INSTITUT D'ASTROPHYSIQUE DE PARIS

Unité mixte de recherche 7095 CNRS - Université Pierre et Marie Curie

International Center for Theoretical Physics
South American Institute for Fundamental Research

# GRAVITATIONAL WAVES & GENERAL RELATIVITY

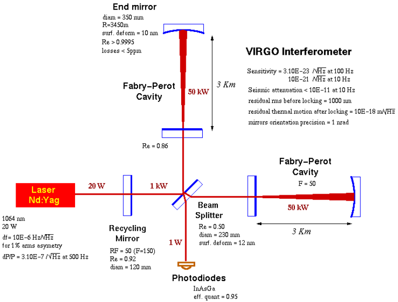
Luc Blanchet

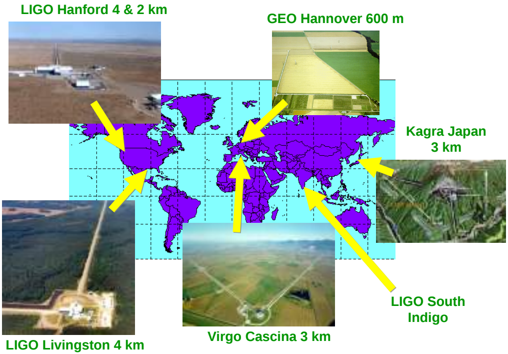
Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

28 novembre 2018

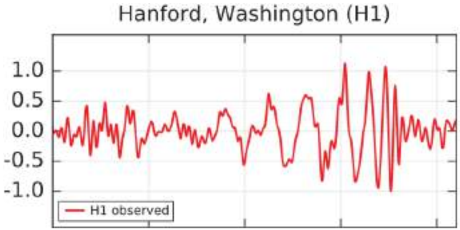
# World-wide network of gravitational wave detectors

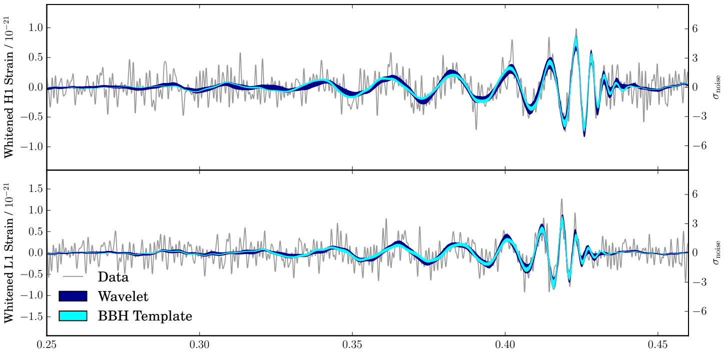
[Rainer Weiss, Barry Barish & Kip Thorne, Nobel prize 2017]



