Lecture IV: validating many-body quantum technologies (I)

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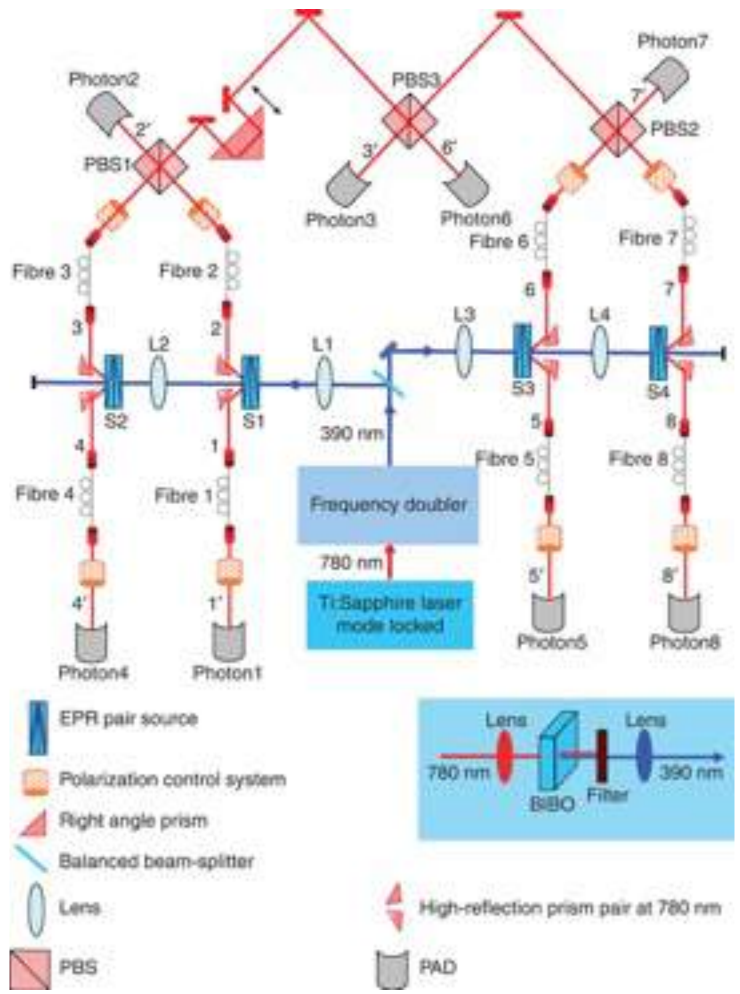
Mini-course on Quantum Computation and Simulability

ICTP/SAIFR-UNESP, October 2018

Impressive experimental progress on many-body quantum technologies

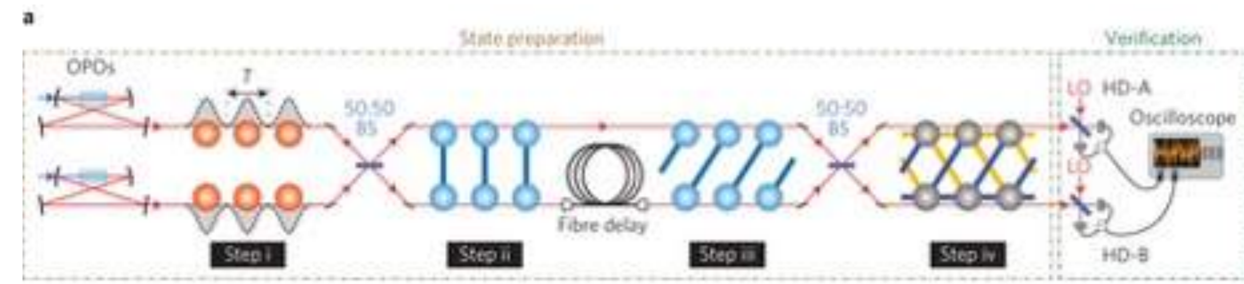
Photonic multi-qubit entangled states

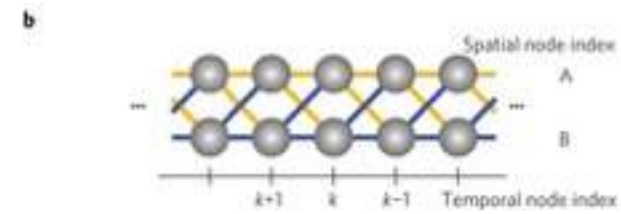
W, Dicke, GHZ, cluster states, ect.



J. W. Pan et al., G. C. Guo et al., P. Walther et al.; H. Weinfurther et al.; S. P Walborn and P. H. S. Ribeiro et al.; C. Monken and S. Padua et al.; etc.

Multi-mode squeezed Gaussian states



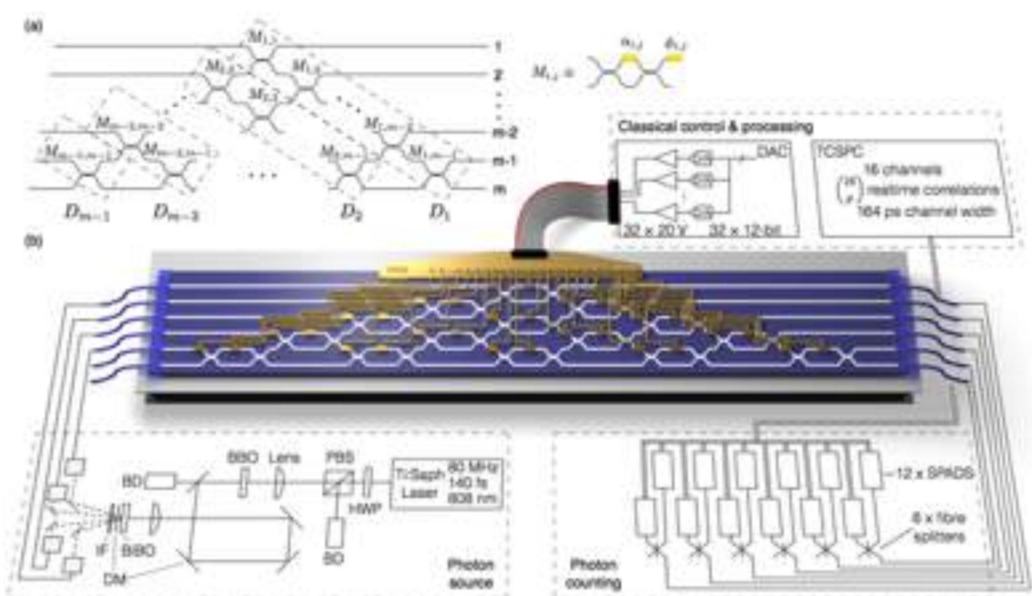


Gaussian cluster states.

A. Furusawa et al.; O. Pfister et al.; R. Schnabel et al.; N Treps et al; etc.

On-chip integrated linear-optical networks

Small-sized simulations of
- Boson-Sampling,
- Anderson localisation,
- quantum walks, etc.

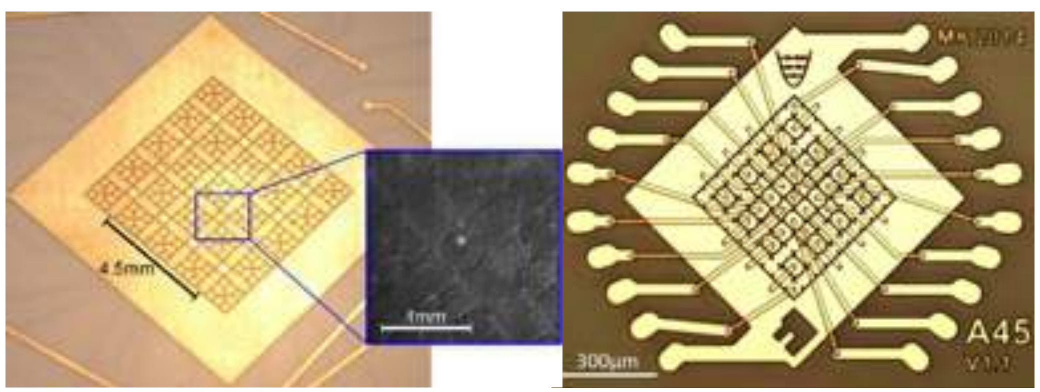


J. O'Brien et al.; I. Walmsley et al.; P. Walther et al.; F. Sciarrino et al.; A. White et al; etc.

Impressive experimental progress on many-body quantum technologies

Micro-fabricated trapped-ion architectures

(Tens of ions):
•Multi-qubit entangled states,
•Digital universal quantum computation,
•Quantum-error correction codes,
•Analogue spin-chain quantum simulations,
•Bosonic quantum information processing, etc.

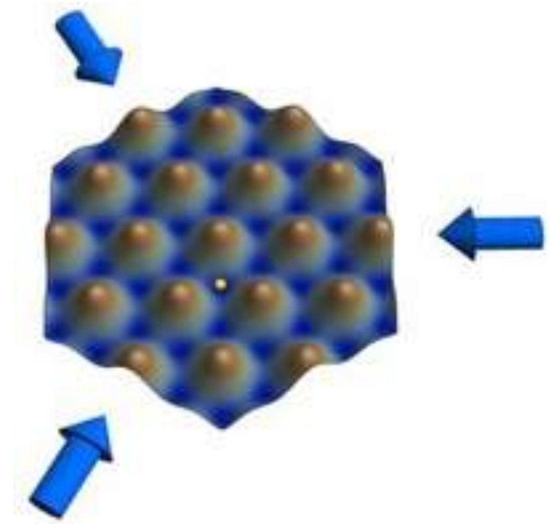


D. Wineland; R. Blatt; C. Monroe; D. D. Leibfried; C. F. Roos; H. Häffner; F. Schmidtkaler; T. Schätz; K. Kim; etc.

and many others!!!

Cold atoms in optical lattices

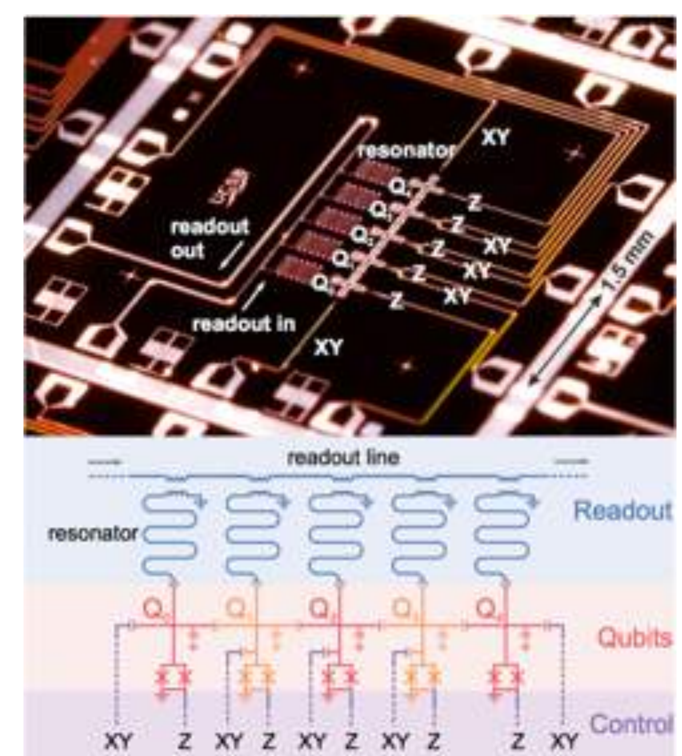
(From hundreds to thousands of atoms):
•Bose-Einstein condensates;
•Bose- and Fermi-Hubbard models;
•Many-body thermalisation;
•Quantum phase transitions; etc.



I. Bloch.; J. Dalibard; W. Zwerger; T. Hänsch; T. Eisslinger; M. Greiner; W. D. Phillips, R. G. Hulet; J. V. Porto; etc.

On chip super-conducting qubit circuits

(Tens of qubits):
•Quantum-error correction codes,
•Digital universal quantum computation;
•Quantum anhealing problems, etc.



