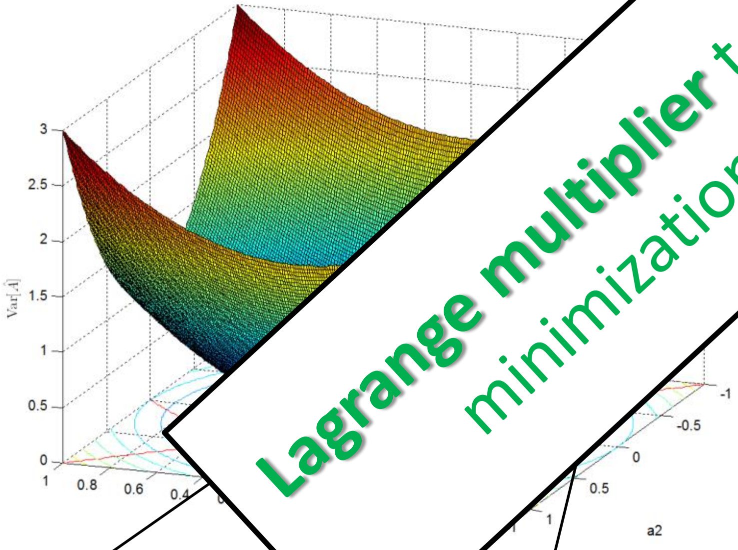
plot(a1, a1, '--r');

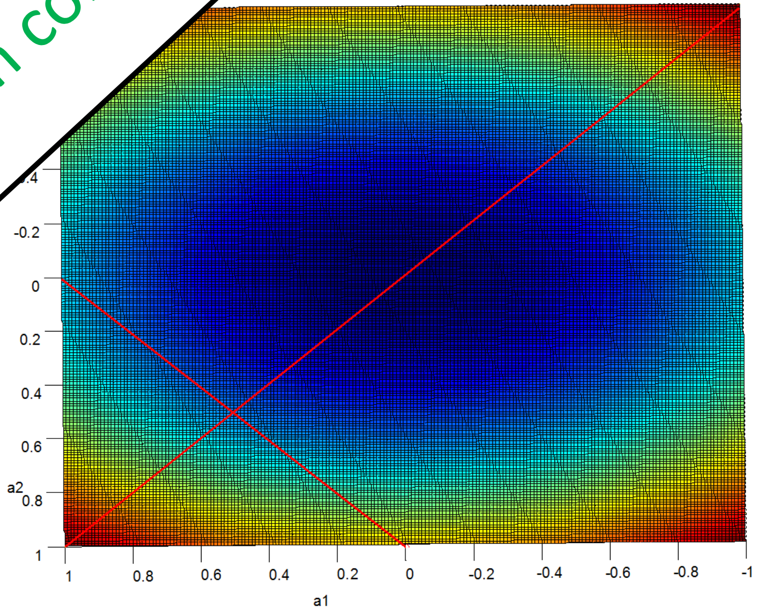


# Visualizing the cost function

- Now applying restriction...
  - **C ≠ I** → $Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21}$ $[\hat{A}] = 2a_1^2 + a_2^2$
  - Lets suppose the following restricti
  - Now a second restriction: $a_1 -$

**Lagrange multiplier** technique is used for minimization with constrains





$a_1 + a_2 = 1$

plot(a1, 1-a1, '--r');

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# Lagrange multiplier review

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Candidates for absolute maximum e minimum of $f(x, y)$ subjected to the constraint $g(x, y)$ are the points on $g(x, y) = 0$ where gradients of $f(x, y)$ and $g(x, y)$ are parallel.

To solve for these points, we find all x, y and λ such that $\nabla f(x, y) = \lambda \nabla g(x, y)$ *and* $g(x, y)$ hold simultaneously

Lagrange equation is given by $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ *, where* $\nabla L(x, y, \lambda) = 0$