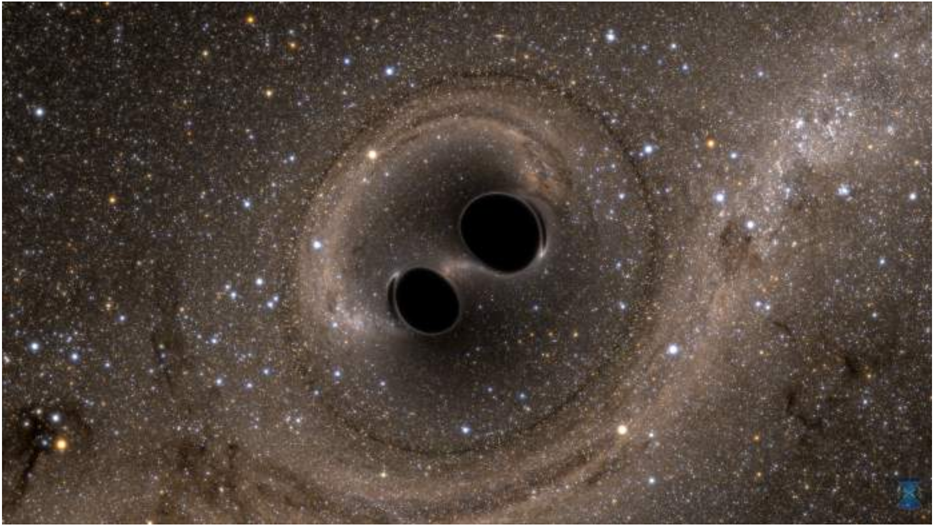



# Binary black-hole event GW150914 [LIGO/Virgo collaboration 2016]

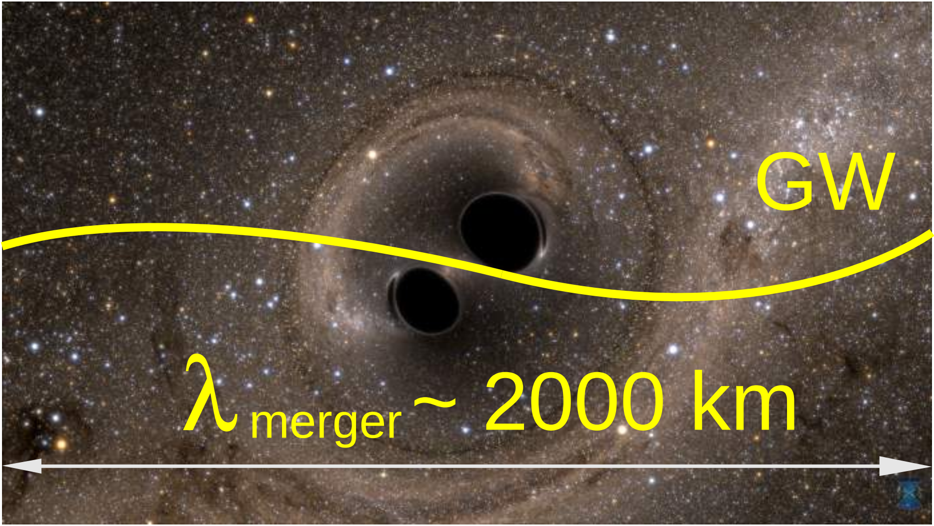


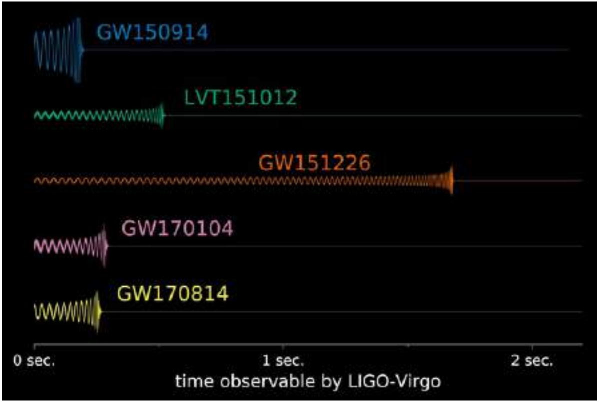
# The Sound of Space-Time



# The Sound of Space-Time

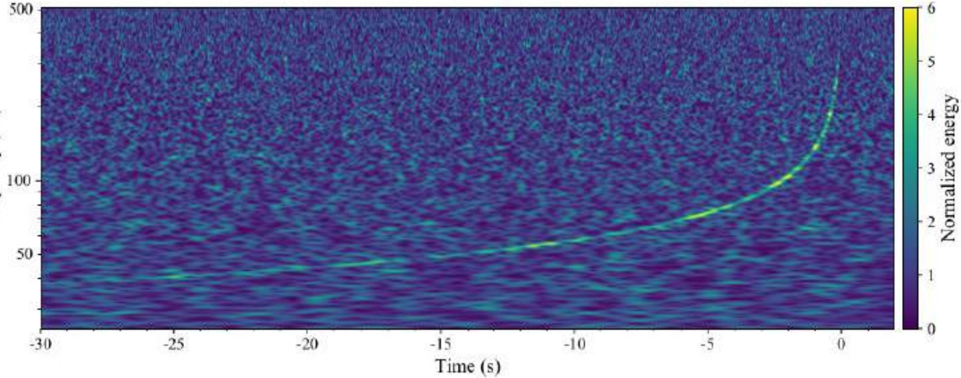


# Gravitational wave events [LIGO/Virgo 2016, 2017]




- For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence
- For NS binaries the detectors will be sensitive to the inspiral phase prior the merger and thousands of cycles are observable