J. M. Martinis; F. Nori; R. J Schoelkopf; A. Houk; H. E. Türeci; A. Blais; A. Wallraf; J. Q. You; Y. Nakamura; M. Weides; R. W. Simmons; J. Koch; etc.

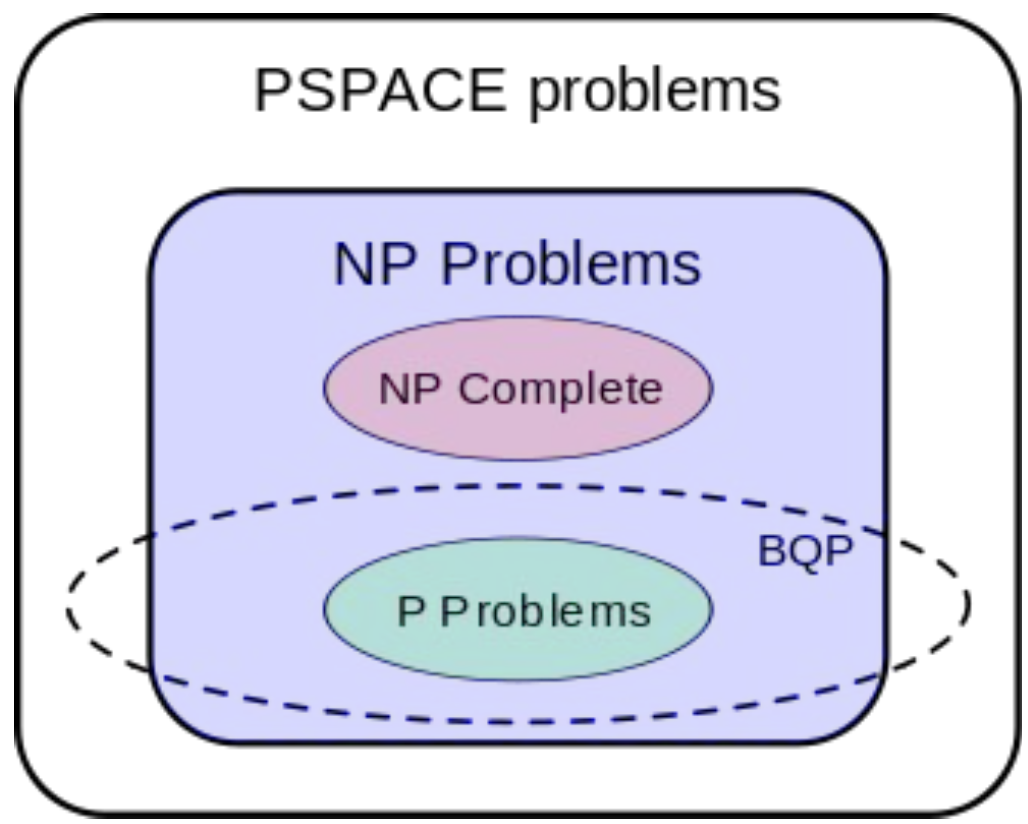
... but how do we trust the quantum devices we build?

With a **universal** quantum computer there would be some strategies for benchmarking…

- E.g.: solve an NP problem efficiently with a quantum computer.

FACTORING any n-bit integer using just over 2n qubits in Poly(n) time

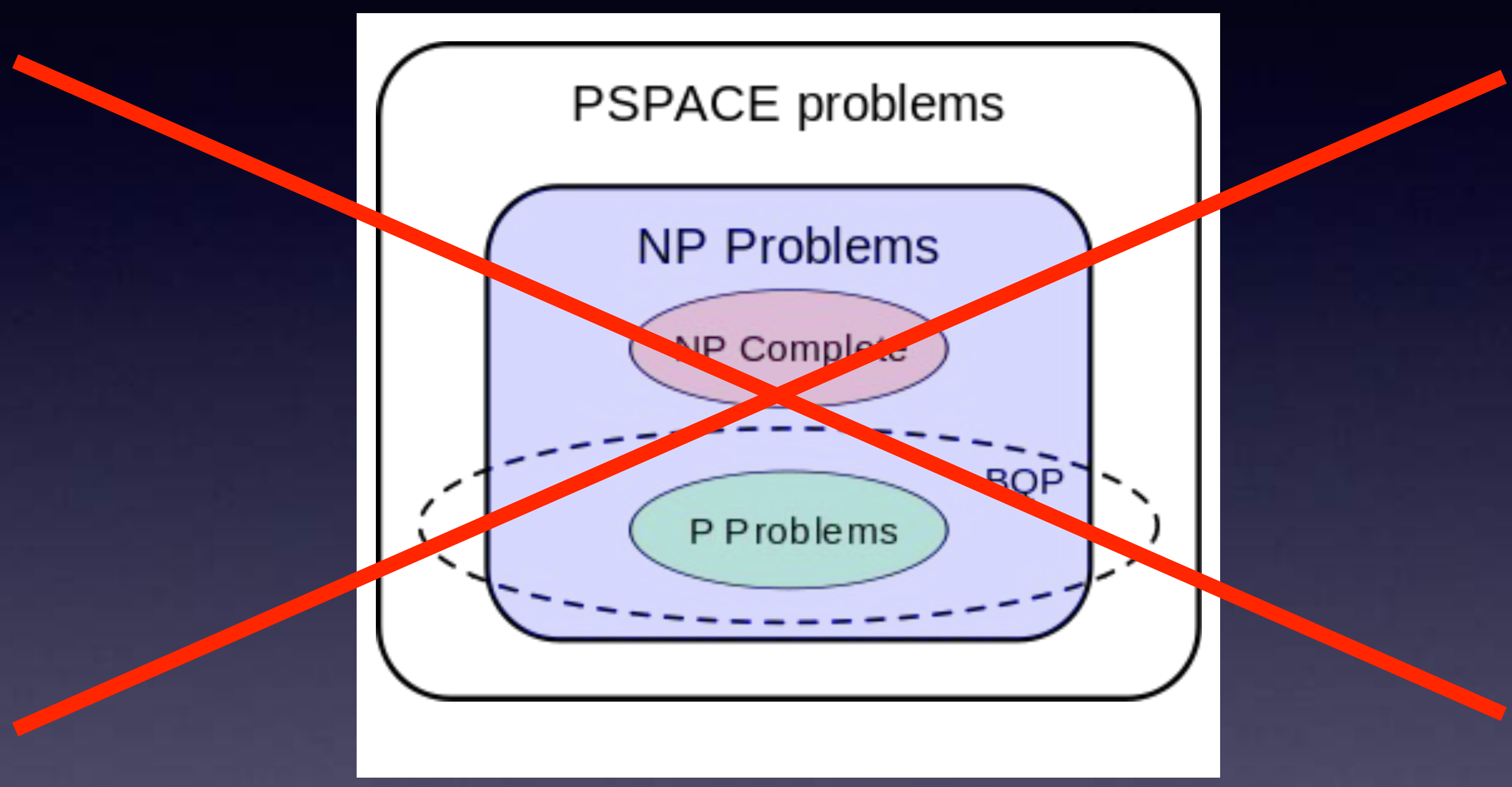
Peter Shor's 1994 algorithm



Suspected relation between BQP and other classes of decision problems

- P solvable by a classical computer in polynomial time.
- NP: verifiable by a classical computer in polynomial time.
- BQP: solvable by a quantum computer in polynomial time.
- PSPACE: solvable by a classical computer using a polynomial amount of space.

But how to validate the performance of a ***non-universal*** *quantum simulator?*



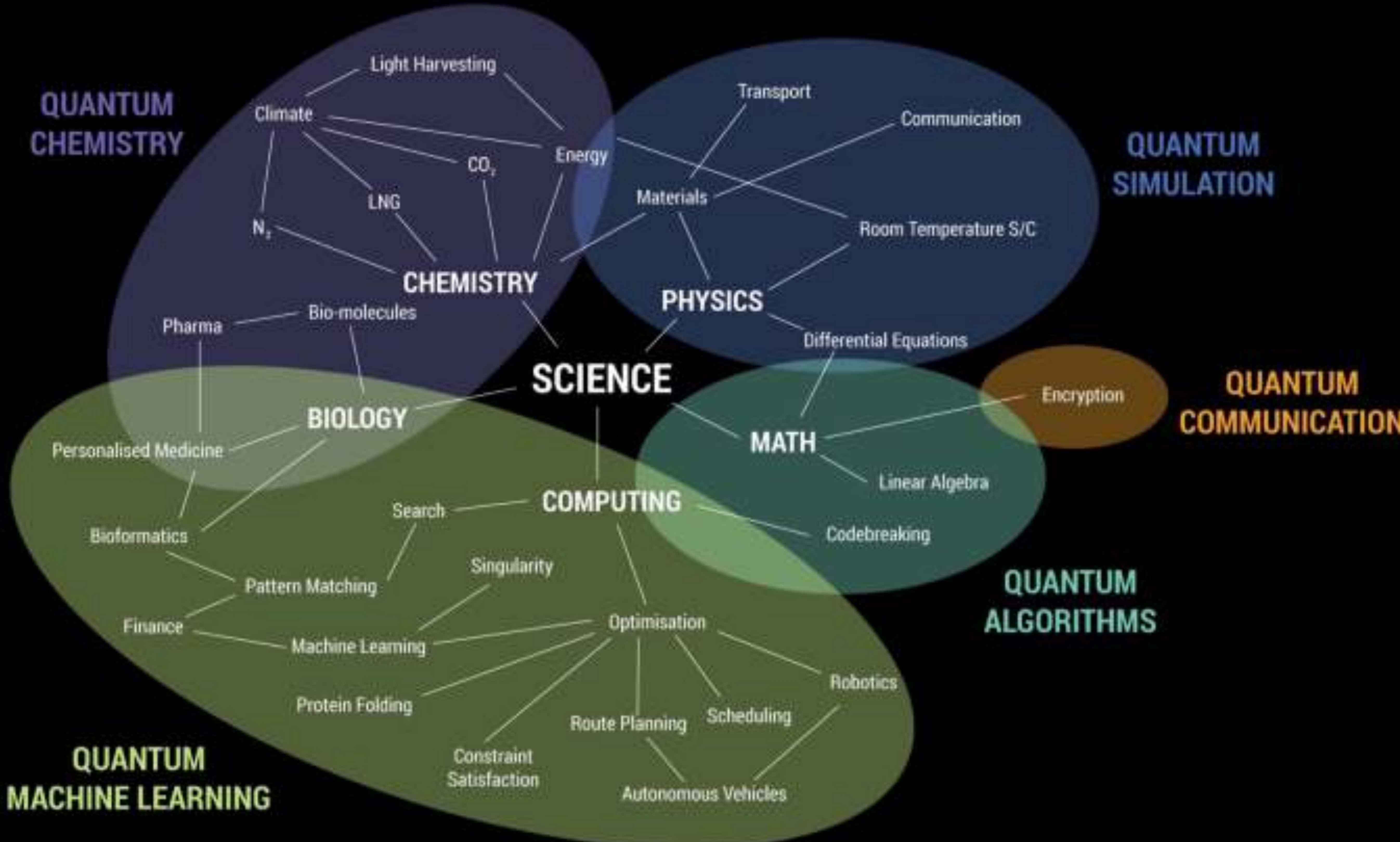
- Quantum simulators can solve problems that, despite classically hard, do not even fall into the decision problem classification (sampling problems).
- In addition, we need to certify non-universal simulators before universal QCs appear.

- Certifying many-body quantum technologies is ultimately about **testing quantum mechanics in unexplored (high computational-complexity) regimes**.

Is Quantum Mechanics falsifiable?
Quantum computation teaches us that quantum mechanics exhibits exponential complexity… the standard scientific paradigm of **"predict and verify" cannot be applied** to testing quantum

D. Aharonov and U. Vazirani,"Is Quantum Mechanics falsifiable? A computational perspective on the foundations of Quantum Mechanics", arXiv: 1206.3686 (2012).

- Further **experimental progress on many-body quantum technologies requires practical benchmarking** tools.