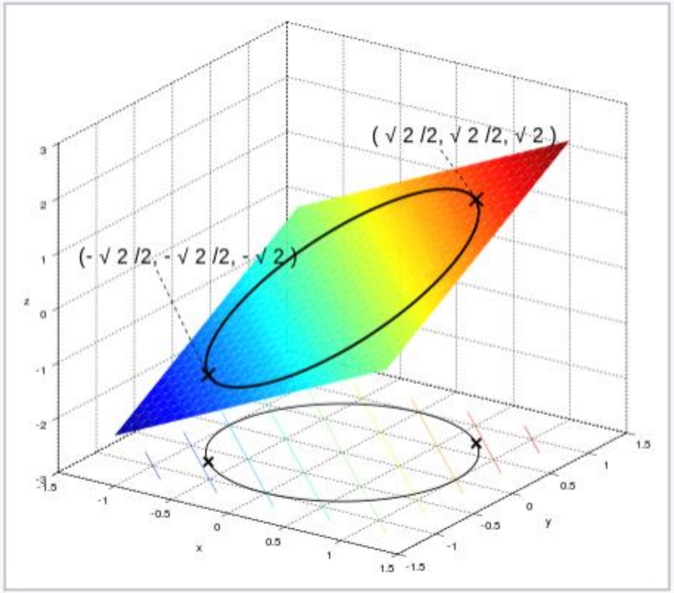

Illustration of the constrained optimization problem

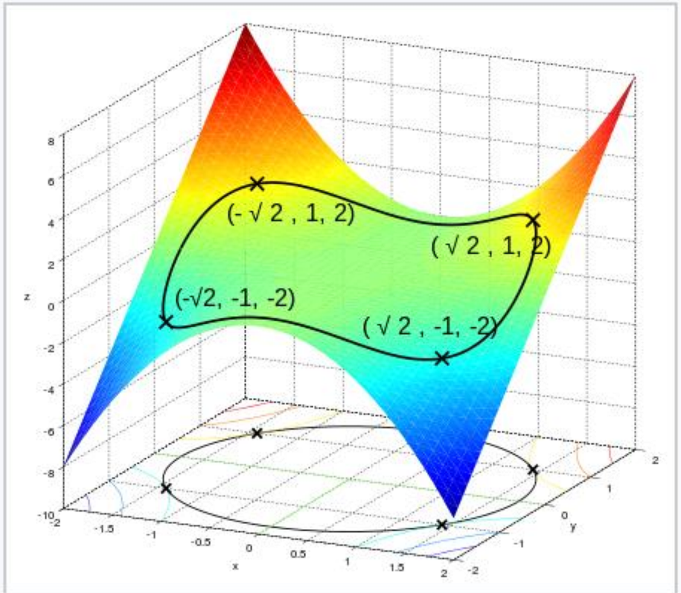

Illustration of the constrained optimization problem

# Lagrange multiplier review

- **C ≠ I** → $Var[\hat{A}] = a_1^2 C_{11} a_2 C_{12} + a_1 C_{21} a_2^2 C_{22}$ ➡ $Var[\hat{A}] = 5a_1^2 + a_2^2$
- Lets suppose the following restriction: $a_1 + a_2 = 1$

```
a1 = -1:0.01:1;
a2 = -1:0.01:1;
C = [5 0; 0 1];

for i=1:length(a1)
  for j=1:length(a2)
    a = [a1(i);a2(j)];
    Var(i,j) = a'*C*a;
  end
end

figure;
hold on;
[X Y] = meshgrid(a1,a2);
contour(X, Y, Var');
plot(a1, 1-a1, '--r');
axis([min(a1) max(a1) min(a2) max(a2)]);
xlabel('a1')
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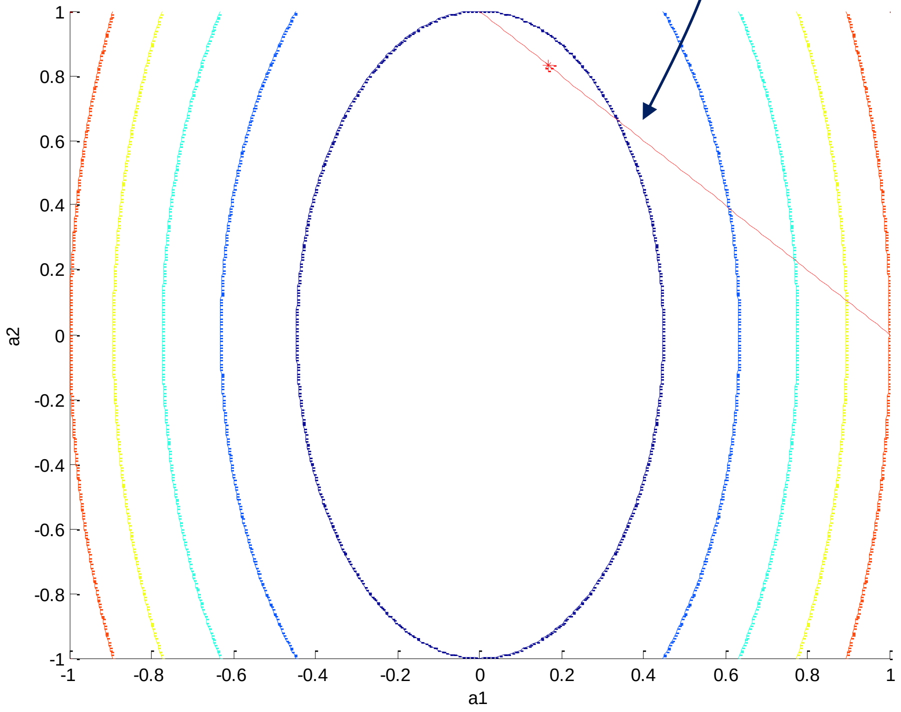
zlabel('Var[$\hat{A}$]','interpreter','latex');
```



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- Lets suppose the following restriction: $a_1 + a_2 = 1$

$$\nabla(5a_1^2+a_2^2) = \lambda\nabla(a_1+a_2-1)$$

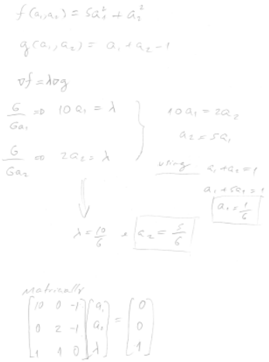
$$10a_1 = \lambda$$
$$2a_2 = \lambda$$
$$a_1 + a_2 = 1$$

$$\begin{bmatrix} 10 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2C_{11} & C_{12} & -1 \\ C_{21} & 2C_{22} & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

```
>> M = [10 0 -1; 0 2 -1; 1 1 0];
>> R = [0; 0; 1];
>> A = inv(M)*R

A =
   0.1667
   0.8333
   1.6667
>> plot(A(1),A(2),'r*')
```



$f(a_1,a_2) = 5a_1^2 + a_2^2$

$g(a_1,a_2) = a_1 + a_2 - 1$

$\nabla f = \lambda \nabla g$

$\frac{\partial}{\partial a_1} \Rightarrow 10a_1 = \lambda$

$\frac{\partial}{\partial a_2} \Rightarrow 2a_2 = \lambda$

$10a_1 = 2a_2$

$a_2 = 5a_1$

using $a_1 + a_2 = 1$

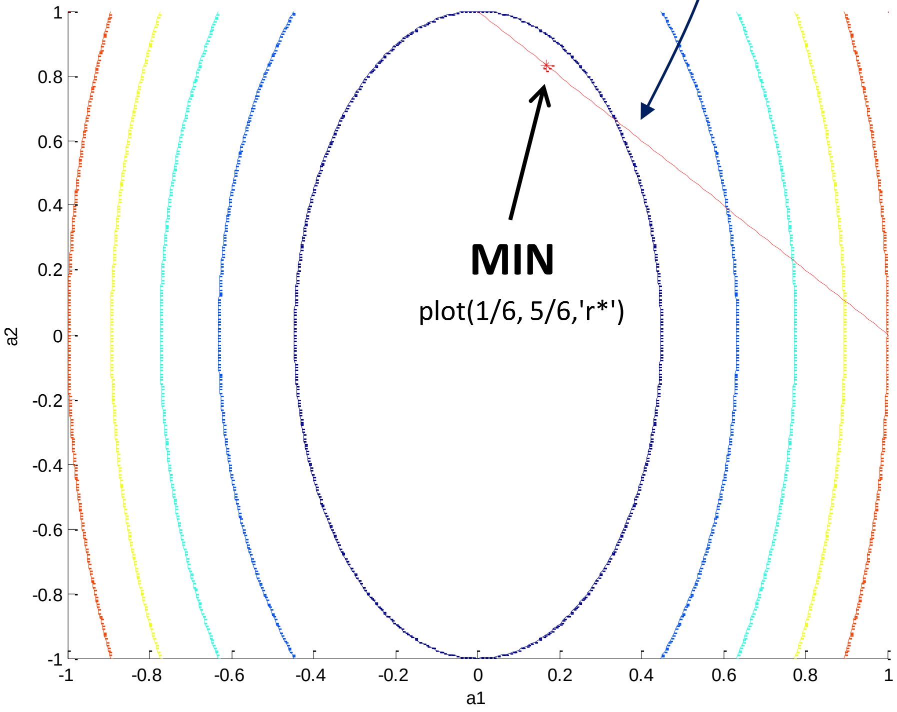
$a_1 + 5a_1 = 1$

$\boxed{a_1 = \frac{1}{6}}$

$\lambda = \frac{10}{6}$ e $\boxed{a_2 = \frac{5}{6}}$

Matrically

$$\begin{bmatrix} 10 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



**MIN**

plot(1/6, 5/6,'r*')

# Lagrange multiplier review

- **C ≠ I** → $Var[\hat{A}] = a_1^2C_{11}a_2C_{12} + a_1C_{21}a_2^2C_{22}$ ➡ $Var[\hat{A}] = 5a_1^2 + a_2^2$
- Lets suppose the following restriction: $a_1 + a_2 = 1$

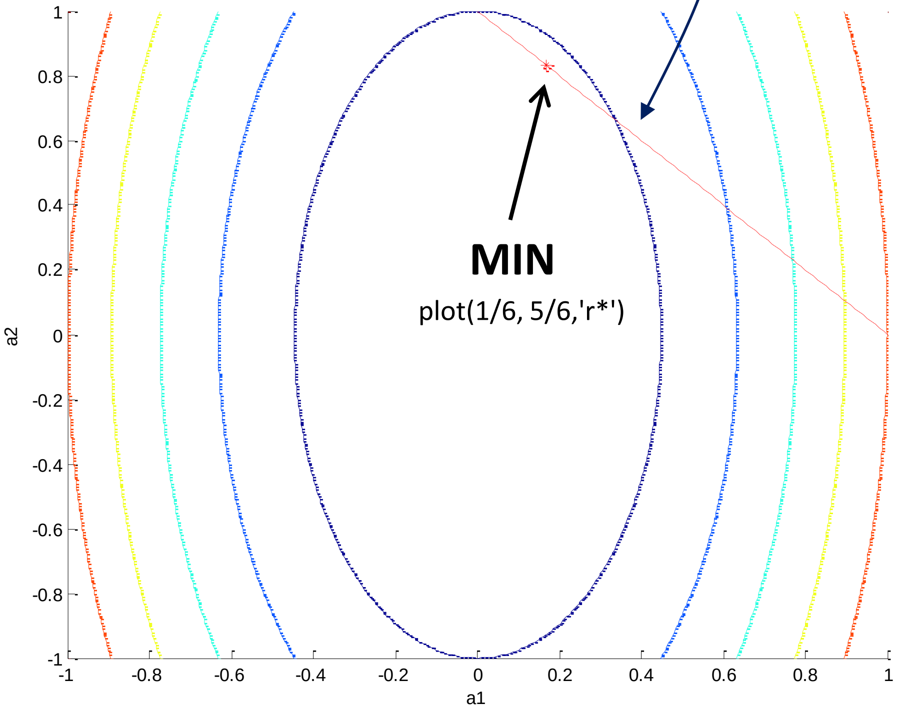
$$\nabla(5a_1^2+a_2^2) = \lambda\nabla(a_1+a_2-1)$$

$$\begin{aligned} 10a_1 &= \lambda \\ 2a_2 &= \lambda \\ a_1 + a_2 &= 1 \end{aligned}$$