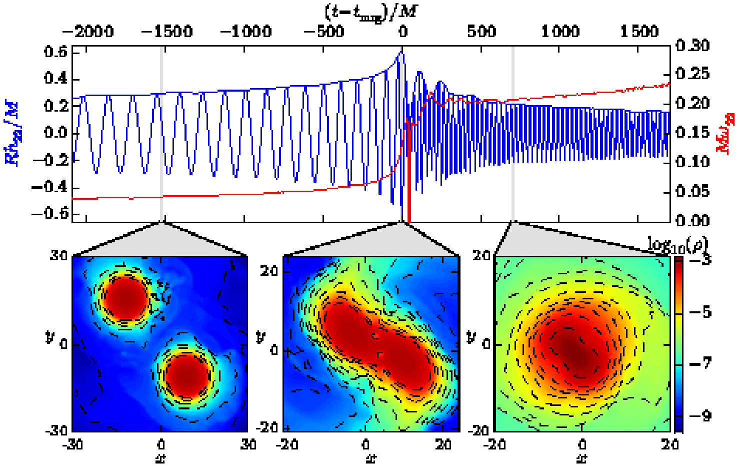
# Binary neutron star event GW170817 [LIGO/Virgo 2017]




- The signal is observed during $\sim 100\,\mathrm{s}$ and $\sim 3000$ cycles and is the loudest gravitational-wave signal yet observed with a combined SNR of 32.4
- The chirp mass is accurately measured to $\mathcal{M} = \mu^{3/5} M^{2/5} = 1.98\,M_\odot$
- The distance is measured from the gravitational signal as $R = 40$ Mpc

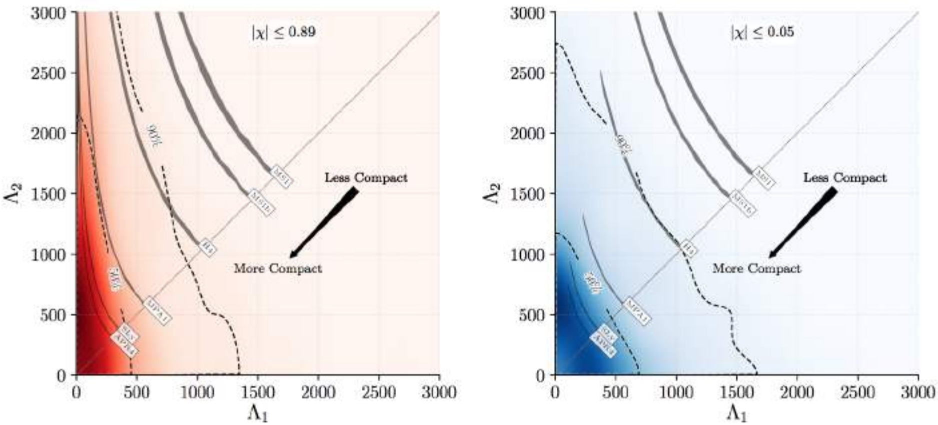
# Post-merger waveform of neutron star binaries

[Shibata *et al.*, Rezzolla *et al.* 1990-2010s]



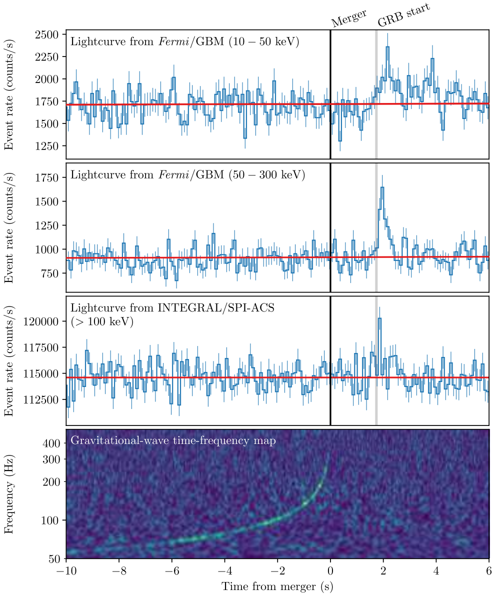
# Constraining the neutron star equation of state

[LIGO/Virgo 2017]