Outline of the two lectures:

- Lecture IV: Certification of non-universal quantum simulators.

- Lecture V: Certification of universal quantum computers.

Outline of Lecture IV:

- The target of quantum certification: What to certify? Sampling problems (weak simulations).

- Quantum state tomography

- Direct fidelity estimation

- Fidelity witnesses

- Partial conclusions.

What to certify?

A "simple" quantum simulation: estimate the expectation value $\mathrm{Tr}[\hat{A}\,\varrho(t)]$ of a local observable $\hat{A}$ on an evolved state $\varrho(t)$ at time t.

Large-deviation (Chernoff) bounds:

$$\text{For } A^* \doteq \frac{1}{C}\sum_{i=1}^{C} a_i, \text{ it holds that } \mathbb{P}\left[\left|A^* - \mathrm{Tr}\big[\hat{A}\,\varrho(t)\big]\right| \leq \varepsilon\right] \geq 1-\delta,$$

$$\text{if } C = \Omega\left(\frac{1}{\mathrm{Poly}(\varepsilon,\delta)}\right), \text{ or, in the best hypothesis, } C = \Omega\left(\frac{1}{\mathrm{Poly}\big(\varepsilon, \log(\delta)\big)}\right).$$

Not even an ideal quantum simulator *(or universal quantum computer!)* ***can succeed*** *in the estimation* ***if higher than polynomial precision is required****!!!*

Sampling problems are the natural problems that quantum simulators *(and also quantum computers!)* ***solve!!!***

What to certify: sampling problems

Classical-output sampling problems

Quantum-output sampling problems

For an N-body target state ϱ_{t},

given C classical outputs $\{a_i\}_{i\in[C]}$, drawn from an unknown distribution $\mathbb{P}_{\mathrm{p}}$, certify that $\mathbb{P}_{\mathrm{p}}$ is close to $\mathbb{P}_{\mathrm{t}}$, with $\mathbb{P}_{\mathrm{t}}(a) := \mathrm{Tr}\left[\hat{M}_a\, \varrho_{\mathrm{t}}\right] \forall\, a$.

given C quantum outputs $\{\varrho_i\}_{i\in[C]}$, described by an unknown density matrix ϱ_{p} , certify that ϱ_{p} is close to ϱ_{t} .

(Classical or quantum) state certification from a finite-sized sample

Two reasons for inefficiency of such certifications:

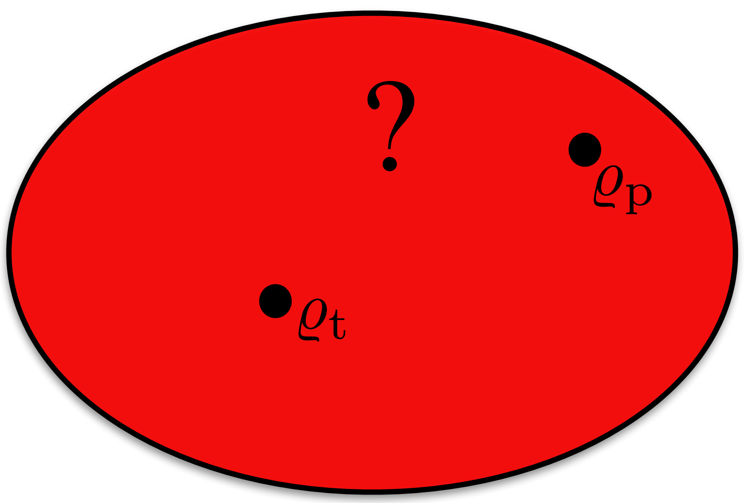
1. **Computational Complexity**: Required classical-computing resources scale exponentially with N.

2. **Sample Complexity**: The required sample size C scales exponentially with N.

Quantum-state certification from finite-size samples

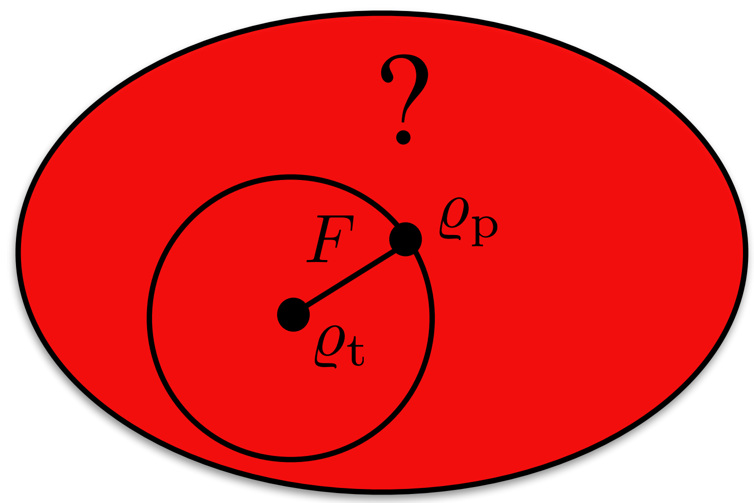
Different certification paradigms

State tomography



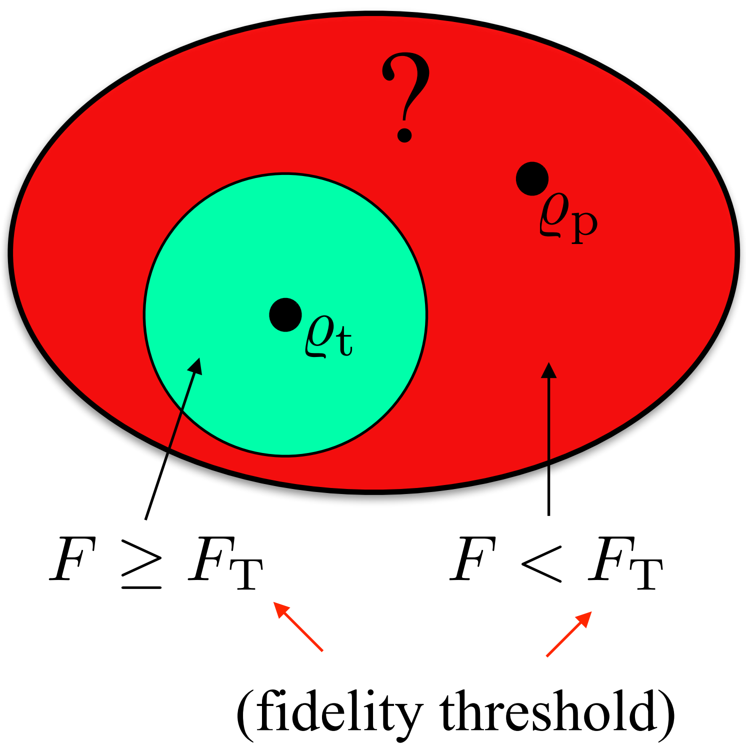
Where is ϱ_{p}*?*

Direct fidelity estimation



How far away is ϱ_{p}*?*

Membership-problem certification



Is ϱ_{p} *far away or close?*

Quantum-state tomography

U. Leonhardt, Phys. Rev. Lett. **74,** 4101 (1995);

A. G. White et al., Phys. Rev. Lett. **83,** 3103 (1999);

C. F. Roos et al., Phys. Rev. Lett. **92,** 220402 (2004);

H. Häffner et al., Nature **438,** 646 (2005).

A long history of quantum state characterisation

• **Quantum state tomography:** reconstructs the **full experimental quantum state**, but requires the measurement of $\mathrm{O}(D^2)$ observables and is thus **exponentially expensive**.

U. Leonhardt, Phys. Rev. Lett. **74,** 4101 (1995); A. G. White, D. F. V. James, P. H. Eberhard, and P. G. Kwiat, Phys. Rev. Lett. **83,** 3103 (1999); C. F. Roos, G. P. T. Lancaster, M. Riebe, H. Häffner, W. Hänsel, S. Gulde, C. Becher, J. Eschner, F. Schmidt-Kaler, and R. Blatt, Phys. Rev. Lett. **92,** 220402 (2004).

• **Compressed sensing:** reconstructs states well approximated by **low-rank density matrices** with **significantly less resources**, but is **still exponentially expensive**. It requires the measurement of $\mathrm{O}(r\, D \log D)$ observables.

D. Gross, Y.-K. Liu, S. T. Flammia, S. Becker, and J. Eisert, Phys. Rev. Lett. **105**, 150401 (2010); D. Gross, IEEE Trans. Info. Theory **57**, 1548 (2011); S. T. Flammia, D. Gross, Y.-K. Liu, and J. Eisert, New J. Phys. **14**, 095022 (2012).

• **Permutationally invariant tomography:** efficiently reconstructs the part of the experimental state that is symmetric with respect to all particle permutations (W states, Dicke states, GHZ states, etc)**.**

G. Toth, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, Phys. Rev. Lett. **105**, 250403 (2010); T. Moroder, P. Hyllus, G. Toth, C. Schwemmer, A. Niggebaum, S. Gaile, O. Gühne, and H. Weinfurter, New J. Phys. **14**, 105001 (2012).

•**Matrix-Product-State (MPS) tomography:** efficiently reconstructs states well approximated by an MPS.

M. Cramer, M. B. Plenio, S. T. Flammia, R. Somma, D. Gross, S. D. Bartlett, O. Landon-Cardinal, D. Poulin, and Y.-K. Liu, Nat. Commun. **1**, 149 (2010); T. Baumgratz, D. Gross, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **111**, 020401 (2013); B. Lanyon, C. Maier, M. Holzäpfel, T. Baumgratz, C. Hempel, P. Jurcevic, I. Dhand, A. Buyskikh, A. Daley, M. Cramer, M. Plenio, R. Blatt, and C. F. Roos, Nat. Phys. **13**, 1158 (2017).

•**Neural network state tomography:** efficiently reconstructs states neural network quantum states using unsupervised learning.

G. Torlai, G. Mazzola, J. Carrasquilla, M. Troyer, R. Melko, and G. Carleo, Nat. Phys. **14**, 447 (2018); G. Torlai and R. Melko, Phys. Rev. Lett. **120**, 240503 (2018); J. Carrasquilla, G. Torlai, R. Melko, and L. Aolita, in preparation (2018).

Full quantum state tomography

Experimental state to reconstruct: ϱ

N-qudit Hilbert space: $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \ldots \mathbb{H}_N$

$$\mathrm{Dim}(\mathbb{H}) = d^N =: D$$

(characteristic function of ϱ)

(orthonormal operator basis of the i-th qudit)

Decompose states as $\varrho = \sum_{\boldsymbol{\alpha}} \chi_{\varrho}(\boldsymbol{\alpha})\, \hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})}$, with $\chi_{\varrho}(\boldsymbol{\alpha}) := \left\langle \hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})} \right\rangle_{\varrho}$

and $\hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})} := \hat{P}_1^{(\alpha_1)} \otimes \ldots \otimes \hat{P}_N^{(\alpha_N)}$, where $\left\{ \hat{P}_i^{(\alpha_i)} \right\}_{\alpha_i \in [d^2]}$.

$$\mathrm{Tr}\left[\hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})}\, \hat{\boldsymbol{P}}^{(\boldsymbol{\alpha}')}\right] = \delta_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$$