*Lagrange expression...*

$$L(a_1,a_2,\lambda) = (5a_1^2+a_2^2) - \lambda(a_1+a_2-1)$$

$$\nabla L(a_1,a_2,\lambda) = 0$$



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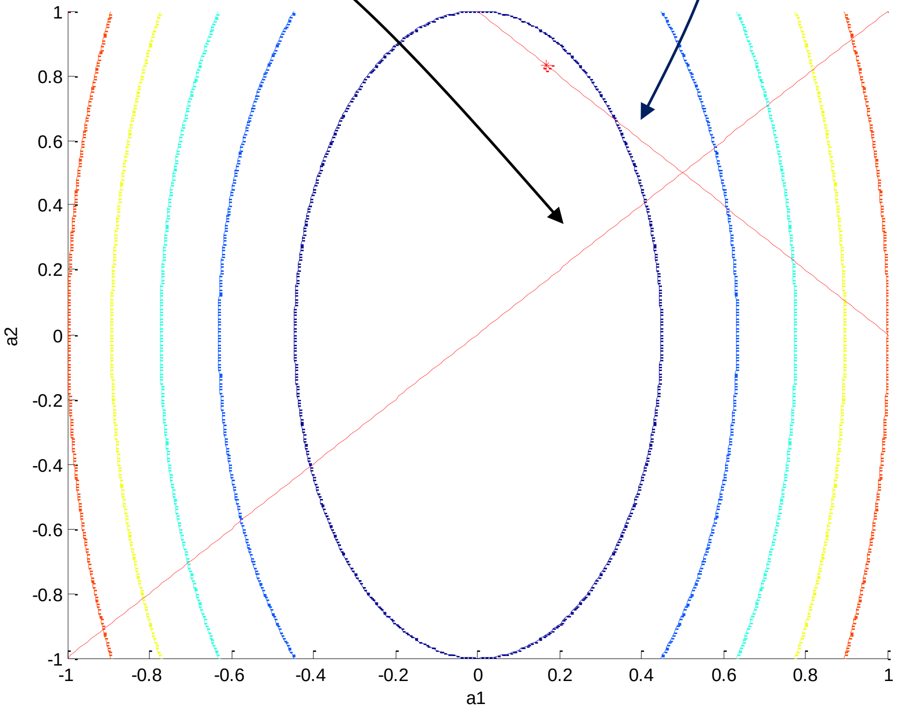
- Lets suppose the following restriction: $a_1 + a_2 = 1$
- And a new restriction: $a_1 = a_2$

$$\nabla(5a_1^2 + a_2^2) = \lambda\nabla(a_1 + a_2 - 1) + k\nabla(a_1 - a_2)$$



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- Lets suppose the following restriction: $a_1 + a_2 = 1$
- And a new restriction: $a_1 = a_2$