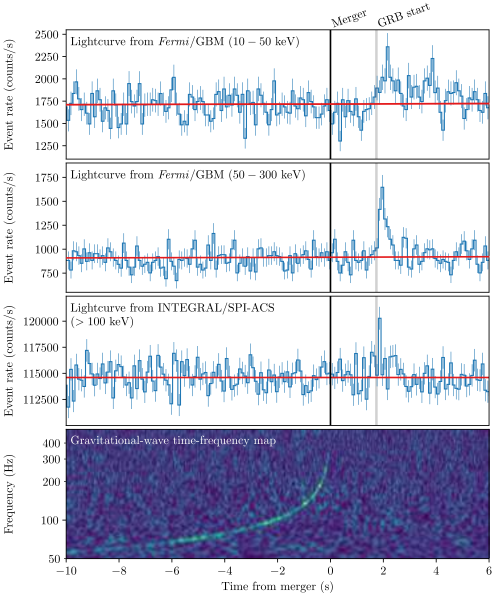

$$\Lambda = \frac{2}{3} k_2 \left( \frac{c^2 a}{G m} \right)^5$$

# The advent of multi-messenger astronomy




- The gamma-ray burst has been detected 1.7 second after the instant of merger
- This is the closest gamma-ray burst whose distance is known and is probably seen off-axis with respect to the relativistic jet

# Speed of gravitational waves versus speed of light




The observed time delay between GW170817 and GRB170817A gives a strong constraint

$$|c_g - c_{em}| \lesssim 10^{-15} c$$

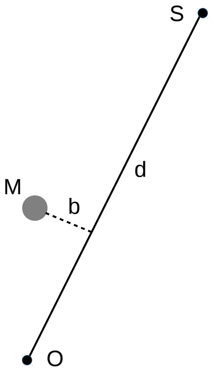
# Test of the strong equivalence principle [Desai & Kahya 2016]

1. Cumulative Shapiro time delay due to the gravitational potential of the dark matter distribution
2. Violation of the EP is quantified by a PPN like parameter $\gamma_a$ with $a = \text{g}, \text{em}$. For a spherical mass distribution

$$\Delta t^a_{\text{Shapiro}} = (1 + \gamma_a) \frac{GM}{c^3} \ln \left( \frac{d}{b} \right)$$

3. Main contributions are from the host galaxy NGC4993 and the Milky Way ($M_{\text{MW}} = 5.6\, 10^{11}\, M_\odot$). Assuming an isothermal density profile for DM the GR delay is $400\,\text{days}$
4. The observed difference in arrival time $\Delta t = 1.7\,\text{s}$ yields

$$\boxed{|\gamma_{\text{g}} - \gamma_{\text{em}}| \lesssim 10^{-7}}$$



# Generalized scalar-tensor theories

1. Traditional scalar-tensor theories [Jordan 1949; Fierz 1956; Brans & Dicke 1961]

$$L\big[\phi, \nabla_\mu \phi\big] = F(\phi) R - Z(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - U(\phi) + \underbrace{L_\mathsf{m}\big[\psi_\mathsf{m}, g_{\mu\nu}\big]}_{\text{universal coupling to } g_{\mu\nu}}$$

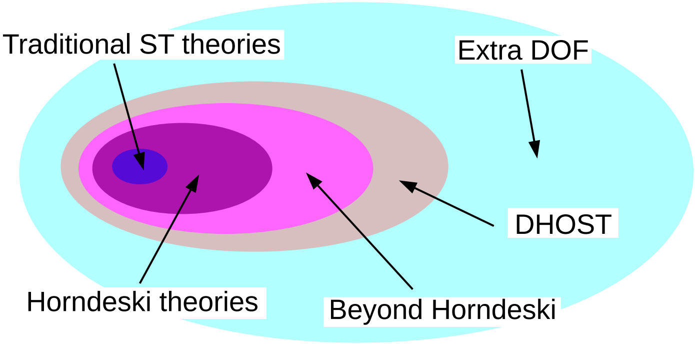
2. Generalized theories with second-order derivatives

$$L\big[\phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi\big]$$

generically contain an extra scalar degree of freedom and lead to instabilities (in particular, the Hamiltonian is not bounded from below) [Ostrogradsky 1850]