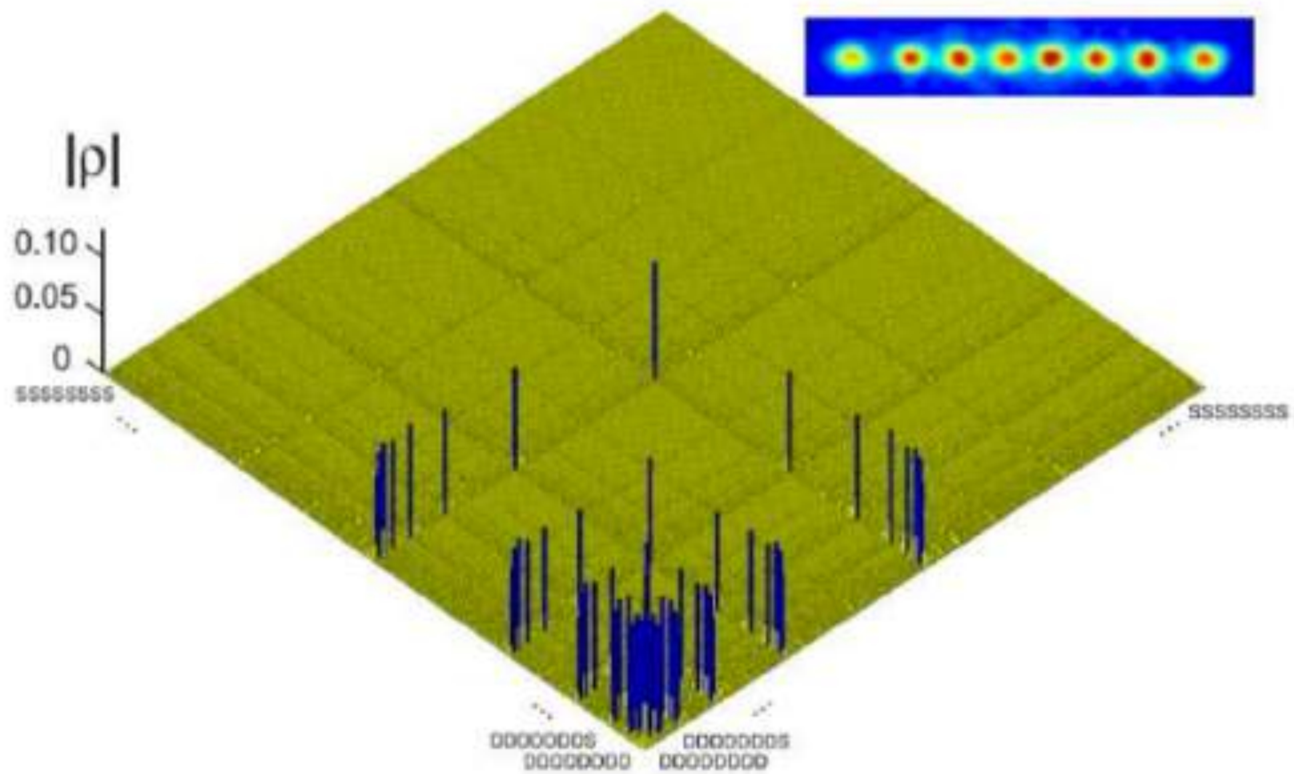
E.g., for d=2, Pauli operators: $\hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})} = \dfrac{\hat{\boldsymbol{\sigma}}^{(\boldsymbol{\alpha})}}{\sqrt{D}}$

Experimental procedure: measure all $\mathrm{O}(D^2)$ *observables!!!!*

Full quantum state tomography: not a scalable option

- Requires the measurement of $\mathrm{O}(D^2)$ observables, with $D^2 = \left(d^N\right)^2$.
- Plus computationally expensive classical post-processing of the experimental data.

Example. Reconstructed experimental W state of N=8 trapped-ion qubits: **storage of and optimisation** over $6^8 = 1679616$ measurement-outcome probabilities.



H. Häffner et al., Nature **438,** 646 (2005).

Not a scalable option :-(

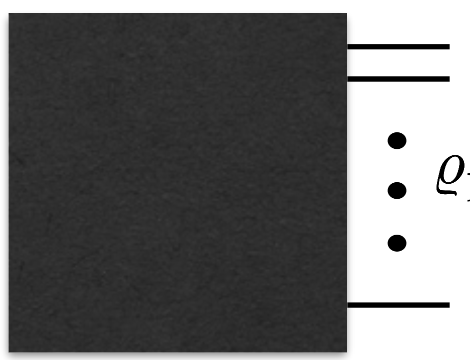
In search for the cheapest certification paradigm…

Direct certification schemes
(i.e. without state reconstruction)

Non-interactive certification mindset

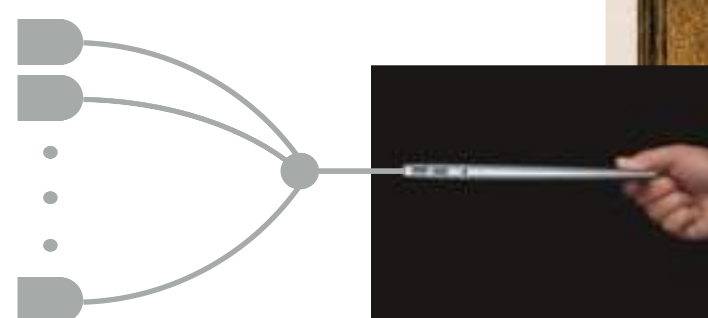
Untrusted quantum prover Merlin

$F(\varrho_{\rm t}, \varrho_{\rm p})$?



$\varrho_{\rm p}$

$\varrho_{\rm t}$ and C



Skeptic almost-classical certifier Arthur

- Similar mindset to interactive proofs but with a single quantum interaction.
- No restriction on type of quantum noise, preparation totally unknown.
- Only assumption: i.i.d. preparations $\Rightarrow \varrho_{\rm p}^{\otimes C}$.

Quantum state fidelity

Figure of merit to "get convinced": $F := F(\varrho_{\mathrm{t}}, \varrho_{\mathrm{p}})$ (fidelity between target and experimental preparation)

- **Pure targets:** $\varrho_{\mathrm{t}} = |\Psi_{\mathrm{t}}\rangle\langle\Psi_{\mathrm{t}}| \Rightarrow F = \mathrm{Tr}\big[|\Psi_{\mathrm{t}}\rangle\langle\Psi_{\mathrm{t}}|\,\varrho_{\mathrm{p}}\big] = 1 - \mathbb{P}_{\mathrm{incorrect}}$,

  where $\mathbb{P}_{\mathrm{incorrect}} := \mathrm{Tr}\left[\left(1 - |\Psi_{\mathrm{t}}\rangle\langle\Psi_{\mathrm{t}}|\right)\varrho_{\mathrm{p}}\right]$.

  $\mathbb{P}_{\mathrm{incorrect}}$: (probability of an incorrect output)

  $1 - |\Psi_{\mathrm{t}}\rangle\langle\Psi_{\mathrm{t}}|$: (projector onto the subspace orthogonal to the ideal output)

- **The fidelity yields also an estimate of the state distance** $\mathrm{D} := \mathrm{D}(\varrho_{\mathrm{t}}, \varrho_{\mathrm{p}})$**:**

$$\text{For } \mathrm{D}(\varrho_{\mathrm{t}}, \varrho_{\mathrm{p}}) := \frac{\mathrm{Tr}\left[|\varrho_{\mathrm{t}} - \varrho_{\mathrm{p}}|\right]}{2} \text{ and } \varrho_{\mathrm{t}} \text{ pure: } 1 - F \leq \mathrm{D} \leq \sqrt{1 - F}.$$

$\frac{\mathrm{Tr}\left[|\varrho_{\mathrm{t}} - \varrho_{\mathrm{p}}|\right]}{2}$: (1-norm distance in state space: trace distance)

Direct fidelity estimation

S. S. T. Flammia and Y.-K. Liu, Phys. Rev. Lett., **106,** 230501 (2011);

M. P. da Silva, O. Landon-Cardinal, and D. Poulin, Phys. Rev. Lett., **107**, 210404 (2011).

Importance sampling in Hilbert space

known pure target state — unknown experimental preparation — (because the target state is pure)

The fidelity to estimate: $F := F(\varrho_{\mathrm{t}}, \varrho_{\mathrm{p}}) := \mathrm{Tr}\left[\left(\sqrt{\varrho_{\mathrm{t}}}\,\varrho_{\mathrm{p}}^{\dagger}\sqrt{\varrho_{\mathrm{t}}}\right)^{1/2}\right]^2 = \mathrm{Tr}[\varrho_{\mathrm{t}}\varrho_{\mathrm{p}}]$

(characteristic function of ϱ)

(orthonormal operator basis of the *i*-th qudit)

Decompose states as $\varrho = \sum_{\boldsymbol{\alpha}} \chi_{\varrho}(\boldsymbol{\alpha})\, \hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})}$, with $\chi_{\varrho}(\boldsymbol{\alpha}) := \left\langle \hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})} \right\rangle_{\varrho}$

and $\hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})} := \hat{P}_1^{(\alpha_1)} \otimes \ldots \otimes \hat{P}_N^{(\alpha_N)}$, where $\left\{\hat{P}_i^{(\alpha_i)}\right\}_{\alpha_i \in [d^2]}$.

$$\mathrm{Tr}\left[\hat{\boldsymbol{P}}^{(\boldsymbol{\alpha})}\, \hat{\boldsymbol{P}}^{(\boldsymbol{\alpha}')}\right] = \delta_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$$

Then, $\mathrm{Tr}[\varrho_{\mathrm{t}}\varrho_{\mathrm{p}}] = \sum_{\boldsymbol{\alpha}} \chi_{\varrho_{\mathrm{t}}}(\boldsymbol{\alpha})\chi_{\varrho_{\mathrm{p}}}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\alpha}} \mathbb{P}_{\mathrm{t}}(\boldsymbol{\alpha}) X_{\mathrm{p}}(\boldsymbol{\alpha})$, where $\mathbb{P}_{\mathrm{t}}(\boldsymbol{\alpha}) := \chi_{\varrho_{\mathrm{t}}}(\boldsymbol{\alpha})^2$ and $X_{\mathrm{p}}(\boldsymbol{\alpha}) := \dfrac{\chi_{\varrho_{\mathrm{p}}}(\boldsymbol{\alpha})}{\chi_{\varrho_{\mathrm{t}}}(\boldsymbol{\alpha})}$.

(relevance distribution) — (random variable)

$$\Rightarrow F = \mathbb{E}\left[X_{\mathrm{p}}\right]_{\mathbb{P}_{\mathrm{t}}}$$

Randomly measure observables according to their relevance for ϱ_{t} ***!!!***

How many observables are required?

$$\text{Estimate } F = \mathbb{E}\left[X_\text{p}\right]_{\mathbb{P}_\text{t}} \text{ with } X_\text{p}^* := \frac{1}{l}\sum_{i=1}^{l} X_\text{p}(\boldsymbol{\alpha_i}), \text{ where } \alpha_i \sim \mathbb{P}_\text{t}.$$

(finite-sample estimate)

(sample size, number of observables)

Sampling from $\mathbb{P}_\text{t}$ ***can be highly non-trivial*** **:-(**

Chebyshev's inequality:

(squared variance)

$$\mathbb{P}\left[\left|F - X_\text{p}^*\right| \geq \varepsilon\right] \leq \frac{\sigma^2}{\varepsilon^2}$$

(constant additive error)

$$\text{Besides: } \sigma^2 \leq \frac{1}{l}$$

(follows directly from normalization)

(constant failure probability)

$$\text{Then, } \mathbb{P}\left[\left|F - X_\text{p}^*\right| \geq \varepsilon\right] \leq \delta, \text{ if } l \geq \left\lceil \frac{1}{\varepsilon^2\,\delta} \right\rceil.$$

F is estimated with a constant (N-independent) number of observables!!!! **:-)**

Sample complexity of direct fidelity estimation

But, how about the total number of preparations required?

Since $X_{\mathrm{p}}(\boldsymbol{\alpha}) := \dfrac{\chi_{\varrho_{\mathrm{p}}}(\boldsymbol{\alpha})}{\chi_{\varrho_{\mathrm{t}}}(\boldsymbol{\alpha})}$, $\chi_{\varrho_{\mathrm{p}}}(\boldsymbol{\alpha})$ must be estimated up to error $\varepsilon = \mathrm{O}\left(\chi_{\varrho_{\mathrm{t}}}(\boldsymbol{\alpha})\right)$.

Decreases in general exponentially in N

Exponential sample complexity
:-(

•Only "**well-conditioned**" target states (with **step-like characteristic functions)** can be efficiently handled: in practice **only W, GHZ, and stabiliser states :-(**

•Particularly critical **for CV systems**, where **there exist no well conditioned states**.