$$\nabla(5a_1^2 + a_2^2) = \lambda\nabla(a_1 + a_2 - 1) + k\nabla(a_1 - a_2)$$

$$\begin{aligned} 10a_1 &= \lambda + k \\ 2a_2 &= \lambda - k \\ a_1 + a_2 &= 1 \\ a_1 - a_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 10 & 0 & -1 & -1 \\ 0 & 2 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

```
>> M = [10 0 -1 -1; 0 2 -1 +1; 1 1 0 0; 1 -1 0 0];
>> R = [0; 0; 1; 0];
>> A = inv(M)*R
A =
   0.5000
   0.5000
   3.0000
   2.0000
>> plot(A(1), A(2), 'r*');
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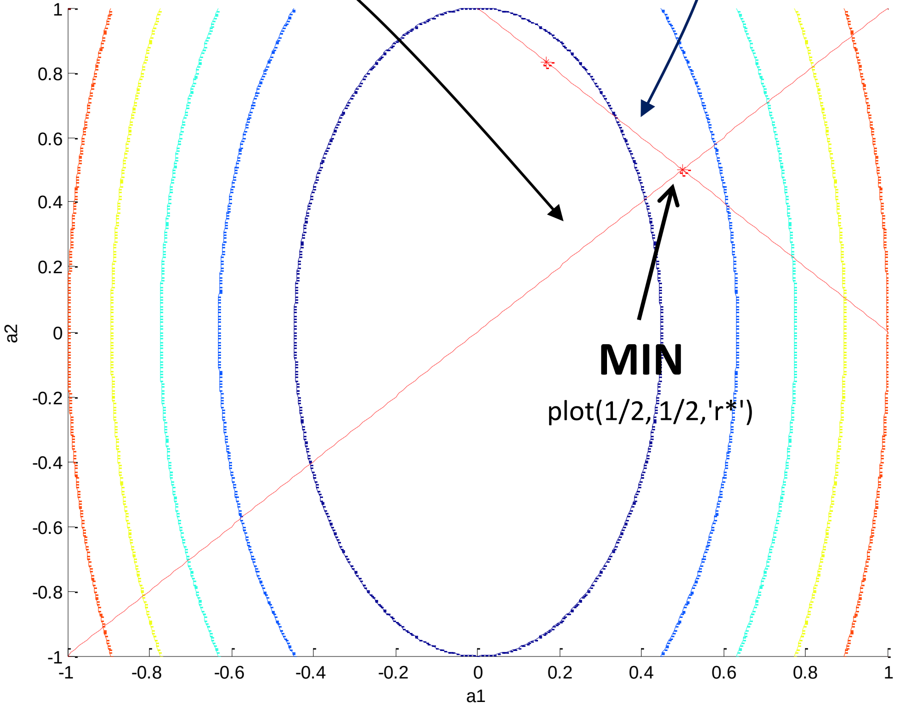
$$\begin{bmatrix} 2C_{11} & C_{12} & -1 & -1 \\ C_{21} & 2C_{22} & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



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plot(1/2,1/2,'r*')
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# Lagrange multiplier review

- **C ≠ I** → $Var[\hat{A}] = a_1^2C_{11}a_2C_{12} + a_1C_{21}a_2^2C_{22}$ ➡ $Var[\hat{A}] = 5a_1^2 + a_2^2$
- Lets suppose the following restriction: $a_1 + a_2 = 1$
- And a new restriction: $a_1 = a_2$

$$\nabla(5a_1^2+a_2^2) = \lambda\nabla(a_1+a_2-1)+k\nabla(a_1-a_2)$$

$$\boxed{\begin{aligned} 10a_1 &= \lambda + k \\ 2a_2 &= \lambda - k \\ a_1 + a_2 &= 1 \\ a_1 - a_2 &= 0 \end{aligned}}$$

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   0.5000
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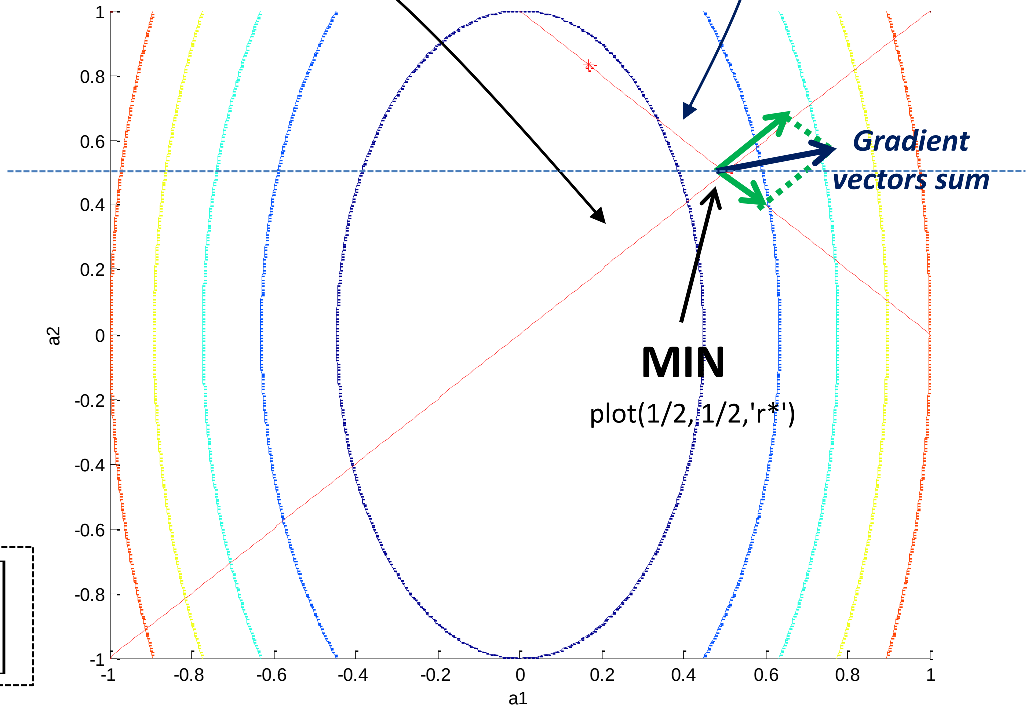

# Lagrange multiplier review

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$$\nabla(5a_1^2 + a_2^2) = \lambda\nabla(a_1 + a_2 - 1) + k\nabla(a_1 - a_2)$$

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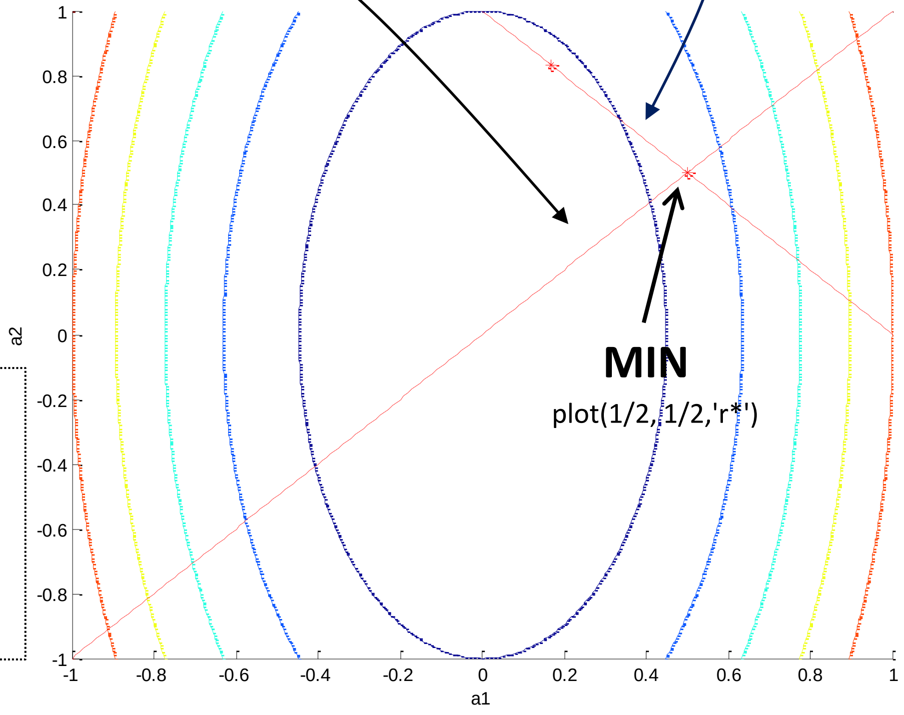

**MIN**

plot(1/2,1/2,'r*')

*Lagrange expression...*

$$L(a_1, a_2, \lambda, k) = (5a_1^2 + a_2^2) - \lambda(a_1 + a_2 - 1) - k((a_1 - a_2)$$

$$\nabla L(a_1, a_2, \lambda, k) = 0$$

# Lagrange multiplier review

- Previous example

$C_{12} = C_{21} = 0$

$$Var[\hat{A}] = a_1^2C_{11}a_2C_{12} + a_1C_{21}a_2^2C_{22} = 5a_1^2 + a_2^2$$

$$\nabla(5a_1^2+a_2^2) = \lambda\nabla(a_1+a_2-1)+k\nabla(a_1-a_2)$$

$$\boxed{\begin{array}{r} 10a_1 = \lambda + k \\ 2a_2 = \lambda - k \\ \text{constrains}\left\{\begin{array}{l} a_1 + a_2 = 1 \\ a_1 - a_2 = 0 \end{array}\right. \end{array}} \Longrightarrow \begin{bmatrix} 2C_{11} & C_{12} & -1 & -1 \\ C_{21} & 2C_{22} & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Generalization….

$$\text{constrains}\left\{\begin{array}{l} f_{c1} = a_1 + a_2 - 1 \\ f_{c2} = a_1 - a_2 \end{array}\right.$$

$$\begin{array}{l} a_1 + a_2 = 1 \longrightarrow \\ a_1 - a_2 = 0 \longrightarrow \end{array} \begin{bmatrix} 2C_{11} & C_{12} & -\frac{\partial f_{c1}}{\partial a_1} & -\frac{\partial f_{c2}}{\partial a_1} \\ C_{21} & 2C_{22} & -\frac{\partial f_{c1}}{\partial a_2} & -\frac{\partial f_{c2}}{\partial a_2} \\ \frac{\partial f_{c1}}{\partial a_1} & \frac{\partial f_{c1}}{\partial a_2} & 0 & 0 \\ \frac{\partial f_{c2}}{\partial a_1} & \frac{\partial f_{c2}}{\partial a_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \lambda \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f_{c1}\Big|_{\substack{a_1=0 \\ a_2=0}} \\ -f_{c2}\Big|_{\substack{a_1=0 \\ a_2=0}} \end{bmatrix}$$

*Considering linear equations…*

# Lagrange multiplier review **MATLAB code**

- *Considering known* ***C***

```
C = [5 0; 0 1]; %noise covariance matrix
```

$$
\left.\begin{aligned} f_{c1} &= a_1 + a_2 - 1 \\ f_{c2} &= a_1 - a_2 \end{aligned}\right\}
$$

*constrains*

```
%DEFINE CONSTRAINT FUNCTIONS
Ncoef = length(diag(C)); %number of dimensions/coeffs (a1, a2, ...)
a = sym('a', [1 Ncoef]);
Ncons = 2; %number of constraint functions
fc1 = a(1)+a(2)-1;  %constraint function 1
fc2 = a(1)-a(2);    %constraint function 2

% BUILD-UP MATRIX AND VECTOR TO FIND COEFFs.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%% LEFT-HAND MATRIX
% COVARIANCE PART
C = C*diag(2*ones(1, length(diag(C)))); %covariante matrix but diag*2

% CONSTRAINTS FUNCTIONS PART
for j = 1:Ncons
  s = sprintf('fc%d', j);
  AA = diff(eval(s),a(1));
  for i = 2:Ncoef
    AA = [AA diff(eval(s),a(i))];
  end
  if j==1 Mconst = AA;
  else Mconst = [Mconst; AA]; end
end
Mconst = double(Mconst); %transform symbolic to numeric

% FINAL MATRIX
M = [C -Mconst; Mconst zeros(Ncons, Ncons)]; %build up final matrix

%%%%%%%%% RIGHT-HAND VECTOR
for i = 1:Ncoef
  eval(sprintf('a%d=0', i));
end
vector = [0 0 -subs(fc1) -subs(fc2)]';

% OBTAIN COEFFs.
A = inv(M)*vector
```

$$
\begin{array}{r}
\\ \\
a_1 + a_2 = 1 \rightarrow \\
a_1 - a_2 = 0 \rightarrow
\end{array}
\begin{bmatrix}
2C_{11} & C_{12} & -\dfrac{\partial f_{c1}}{\partial a_1} & -\dfrac{\partial f_{c2}}{\partial a_1} \\
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\dfrac{\partial f_{c1}}{\partial a_1} & \dfrac{\partial f_{c1}}{\partial a_2} & 0 & 0 \\
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\end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ \lambda \\ k \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ -f_{c1}\Big|_{\substack{a_1=0\\a_2=0}} \\ -f_{c2}\Big|_{\substack{a_1=0\\a_2=0}} \end{bmatrix}
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*Considering linear equations…*

# Lagrange multiplier review **MATLAB code**

- *Considering known **C***

`C = [5 0; 0 1]; %noise covariance matrix`

$$\left.\begin{aligned} f_{c1} &= a_1 + a_2 - 1 \\ f_{c2} &= a_1 - a_2 \end{aligned}\right\}$$

*constrains*

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*constrains*

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*Considering linear equations…*

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% OBTAIN COEFFs.
A = inv(M)*vector
```