3. It is however possible to avoid the instabilities for special choices of second-order Lagrangians

# Generalized scalar-tensor theories




- Horndeski theories [Horndeski 1974]
- Beyond Horndeski [Gleyzes, Langlois, Piazza & Vernizzi 2014]
- Degenerate Higher Order Scalar Tensor (DHOST) [Langlois & Noui 2016]

# Horndeski theories [Horndeski 1974]

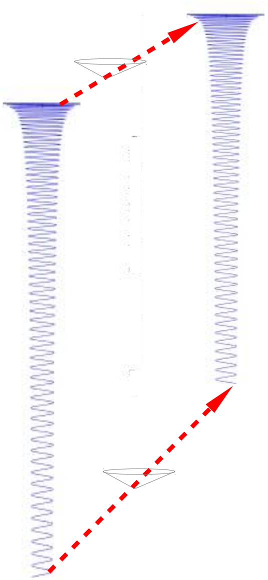
1. Most general theory with at most second-order Euler-Lagrange equations
2. It involves four functions of $\phi$ and the kinetic term $X = \nabla_\mu\phi\nabla^\mu\phi$

$$\begin{aligned}L = &\, G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi, X)R \\ &- 2G_{4,X}(\phi, X)\left(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right) + G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} \\ &+ \frac{1}{3}G_{5,X}(\phi, X)\left(\Box\phi^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi^{\mu\nu}\phi_{\mu\rho}\phi^\rho_\nu\right)\end{aligned}$$

3. Imposing the speed of GWs to be one drastically reduces the space of allowed theories (with $B_4 = G_4 + \frac{X}{2}G_{5,\phi}$)

$$\boxed{L_{c_g=1} = G_2(\phi, X) + G_3(\phi, X)\Box\phi + B_4(\phi)R}$$

# Bounding the mass of the graviton [Will 1998]



- Dispersion relation for a massive graviton

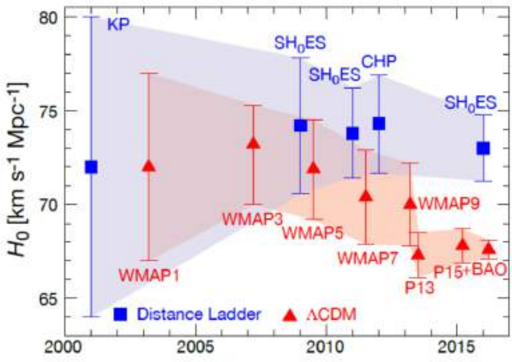
$$\boxed{\frac{v_g^2}{c^2} = 1 - \frac{m_g^2 c^4}{E_g^2}} \quad \text{with} \quad E_g = \hbar\omega_g$$

- The frequency of GW sweeps from low to high frequency during the inspiral and the speed of GW varies from lower to higher (close to c) speed at the end
- The constraint is [LIGO/Virgo 2016]

$$\boxed{m_g \lesssim 10^{-22}\,\text{eV}} \quad \Leftrightarrow \quad \lambda_g \gtrsim 0.02\ \text{ly}$$

# EM measurement of the Hubble-Lemaître constant

[Planck collaboration 2016; $SH_0ES$ 2016]

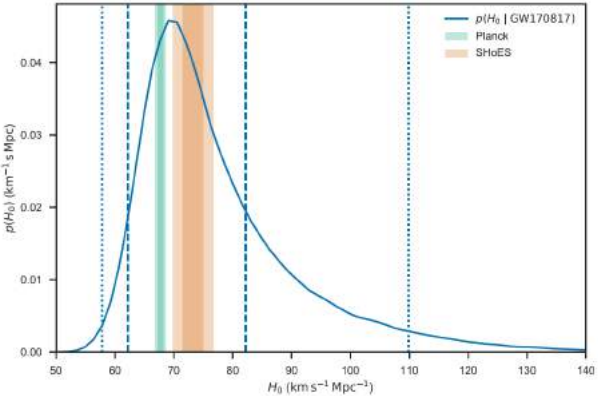



In the concordance model of cosmology $\Lambda$CDM the luminosity distance $D_\mathsf{L}$ is

$$D_\mathsf{L}(z) = \frac{1+z}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_\mathsf{M}(1+z')^3 + \Omega_\mathsf{DE}(1+z')^{3(1+w)}}}$$