Even for pure N-mode coherent states, $\chi_{\varrho_{\mathrm{t}}}(\boldsymbol{\alpha}) = \mathrm{O}\left(\mathrm{Exp}(-N)\right)$ **:-(**

Fidelity witnesses

M. Cramer, M. B. Plenio, S. T. Flammia, R. Somma, D. Gross, S. D. Bartlett, O. Landon-Cardinal, D. Poulin, and Y.-K. Liu, Nat. Comms. **1**, 149 (2010);

L. Aolita, C. Gogolin, M. Kliesch, and J. Eisert, Nat. Comms. **6**, 8498 (2015);

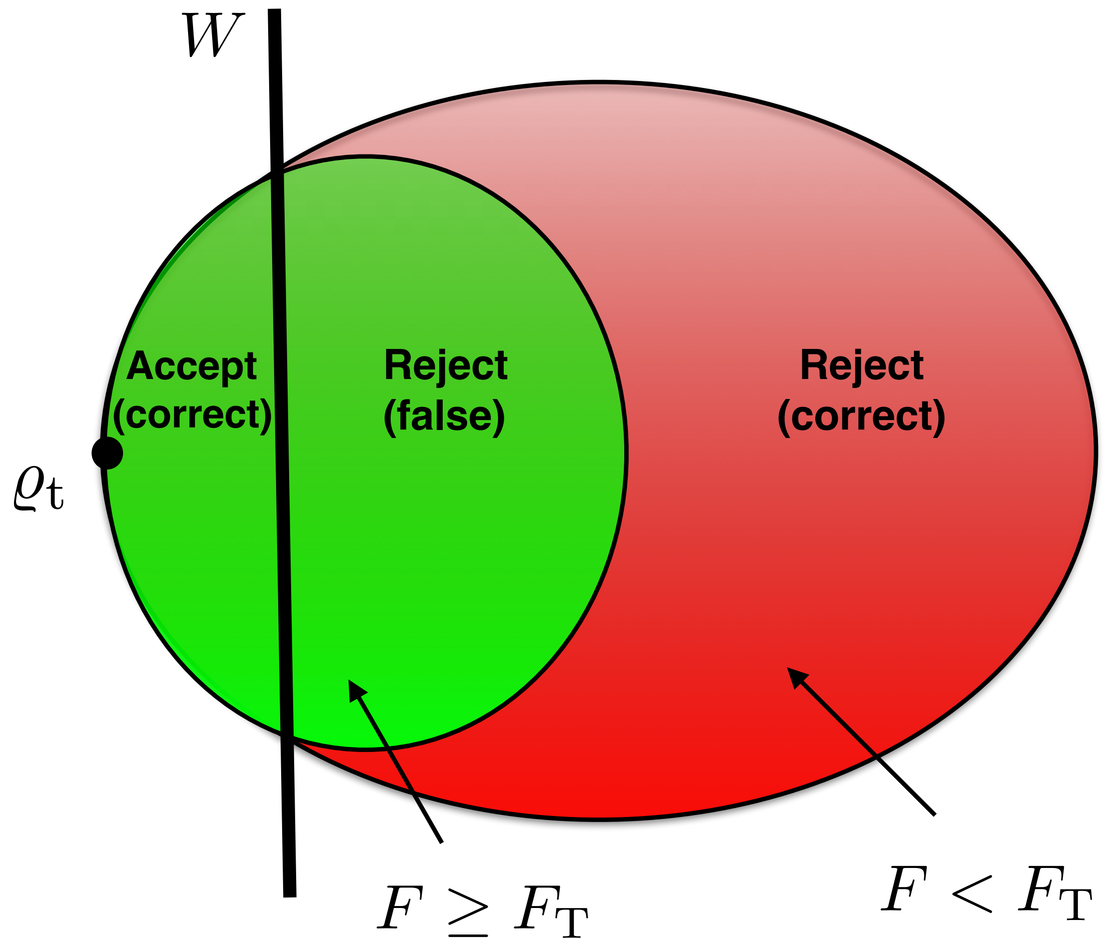
M. Gluza, M. Kliesch, J. Eisert, and L. Aolita, Phys. Rev. Lett. **120**, 190501 (2018).

Fidelity witnesses (certification as weak-membership problem)

Definition 1 (Fidelity witnesses)**.** *An observable $\mathcal{W}$ is a* fidelity witness *for ϱ_t if, for $F_{\mathcal{W}}(\varrho_p) := \mathrm{tr}[\mathcal{W}\,\varrho_p]$, it holds that*

i) $F_{\mathcal{W}}(\varrho_p) = 1$ *if, and only if,* $\varrho_p = \varrho_t$, *and*

ii) $F_{\mathcal{W}}(\varrho_p) \leq F$ *for all states* ϱ_p.

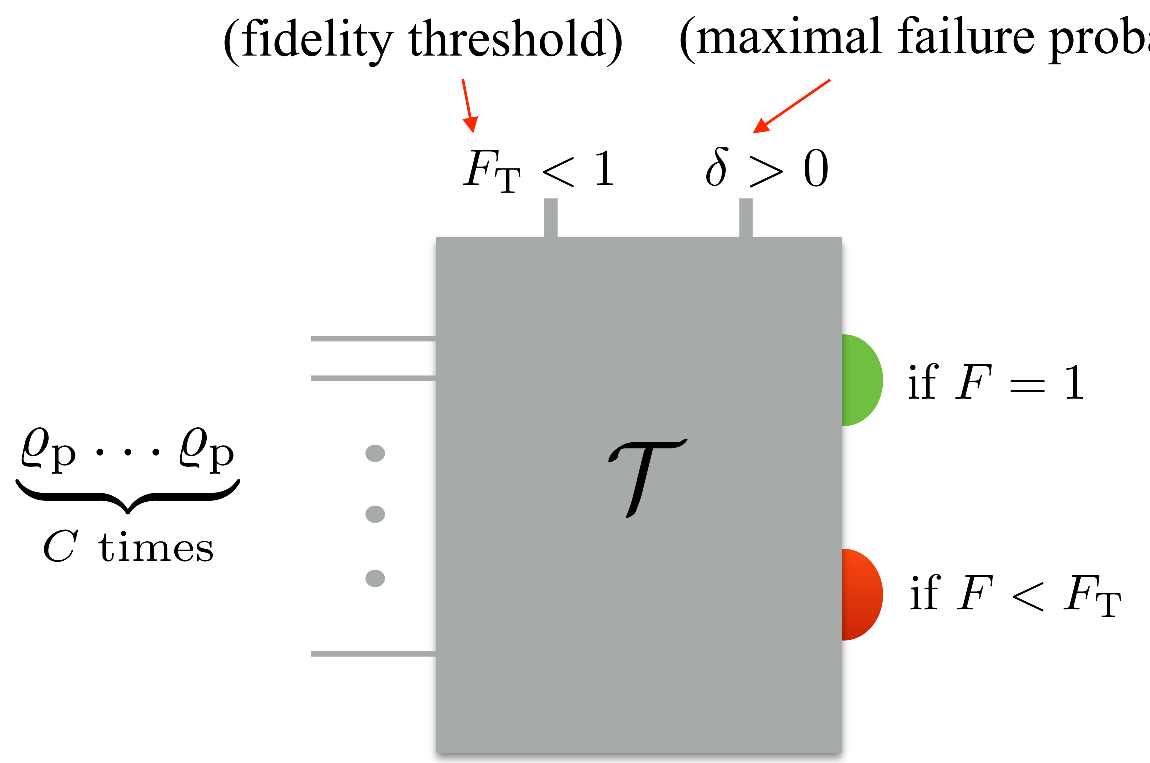


A significant subset of valid states is sacrificed, but the experimental estimation is considerably more efficient

M. Gluza, M. Kliesch, J. Eisert, and L. Aolita, Phys. Rev. Lett. **120**, 190501 (2018).

Fidelity-witness-based certification tests

The desired test as a black box:

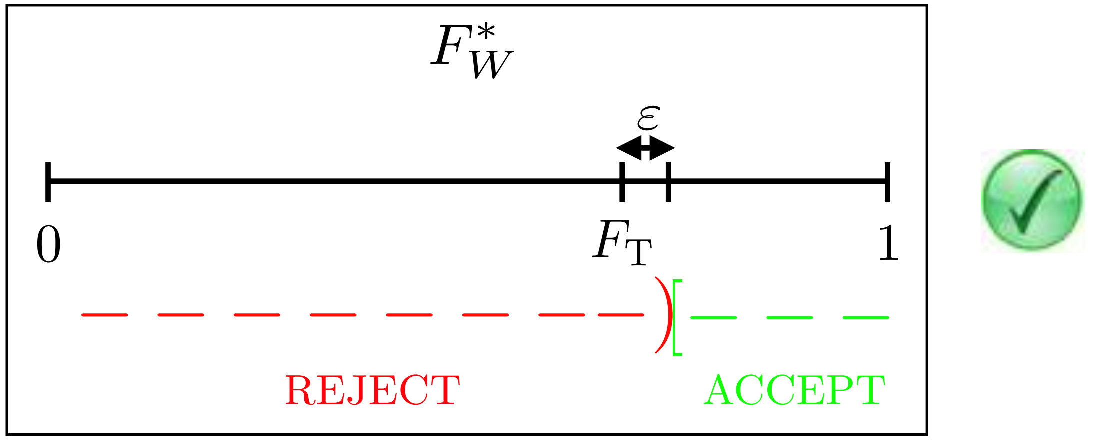


The experimental estimation must be s. t.

$$\mathbb{P}\Big[\big|F_W - F_W^*\big| > \varepsilon\Big] < \delta$$

(constant additive error)

Then, the accept/reject criterion:



Generic expression of fidelity witnesses

Proposition 3 (General witness construction)**.** *Let* ϱ_t *be any pure target state,* $0 < \Delta = \lambda_1 \leq \ldots \leq \lambda_N$*, and* P_1*,* P_2*, …, and* P_N *positive-semidefinite operators such that* $\varrho_t + \sum_{l=1}^{N} P_l = \mathbb{1}$ *and* $\mathrm{tr}(\varrho_t P_l) = 0$ *for all* $l = 1, \ldots, N$*. Then,*

$$\mathcal{W} := \mathbb{1} - \Delta^{-1} \sum_{l=1}^{N} \lambda_l P_l$$

is a fidelity witness for ϱ_t*.*

Proof.

$$i)\ \mathrm{Tr}\Big[\sum_{l=1}^{N} \lambda_l P_l \varrho_{\mathrm{p}}\Big] = 0 \Leftrightarrow \varrho_{\mathrm{p}} = \varrho_{\mathrm{t}}.$$

$$\begin{aligned} ii)\ \mathrm{Tr}\Big[\Big(1 - \Delta^{-1} \sum_{l=1}^{N} \lambda_l P_l\Big) \varrho_{\mathrm{p}}\Big] &\leq 1 - \mathrm{Tr}\Big[\sum_{l=1}^{N} P_l \varrho_{\mathrm{p}}\Big] \\ &= 1 - \Big(1 - \mathrm{Tr}\Big[\varrho_{\mathrm{t}} \varrho_{\mathrm{p}}\Big]\Big) \\ &= F. \end{aligned}$$

M. Gluza, M. Kliesch, J. Eisert, and L. Aolita, Phys. Rev. Lett. **120**, 190501 (2018).

Ground-state witnesses

M. Cramer et al., Nat. Commun. **1**, 149 (2010); G. Toth et al., Phys. Rev. Lett. **105**, 250403 (2010); D. Hangleiter, M. Kliesch, M. Schwarz, and J. Eisert, Quantum Sci. Technol. **2**, 015004 (2017).

Local, gapped Hamiltonians

Defs.