**Final result (coefficients)**

# Example based on simulation

# Example based on simulation

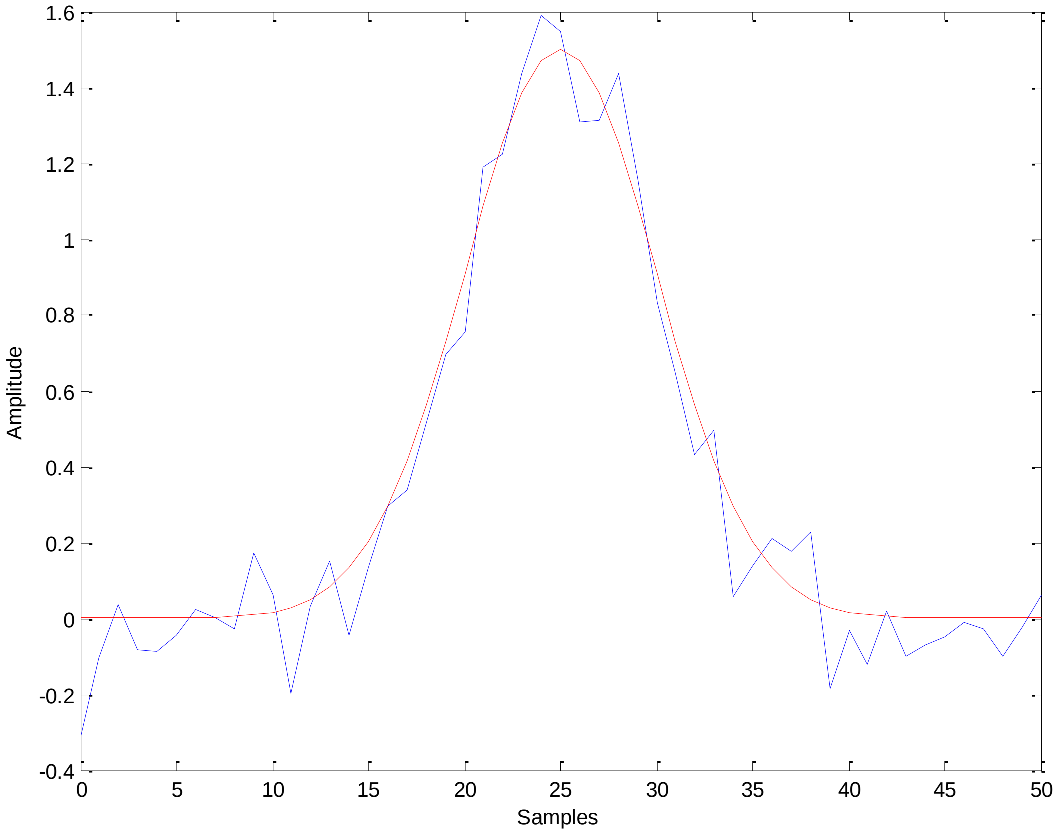
```
clear all; close all;

% Signal Generation
%%%%%%%%%%%%%%%

x = 0:1:50;
y = gaussmf(x,[5 25]);
N = 10000;

for i=1:N;
  truePeak(i) = rand+0.5;
  noise(i,:) = randn(1,51)*0.1;
  yt (:, i) = y*truePeak(i);
  yy(:, i) = y*truePeak(i)+noise (i,:);
end

plot(x,yy(:,1)); %signal with noise
hold on;
plot(x,yt(:,1), 'r'); %true signal
xlabel('Samples')
ylabel('Amplitude')
```



# Example based on simulation

```
plot(x,yy(:,1)); %signal with noise
hold on;
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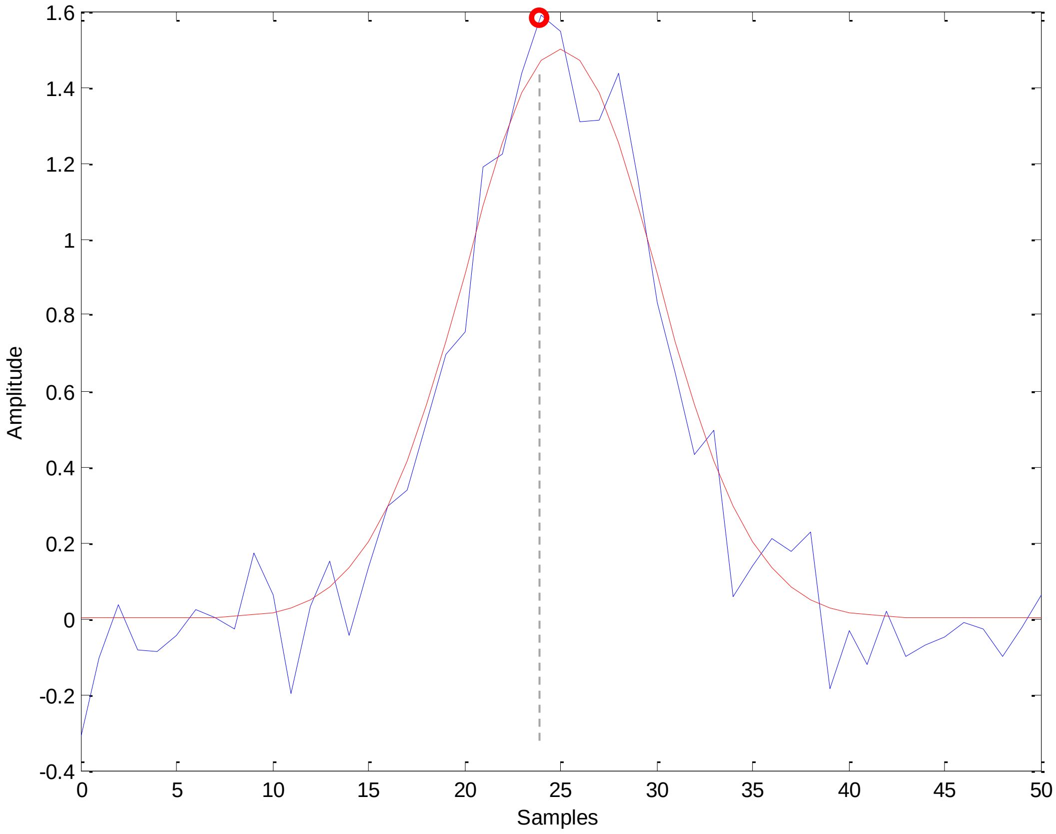
xlabel('Samples')
ylabel('Amplitude')
```

```
% peak estimator based on max.
peakEst1 = max(yy);
```

```
% peak estimator based on position
peakEst2 = yy(25,:);
```