A Hamiltonian $\hat{H} := \sum_{i=1}^{N} \hat{H}_i$, with a ground state $|\phi_0\rangle$ s.t. $\hat{H}|\phi_0\rangle = 0$, is:

i) k-local if, $\forall\ i \in [N]$, $\hat{H}_i$ acts non-trivially on at most k sites;

ii) short-ranged if these k sites are contiguous;

iii) gapped if $|\phi_0\rangle$ has spectral gap $\Delta E > 0$ to the first excited state(s); and gapless, or critical, otherwise;

iv) frustration-free if, $\forall\ i \in [N]$, $\hat{H}_i|\phi_0\rangle = 0$;

We call the ground state unique if $\hat{H}|\phi\rangle = 0 \Rightarrow |\phi\rangle = |\phi_0\rangle$, and degenerate otherwise.

Finally, we refer to $\hat{H}$ as the parent Hamiltonian of $|\phi_0\rangle$.

- Every gapped, local, short-ranged Hamiltonian is approximated by a gapped, local, short-ranged, frustration-free one.

M. B. Hastings, Phys. Rev. B **73**, 085115 (2006).

Local, gapped Hamiltonians as efficient ground-state witnesses

On the other hand, in its eigenbasis, $\hat{H} = \sum_{n=0}^{d^N-1} E_n |\phi_n\rangle\langle\phi_n|.$

(eigen-energies) (eigen-states)

Then, for $\varrho_{\rm t} = |\phi_0\rangle\langle\phi_0|$, $F = \mathrm{Tr}\left[|\phi_0\rangle\langle\phi_0|\, \varrho_{\rm p}\right] \geq 1 - \frac{1}{\Delta E}\mathrm{Tr}\left[\hat{H}\, \varrho_{\rm p}\right].$

The Hamiltonian acts as a witness for its ground state!!!

Now, $\mathrm{Tr}\left[\hat{H}\, \varrho_{\rm p}\right] = \mathrm{Tr}\left[\sum_{i=1}^{N} \hat{H}_i\, \varrho_{\rm p}\right] = \sum_{i=1}^{N} \mathrm{Tr}\left[\hat{H}_i\, \varrho_{\rm p}\right] = \sum_{i=1}^{N} \mathrm{Tr}\left[\hat{H}_i\, \varrho_{{\rm p}_i}\right].$

Only the reductions of $\varrho_{\rm p}$ are required!!!

The fidelity lower bound:

$$F \geq F_W := 1 - \frac{1}{\Delta E} \sum_{i=1}^{N} \mathrm{Tr}\left[\hat{H}_i \, {\varrho_{\mathrm{p}}}_i\right]$$

reduced state of ϱ_{p} on the sites where $\hat{H}_i$ acts non-trivially

Upsides:

- Requires only the expectation values of the local interaction terms $\hat{H}_i$ (linear overhead scaling!).
- Only locality + gap required (no frustration-freeness).
- Covers all MPSs (all local, gapped Hamiltonians satisfy an entanglement area law).

Downsides:

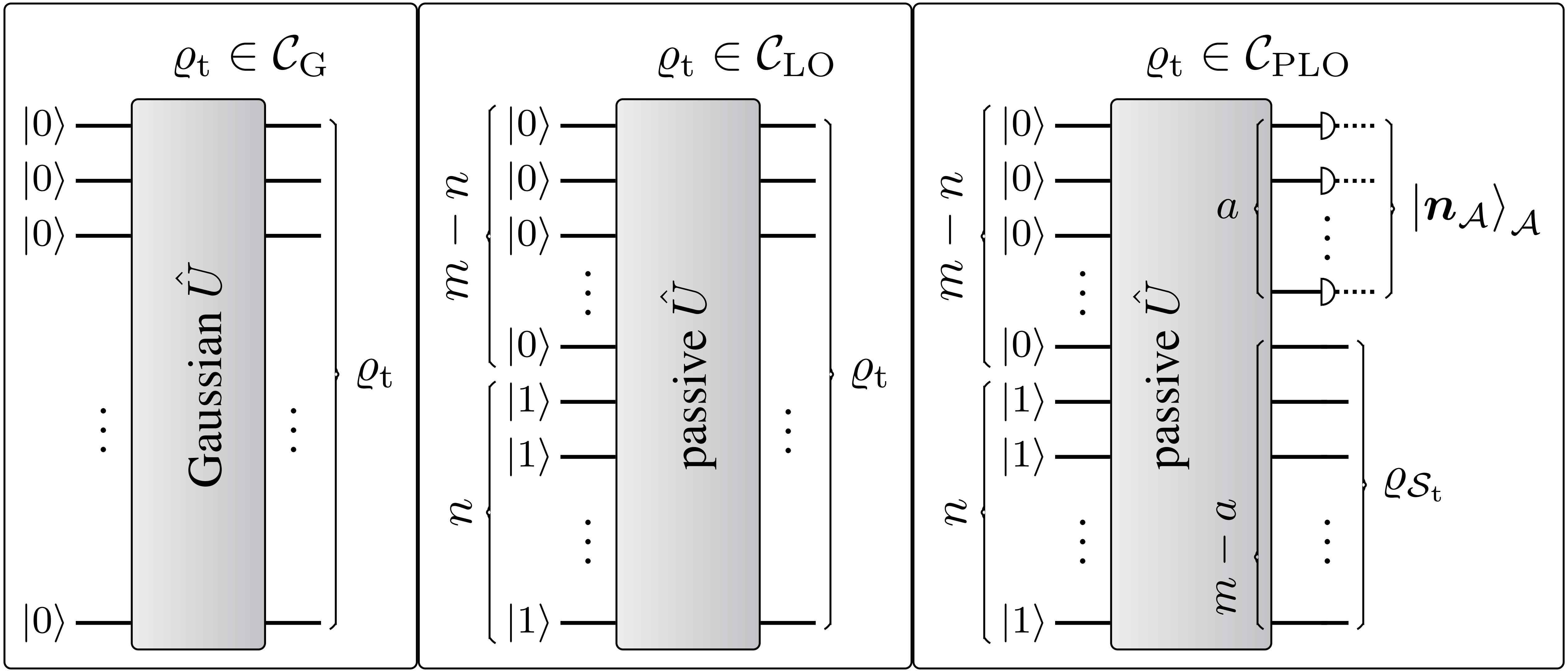
- Must know spectral gap (not trivial in general) :-(
- In practice limited to MPSs (no long-range entanglement, only short-time evolutions) :-(

M. Cramer, M. B. Plenio, S. T. Flammia, R. Somma, D. Gross, S. D. Bartlett, O. Landon-Cardinal, D. Poulin, and Y.-K. Liu, Nat. Commun. **1**, 149 (2010).

Fidelity witnesses for bosonic quantum simulations

L. Aolita, C. Gogolin, M. Kliesch, and J. Eisert, Nat. Comms. **6**, 8498 (2015).

Classes of target states covered



Gaussian states

Linear-optical network output states (of constant n!)

Number post-selected linear-optical network output states (of constant n!)

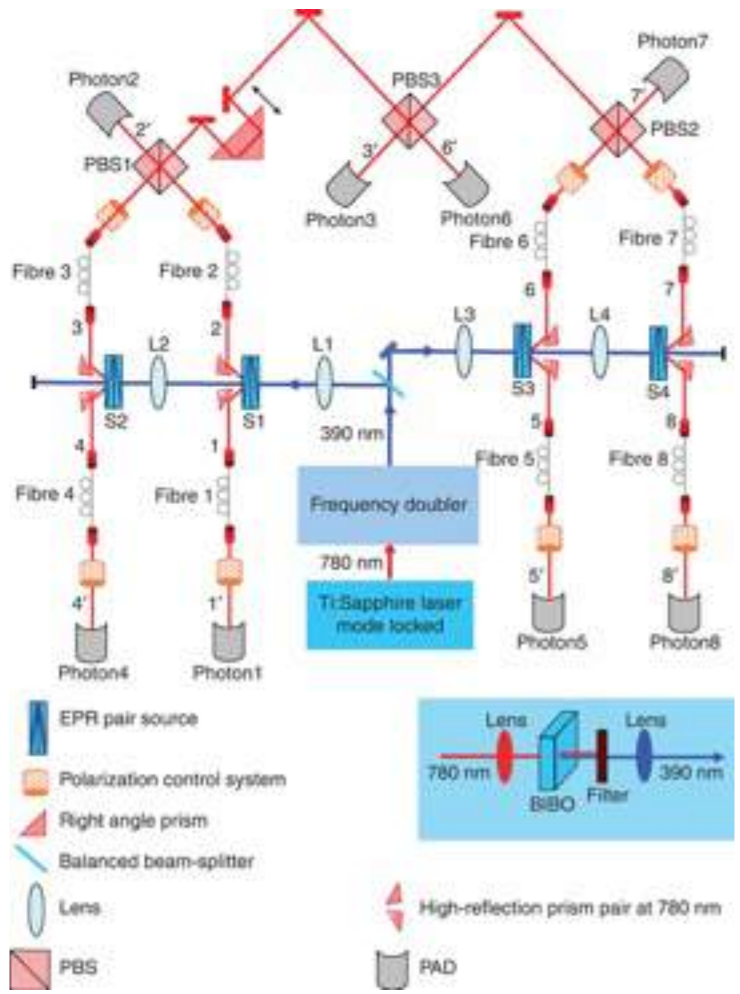
Covers most current photonic setups:

- Gaussian, squeezed, and non-Gaussian.
- on-chip photonic quantum simulations (Boson-Sampling, quantum walks, Anderson-localisation, etc.).
- photonic qubit encodings (polarization qubits, KLM scheme, etc.).

Classes of experimental platforms covered

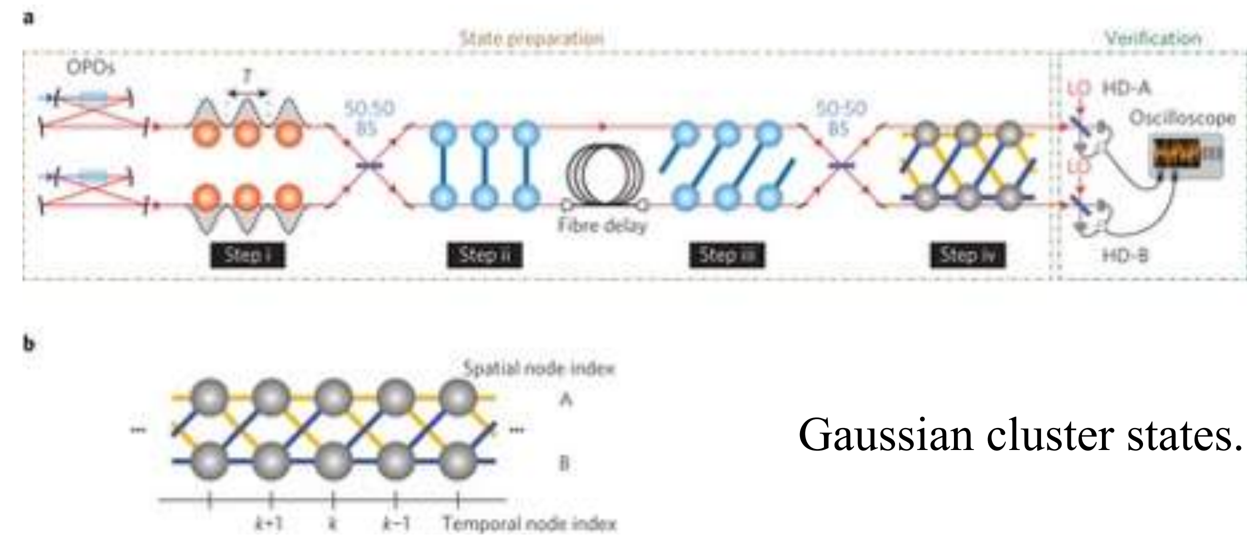
Photonic multi-qubit entangled states

W, Dicke, GHZ, cluster states, ect.



J. W. Pan et al., G. C. Guo et al., P. Walther et al.; H. Weinfurther et al.; S. P Walborn and P. H. S. Ribeiro et al.; C. Monken and S. Padua et al.; etc.