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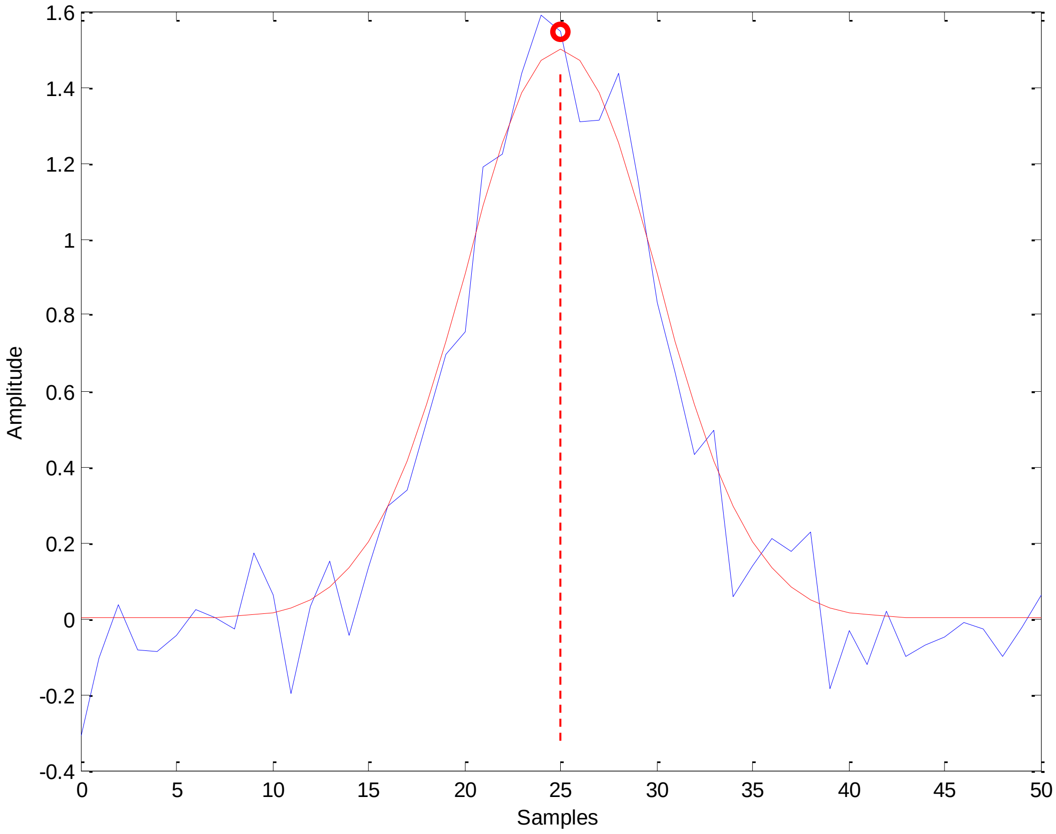

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  yt (:, i) = y*truePeak(i);
  yy(:, i) = y*truePeak(i)+noise (i,:);
end

plot(x,yy(:,1));
hold on;
plot(x,yt(:,1), 'r');
xlabel('Samples')
ylabel('Amplitude')

% Two simple peak estimators
%%%%%%%%%%%%%%%%%%

% peak estimator based on max.
peakEst1 = max(yy);

% peak estimator based on position
peakEst2 = yy(25,:);
```

```
% Performance comparison
%%%%%%%%%%%%%%%%%
figure;
[n1,x1]=hist(peakEst1-truePeak, -0.5:0.02:0.5);
h1=bar(x1,n1,'hist');
set(h1,'FaceColor','no','EdgeColor','g')

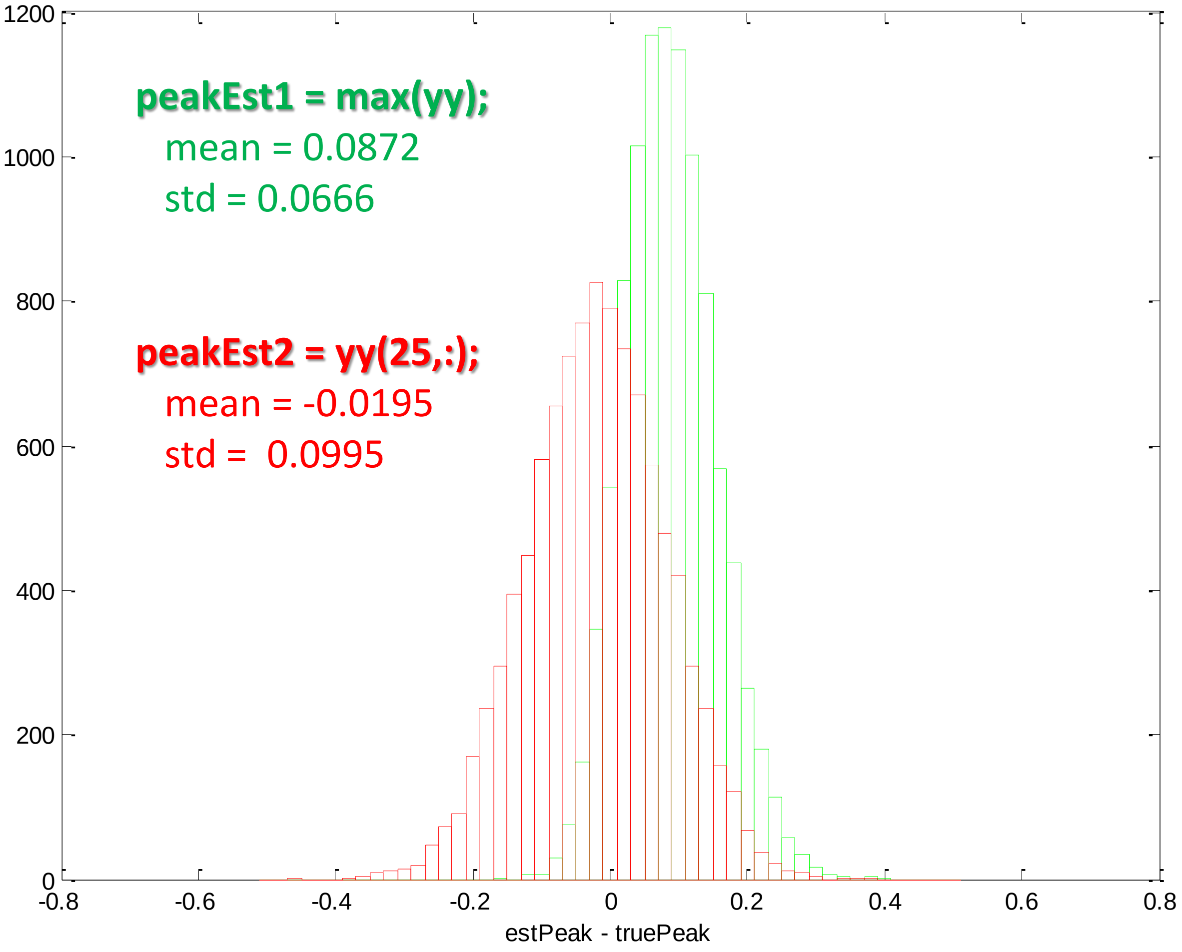
hold on;
[n2,x2]=hist(peakEst2-truePeak, -0.5:0.02:0.5);
h2=bar(x2,n2,'hist');
set(h2,'FaceColor','no','EdgeColor','r')

xlim([-0.8 0.8]);
xlabel('estPeak - truePeak');

% mean results
mean(peakEst1-truePeak)
std(peakEst1-truePeak)

mean(peakEst2-truePeak)
std(peakEst2-truePeak)
```

# Example based on simulation



**peakEst1 = max(yy);**

mean = 0.0872

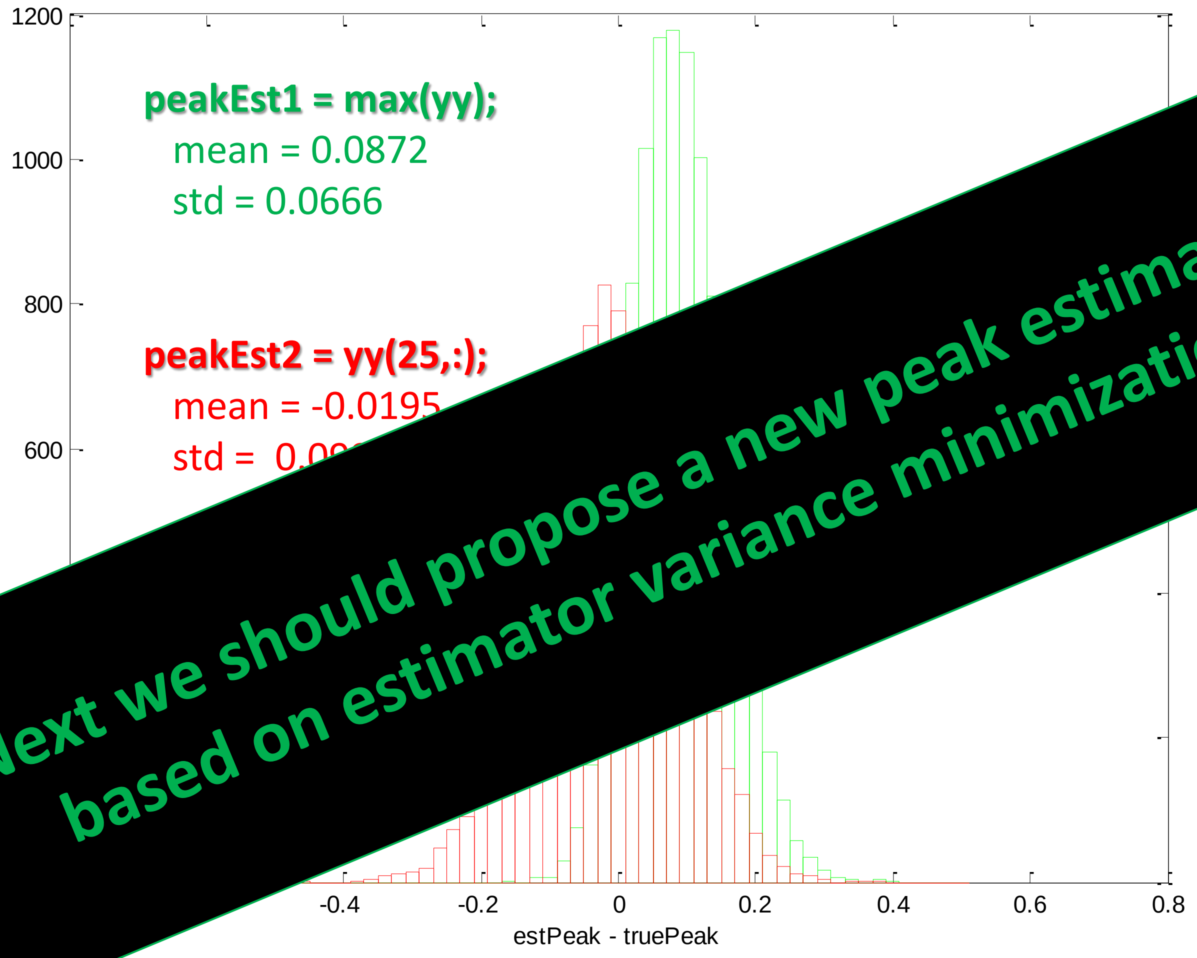
std = 0.0666

**peakEst2 = yy(25,:);**

mean = -0.0195

std =  0.0995

# Example based on simulation



**peakEst1 = max(yy);**

mean = 0.0872

std = 0.0666

**peakEst2 = yy(25,:);**

mean = -0.0195

std =  0.0

**Next we should propose a new peak estimator based on estimator variance minimization**

# Example based on simulation

- To find the estimator coefficients we need to know the signal shape and the noise covariance matrix since variance minimization depends on the following equations

$$Var[\hat{A}] = Var[\boldsymbol{a}^T\boldsymbol{n}] = \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} \qquad \textit{minimization}$$

$$\left.\begin{aligned} \sum_{k=1}^{N} a[k]g[k] &= 1 \\ \sum_{k=1}^{N} a[k] &= 0 \end{aligned}\right\} \quad \textit{constrains}$$

- Usually, both $g[k]$ and $\boldsymbol{C}$ should be estimated from data…

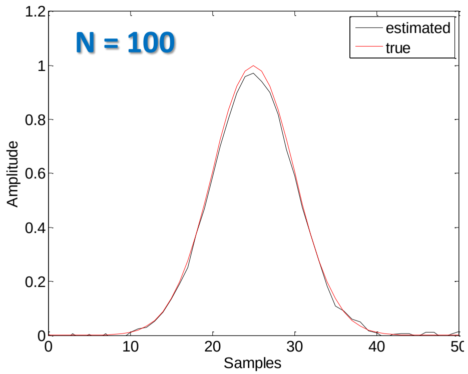
# Example based on simulation

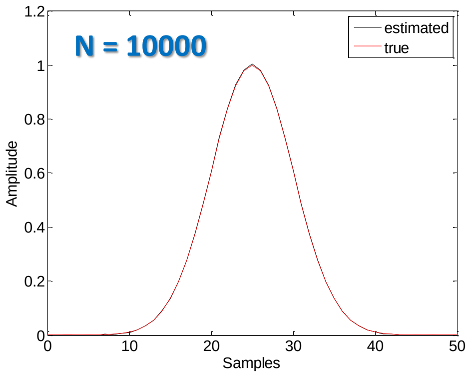
- Signal shape estimation *g[k]*

yMean = mean(yy');

*A simple solution. But it might be more complex….*

- *Timing synchronization*
- *Curve fitting*
- *Pedestal removal*
- *…*



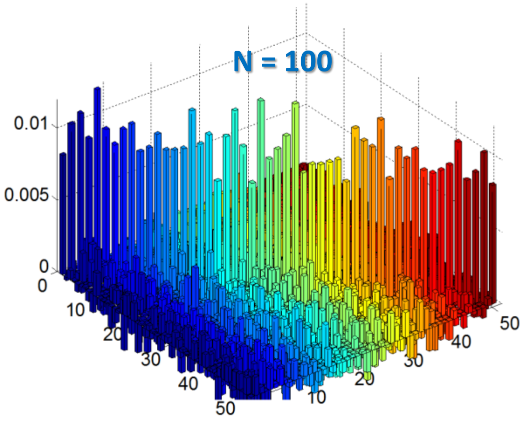


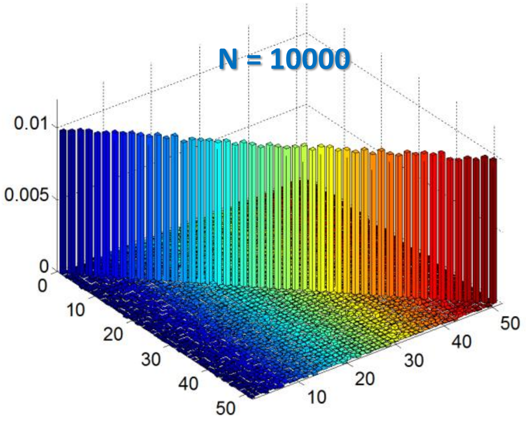
- Noise covariance matrix estimation *C*

C = cov(noise);
bar3(C);

↓

*Supposing we have a noise database*





# Example based on simulation

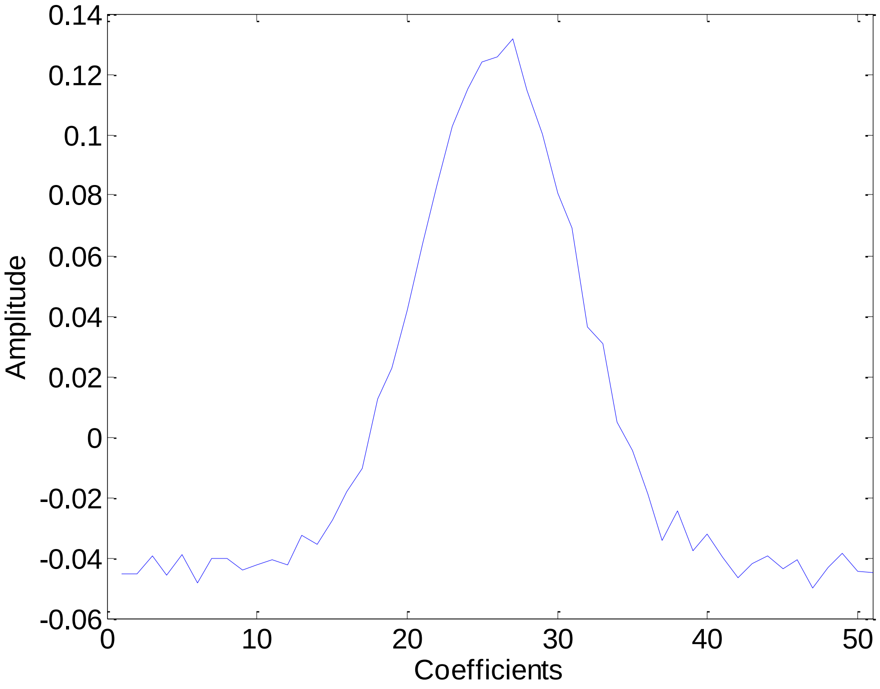
- *Finding the coefficients vector* ***a*** …