Multi-mode squeezed Gaussian states

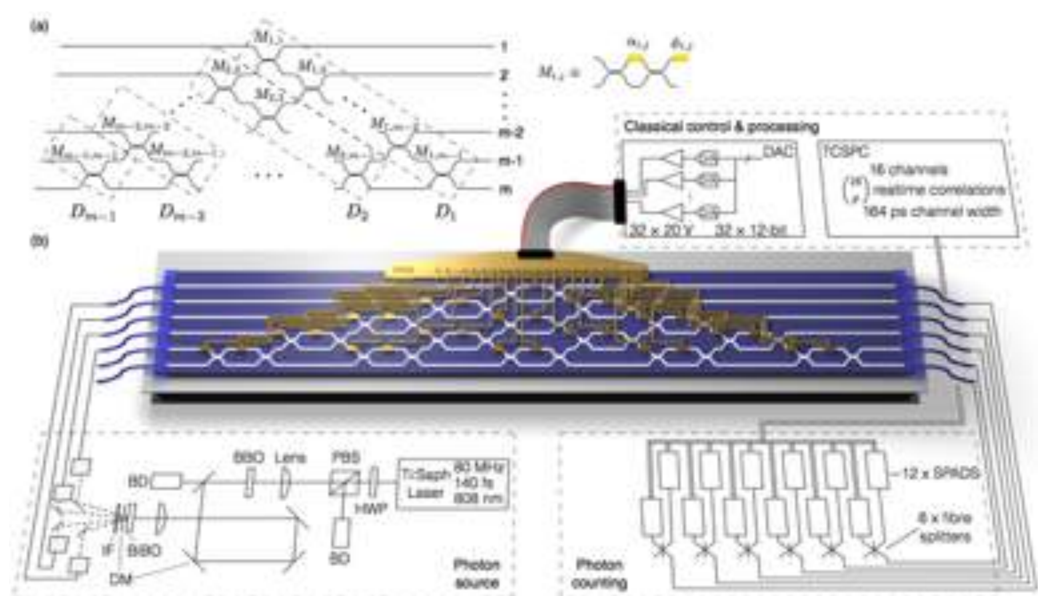


Gaussian cluster states.

A. Furusawa et al.; O. Pfister et al.; R. Schnabel et al.; N Treps et al; etc.

On-chip integrated linear-optical networks

Small-sized simulations of
- •Boson-Sampling,
- •Anderson localisation,
- • quantum walks, etc.

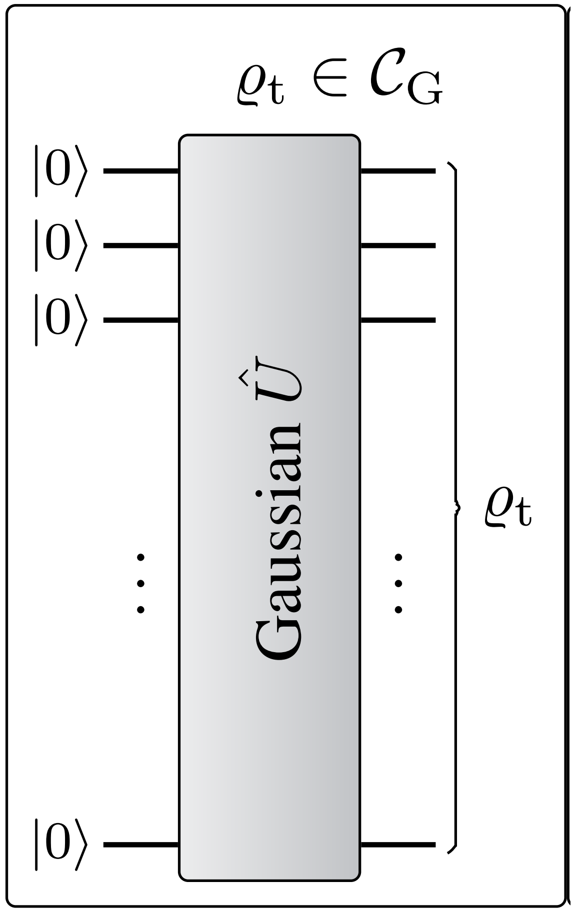


J. O'Brien et al.; I. Walmsley et al.; P. Walther et al.; F. Sciarrino et al.; A. White et al; etc.

Gaussian states

Fidelity witness for multi-mode Gaussian states

$$\mathcal{C}_{\mathrm{G}} := \{\varrho_{\mathrm{t}} = \hat{U}\,|\mathbf{0}\rangle\langle\mathbf{0}|\,\hat{U}^{\dagger} : \hat{U} \text{ Gaussian unitary}\}$$



$$\Rightarrow F = F(\varrho_{\mathrm{t}}, \varrho_{\mathrm{p}}) = \mathrm{Tr}\big[\hat{U}|\mathbf{0}\rangle\langle\mathbf{0}|\hat{U}^{\dagger}\,\varrho_{\mathrm{p}}\big] = \mathrm{Tr}\big[|\mathbf{0}\rangle\langle\mathbf{0}|\,\tilde{\varrho}_{\mathrm{p}}\big]$$

$$\tilde{\varrho}_{\mathrm{p}} := \hat{U}^{\dagger}\varrho_{\mathrm{p}}\hat{U}$$

(Heisenberg representation with respect to $\hat{U}^{\dagger}$)

(photon-number operator of mode j)

$$\hat{n} := \sum_{j=1}^{m} \hat{n}_j$$

(total photon-number operator)

Now, $|\mathbf{0}\rangle\langle\mathbf{0}| \geq \mathbf{1} - \hat{n}$ and $\tilde{\varrho}_{\mathrm{p}} \geq 0$ imply

$$F \geq \mathrm{Tr}\big[(\mathbf{1} - \hat{n})\tilde{\varrho}_{\mathrm{p}}\big]$$

$$= \mathrm{Tr}\Big[\hat{U}(\mathbf{1} - \hat{n})\hat{U}^{\dagger}\varrho_{\mathrm{p}}\Big] \rightarrow \hat{\mathcal{W}}$$

Our Gaussian fidelity witness

But how can we measure $\hat{\mathcal{W}} = \hat{U}(\mathbf{1} - \hat{n})\hat{U}^\dagger$***?***

Idea of the measurement scheme: go phase space!

Fidelity witness: $\hat{\mathcal{W}} = \hat{U}\left(1 - \hat{n}\right)\hat{U}^\dagger = 1 - \hat{\hat{n}}$ → (Heisenberg representation with respect to $\hat{U}^\dagger$)

Phase-space quadrature operators:

$\hat{q}_j$ and $\hat{p}_{j'}$, with $[\hat{q}_j, \hat{p}_{j'}] = i\,\delta_{j,j'}$.

Phase-space quadrature operator vector:

$\hat{\mathbf{r}}$: $\hat{r}_{2j-1} := \hat{q}_j$ and $\hat{r}_{2j} := \hat{p}_j$

$$\Rightarrow \hat{\mathcal{W}} = 1 - \hat{\hat{r}}^2 + \frac{m}{2}$$

↓

$$\hat{\hat{\mathbf{r}}} := \hat{U}\,\hat{\mathbf{r}}\,\hat{U}^\dagger$$

(Heisenberg representation with respect to $\hat{U}^\dagger$)

Identities:

$$\hat{n}_j = \hat{q}_j^2 + \hat{p}_j^2 - 1/2$$

$$\hat{n} = \sum_{j=1}^{m} \hat{n}_j = \sum_{j=1}^{m} (\hat{q}_j^2 + \hat{p}_j^2 - \frac{1}{2}) = \hat{r}^2 - \frac{m}{2}$$

Hence, we must measure the observable $\hat{\tilde{r}}^2$

For a **Gaussian transformation**, phase-quadratures transform under an affine **linear map**:

Symplectic matrix $\mathbf{S} \in \mathrm{Sp}(2m, \mathbb{R})$

Displacement vector $\mathbf{x} \in \mathbb{R}^{2m}$

$$\hat{\mathbf{r}} \mapsto \hat{U}^\dagger \hat{\mathbf{r}} \hat{U} = \mathbf{S}\hat{\mathbf{r}} + \mathbf{x}$$

$$\Rightarrow \hat{\tilde{\mathbf{r}}} := \hat{U} \hat{\tilde{\mathbf{r}}} \hat{U}^\dagger = \mathbf{S}^{-1}(\hat{\mathbf{r}} - \mathbf{x})$$

$\Rightarrow \hat{\tilde{r}}^2$ is a quadratic function of the single-mode observables $\hat{q}_j$ and $\hat{p}_j$!!!

(so that at most $O(m^2)$ observables must be measured only, equivalent to measuring the covariance matrix)

- Affine linear maps computationally highly efficient to handle. ✓
- Only homodyne detection required (no number measurements!). ✓
- Bound highly optimisable for each circuit, polynomial scaling with the squeezing too! ✓

Sample complexity of the estimation

Denote by $\mathcal{N}_{\epsilon,\delta}(\mathcal{W})$ the number of experimental runs needed to obtain F^*_W such that:

(statistical error)

$$\mathbb{P}\left(|F_{\mathcal{W}}(\varrho_{\mathrm{p}}) - F^*_{\mathcal{W}}(\varrho_{\mathrm{p}})| \leq \epsilon\right) \geq 1 - \delta$$

(failure probability)

(maximal single-mode squeezing)

Then, theorem: $$\mathcal{N}_{\epsilon,\delta}(\mathcal{W}) \leq O\left(\frac{\mathrm{s}^4_{\max} m^4}{\varepsilon^2 \log(\delta)}\right)$$ (number of modes)

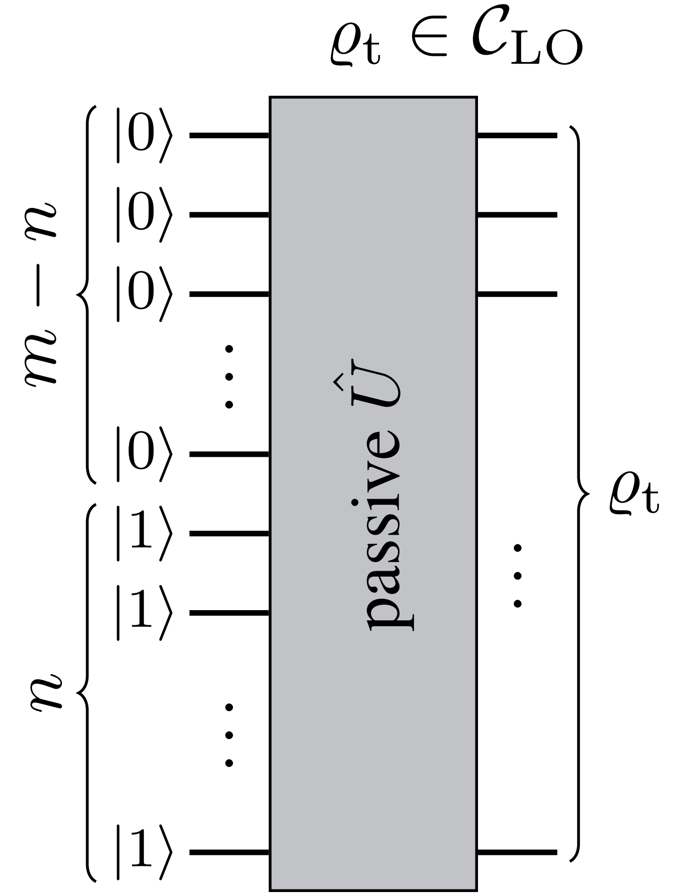
Efficient in the number of modes and in the squeezing!