```
yMean = mean(yy');
C = cov(noise);

%constraint functions matrix
Fcons = [yMean -1; ones(1,length(diag(C))) 0];

%gives the filter coeffs. more lagrange coeffs.
A = lagrangeMin(C, Fcons);

%plot coefficients
plot(A(1:length(diag(C))))
ylabel('Amplitude')
xlabel('Coefficients')
 xlim([0 length(diag(C))+1])
```



# Example based on simulation

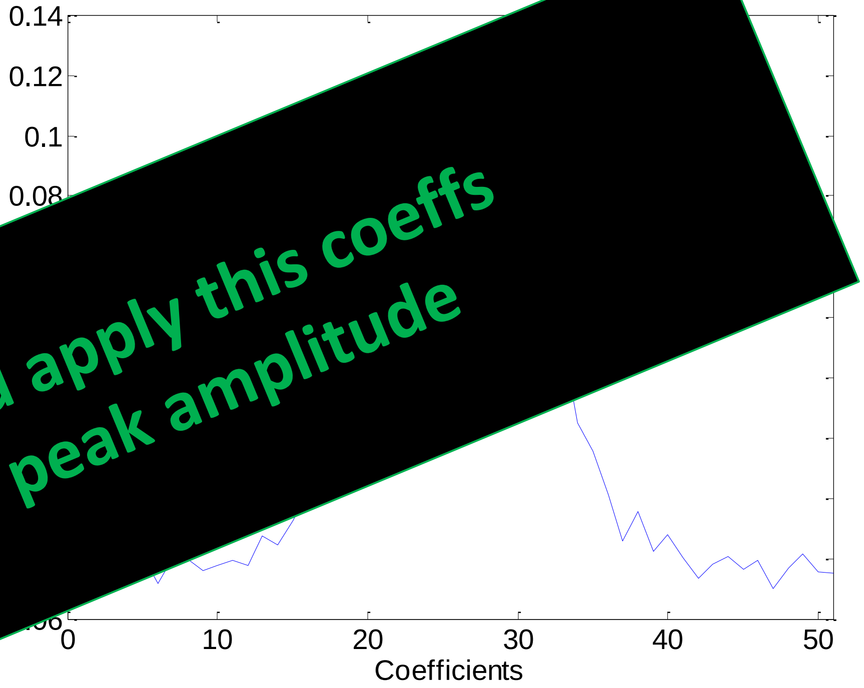
- *Finding the coefficients vector* ***a*** *…*

```
yMean = mean(yy');
C = cov(noise);

%constraint functions matrix
Fcons = [yMean -1; ones(1,length(diag(C))) 0];

%gives the filter coeffs. more lagra
A = lagrangeMin(C, Fcons);

%plot coeffici
plot(A(1
ylabel('A
xlabel('Coe
 xlim([0 leng
```



**Next we should apply this coeffs to estimate peak amplitude**

# Example based on simulation

- Apply coefficients!

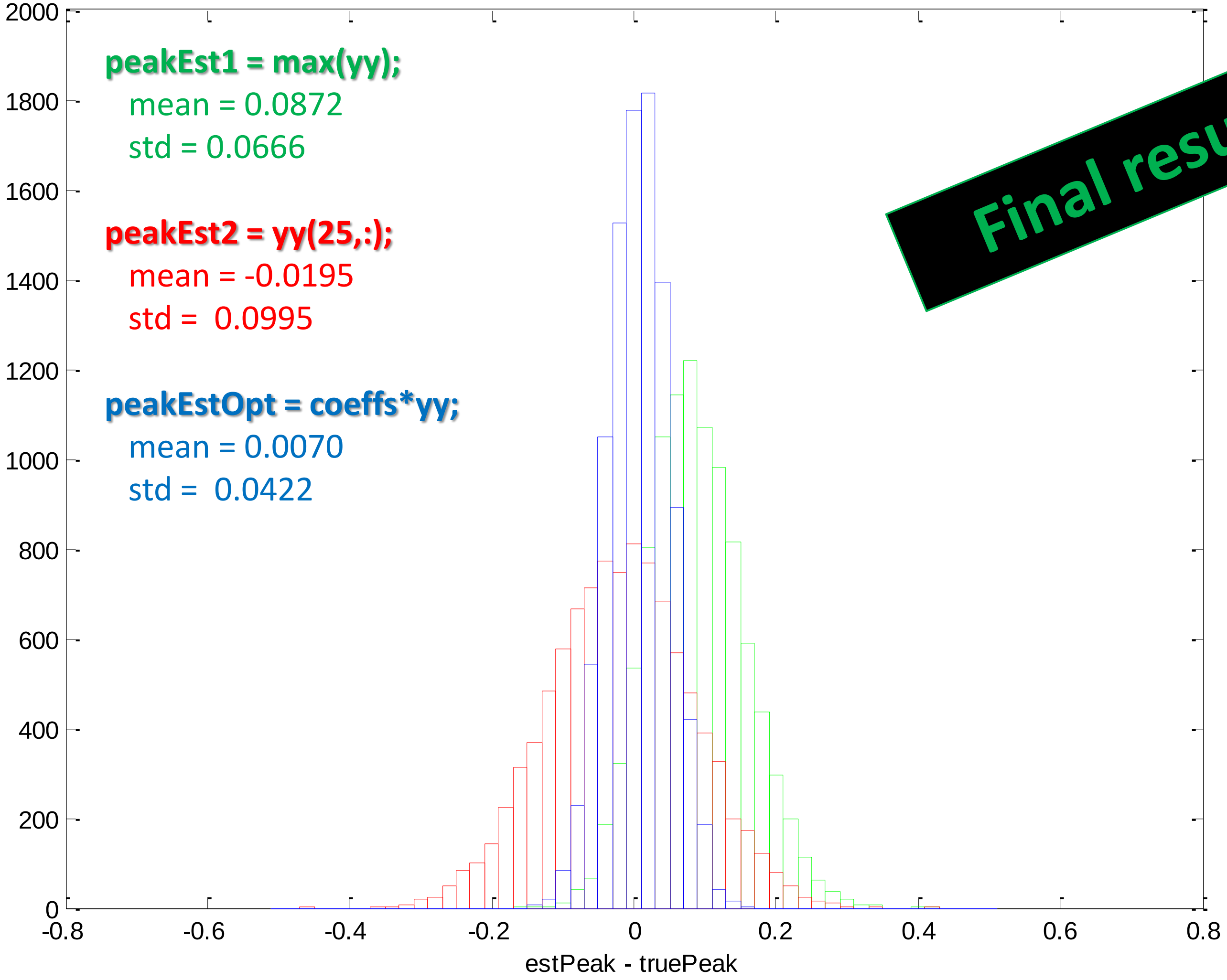
```
%reorganize coefficients
coeffs =  A(1:51,1)';

%estimate peaks
peakEstOpt = coeffs*yy;

%plot histogram
[n3,x3]=hist(peakEstOpt-truePeak, -0.5:0.02:0.5);
h3=bar(x3,n3,'hist');
set(h3,'FaceColor','no','EdgeColor','b')

%mean and std of error
mean(peakEstOpt-truePeak)
std(peakEstOpt-truePeak)
```

# Example based on simulation



**peakEst1 = max(yy);**

mean = 0.0872

std = 0.0666

**peakEst2 = yy(25,:);**

mean = -0.0195

std = 0.0995

**peakEstOpt = coeffs*yy;**

mean = 0.0070

std = 0.0422

**Final result**