L. Aolita, C. Gogolin, M. Kliesch, and J. Eisert, Nat. Comms. **6**, 8498 (2015).

Multi-mode non-Gaussian states

Fidelity witness for multi-mode non-Gaussian states

$$\mathcal{C}_{\mathrm{LO}} := \{\varrho_{\mathrm{t}} = \hat{U}\,|\mathbf{1}_n\rangle\langle\mathbf{1}_n|\,\hat{U}^\dagger : \hat{U} \text{ passive unitary}\}$$



$$\Rightarrow F = F(\varrho_{\mathrm{t}}, \varrho_{\mathrm{p}}) = \mathrm{Tr}\big[\hat{U}|\mathbf{1}_n\rangle\langle\mathbf{1}_n|\hat{U}^\dagger\,\varrho_{\mathrm{p}}\big] = \mathrm{Tr}\big[|\mathbf{1}_n\rangle\langle\mathbf{1}_n|\,\tilde{\varrho}_{\mathrm{p}}\big]$$

$$\tilde{\varrho}_{\mathrm{p}} := \hat{U}^\dagger\varrho_{\mathrm{p}}\hat{U}$$

(Heisenberg representation with respect to $\hat{U}^\dagger$)

And, since $|\mathbf{1}_n\rangle := \prod_{j=1}^{n} \hat{a}_j^\dagger\,|\mathbf{0}\rangle$, $F = F\left(|\mathbf{0}\rangle\langle\mathbf{0}|, \tilde{\varrho}_{\mathrm{p},n}\right)$.

$$\tilde{\varrho}_{\mathrm{p},n} := \prod_{j'=1}^{n} \hat{a}_{j'}\,\tilde{\varrho}_{\mathrm{p}}\,\prod_{j=1}^{n} \hat{a}_j^\dagger$$

(positive semi-definite but not normalised)

Now, $|\mathbf{0}\rangle\langle\mathbf{0}| \geq \mathbf{1} - \hat{n}$ and $\tilde{\varrho}_{\mathrm{p},n} \geq 0$ imply

$$F \geq \mathrm{Tr}\big[(\mathbf{1} - \hat{n})\tilde{\varrho}_{\mathrm{p},n}\big]$$

$$= \mathrm{Tr}\Big[(n + \mathbf{1} - \hat{n})\prod_{j=1}^{n}\hat{n}_j\,\tilde{\varrho}_{\mathrm{p}}\Big]$$

Hence, the fidelity lower bound:

$$F \geq F_{\mathcal{W}} := \mathrm{Tr}\left[(n+1-\hat{n})\prod_{j=1}^{n}\hat{n}_j\,\tilde{\varrho}_{\mathrm{p}}\right]$$

$\tilde{\varrho}_{\mathrm{p}} := \hat{U}^{\dagger}\varrho_{\mathrm{p}}\hat{U}$

$$= \mathrm{Tr}\left[(n+1-\hat{\hat{n}})\prod_{j=1}^{n}\hat{\hat{n}}_j\,\varrho_{\mathrm{p}}\right]$$

$\hat{\mathcal{W}}$

Our non-Gaussian fidelity witness

$\hat{\hat{n}} := \hat{U}\,\hat{n}\,\hat{U}^{\dagger}$

$\hat{\hat{n}}_j := \hat{U}\,\hat{n}_j\,\hat{U}^{\dagger}$

(Heisenberg representation with respect to $\hat{U}^{\dagger}$)

- Also available for input Fock-basis states with more than one photon per mode. ✓
- Fidelity lower bound potentially interesting in its own right in other scenarios. ✓

But how can we measure $\hat{\mathcal{W}} = \left(n + 1 - \hat{\tilde{n}}\right) \prod_{j=1}^{n} \hat{\tilde{n}}_j$ ***?***

Phase space again!

Fidelity witness: $$\hat{\mathcal{W}} := (n+1-\hat{\hat{n}})\prod_{j=1}^{n}\hat{\hat{n}}_j = \left(n+1+\frac{m}{2}-\hat{\hat{r}}^2\right)\prod_{j=1}^{n}\left(\hat{\hat{q}}_j^2+\hat{\hat{p}}_j^2-\frac{1}{2}\right)$$

$\hat{\hat{\mathbf{r}}} := \hat{U}\,\hat{\mathbf{r}}\,\hat{U}^\dagger$ $\quad \hat{\hat{q}}_j := \hat{U}\,\hat{q}_j\,\hat{U}^\dagger$ $\quad \hat{\hat{p}}_j := \hat{U}\,\hat{p}_j\,\hat{U}^\dagger$

(Heisenberg representation with respect to $\hat{U}^\dagger$)

Since each $\hat{\hat{r}}_j$ is a linear combination of at most $2m$ $\hat{r}_j$'s

$$\Rightarrow\ \hat{\mathcal{W}} = \left(n+1+\frac{m}{2}-\hat{\hat{r}}^2\right)\prod_{j=1}^{n}\left(\hat{\hat{q}}_j^2+\hat{\hat{p}}_j^2-\frac{1}{2}\right)$$ is a linear combination of

at most $\mathrm{O}\left(m^{2(n+1)}\right)$ $2(n+1)$-body quadrature correlators!

Upsides:

- Only homodyne or heterodyne detection required (no number measurements!).

Downside:

- High-order correlations required.
- Exponential scaling with the number of input photons :-(

… avoidable?

Sample complexity of the estimation

Denote by $\mathcal{N}_{\epsilon,\delta}(\mathcal{W})$ the number of experimental runs needed to obtain F^*_W such that:

(statistical error)

$$\mathbb{P}\left(|F_{\mathcal{W}}(\varrho_{\mathrm{p}}) - F^*_{\mathcal{W}}(\varrho_{\mathrm{p}})| \leq \epsilon\right) \geq 1 - \delta$$

(failure probability)

(number of modes)

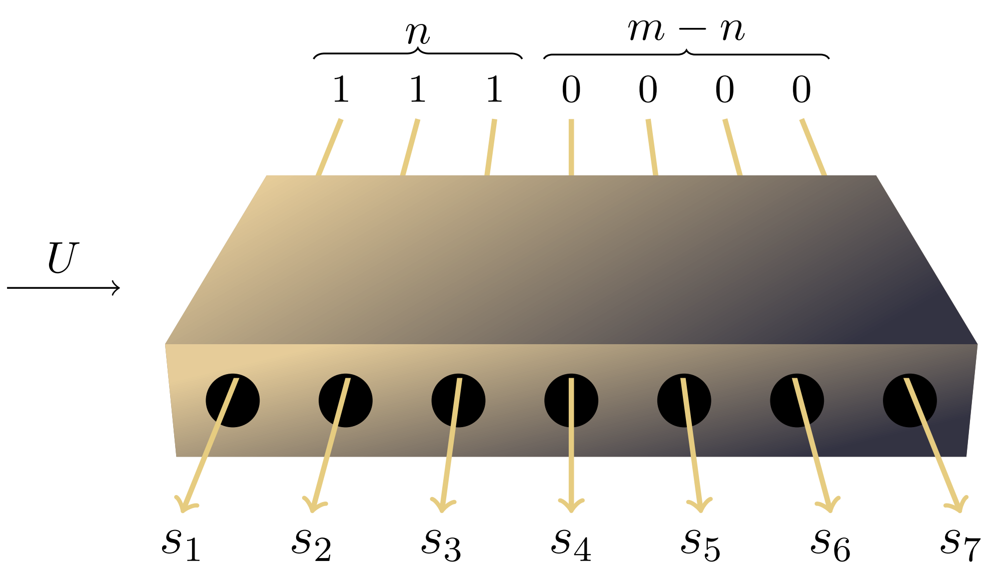
Then, theorem: $\mathcal{N}_{\epsilon,\delta}(\mathcal{W}) \leq O\left(\frac{m^4 (n\, m)^n}{\varepsilon^2 \log(\delta)}\right)$ (number of input photons)

Efficient in the number of modes for every constant n ✓

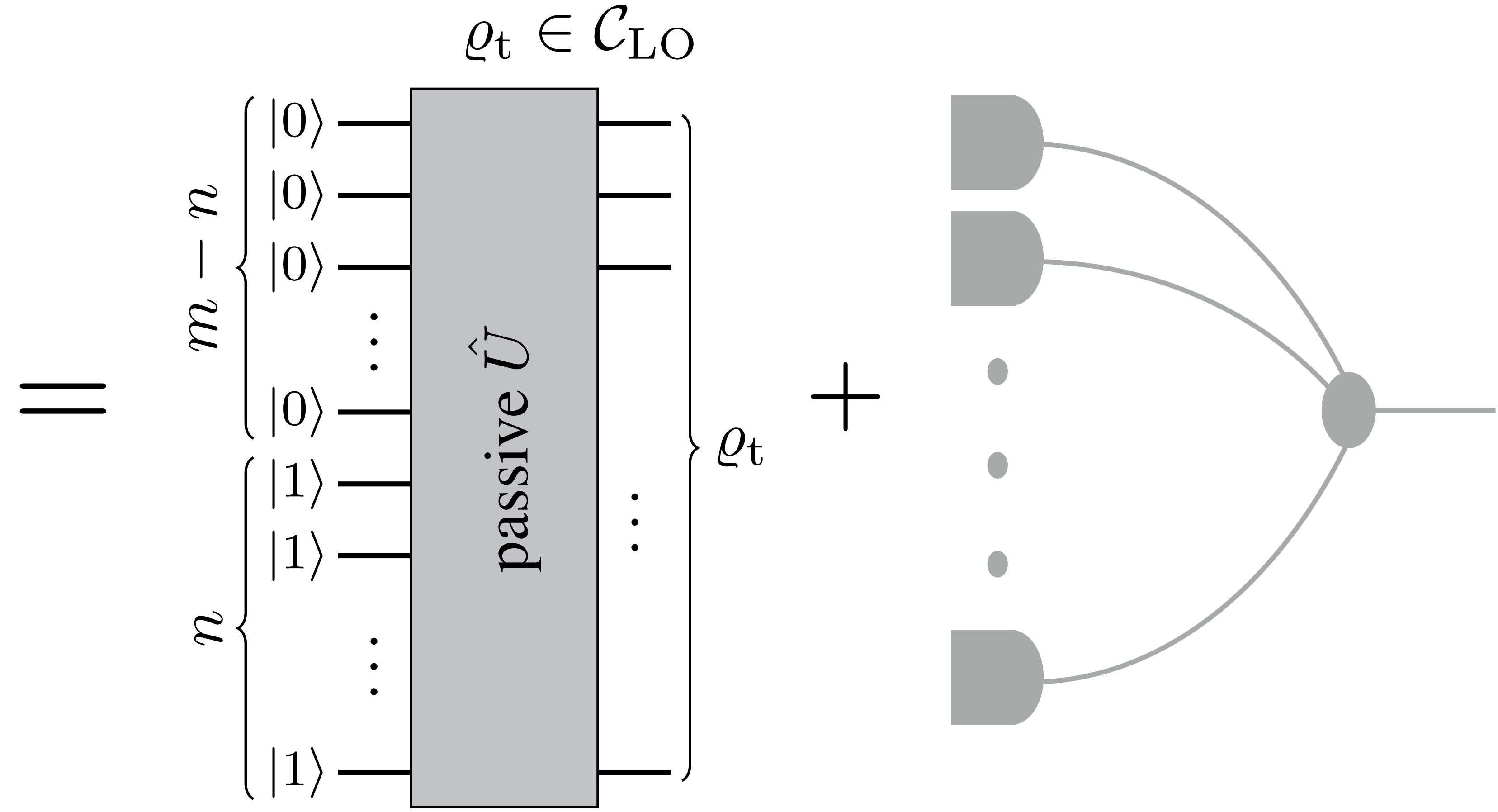
L. Aolita, C. Gogolin, M. Kliesch, and J. Eisert, Nat. Comms. **6**, 8498 (2015).

Certification of Boson-Sampling

Pre-measurement state certification of BS devices



(An *m*-mode *n*-input photon linear-optical network)



If the local measurements are trusted, we can certify BS efficiently for every constant n :-)

Downside:

- Homodyning and heterodyning not compatible with photon number measurement :-(

Conclusions of Lecture IV:

- Quantum state tomography
- In search for the cheapest certification paradigms: direct certification (without tomographic reconstructions)
- Direct (Monte-Carlo) fidelity estimation
- Fidelity witnesses (sacrifice valid experimental preparations for the sake of efficiency).
- Ground-state witnesses: require only local tomography of reductions and cover all MPSs, but are inefficient for states with long-range correlations (no long-time quenches!).
- Bosonic fidelity witness
- Boson-Sampling verification (for constant n!)

Thank you for your attention!