# GW measurement of the Hubble-Lemaître constant

[LIGO/Virgo collaboration 2016]




- The distance $D_\mathsf{L} = 40\,\mathsf{Mpc}$ has been measured from GW170817
- The redshift $z$ of the host galaxy NGC4993 has been measured and its peculiar velocity with respect to the Hubble flow substracted

# 100 years of gravitational radiation [Einstein 1916]

348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

## Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

⟸ small perturbation of Minkowski's metric

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet

# 100 years of gravitational radiation [Einstein 1918]

## Einstein's quadrupole formula

mit $4\pi R^2$ multiplizierte $\bar{S}$ endlich ist der Energieverlust pro Zeiteinheit des mechanischen Systems durch Gravitationswellen. Die Rechnung ergibt

[31]

$$4\pi R^2 \bar{S} = \frac{\varkappa}{80\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right]. \qquad (30)$$

Man sieht an diesem Ergebnis, daß ein mechanisches System, welches dauernd Kugelsymmetrie behält, nicht strahlen kann, im Gegensatz zu dem durch einen Rechenfehler entstellten Ergebnis der früheren Abhandlung.

[32]

Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der früheren Abhandlung ist betont geworden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper infolge der thermischen Agitation verlangen würde, Zweifel an der allgemeinen Gültigkeit der Theorie hervorrufen muß. Es scheint, daß eine vervollkommnete Quantentheorie eine Modifikation auch der Gravitationstheorie wird bringen müssen.

[33]

§ 5. Einwirkung von Gravitationswellen auf mechanische Systeme.

Der Vollständigkeit halber wollen wir auch kurz überlegen, inwiefern Energie von Gravitationswellen auf mechanische Systeme übergehen kann. Es liege wieder ein mechanisches System vor von der

Einstein's quadrupole formula

mit $4\pi R^2$ multiplizierte $\bar{S}$ endlich ist der Energieverlust pro Zeiteinheit des mechanischen Systems durch Gravitationswellen. Die Rechnung ergibt

[31]

$$4\pi R^2 \bar{S} = \frac{\varkappa}{80\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right]. \qquad (30)$$

Man sieht an diesem Ergebnis, daß ein mechanisches System, welches dauernd Kugelsymmetrie behält, nicht strahlen kann, im Gegensatz zu dem durch einen Rechenfehler entstellten Ergebnis der früheren Abhandlung.

[32]

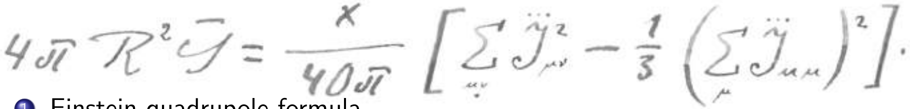
Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der früheren Abhandlung ist betont geworden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper

factor 1/80 should be 1/40 !

§ 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

Der Vollständigkeit halber wollen wir auch kurz überlegen, inwiefern Energie von Gravitationswellen auf mechanische Systeme übergehen kann. Es liege wieder ein mechanisches System vor von der

# Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]



1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathrm{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

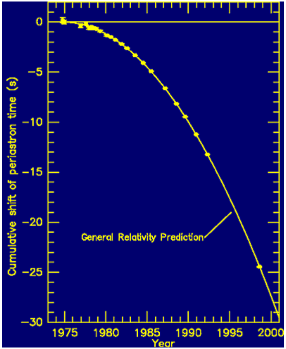
3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\mathrm{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

# The quadrupole formula works for the binary pulsar

[Hulse & Taylor 1974, Taylor & Weisberg 1982]




$$\dot{P} = -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi G M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}$$

[Peters & Mathews 1963, Esposito & Harrison 1975, Wagoner 1975, Damour & Deruelle 1983]

# The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{f} - t)\right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f}\right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6} \left(\frac{100\,\mathsf{Mpc}}{R}\right) \left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $R = 400\,\mathsf{Mpc}$ is measured from the signal itself [Schutz 1986]

# The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi} \left[ \frac{256}{5} \frac{G\mathcal{M}^{5/3}}{c^5} (t_\mathsf{f} - t) \right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[ \frac{5}{96} \frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f} \right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22} \left( \frac{\mathcal{M}}{M_\odot} \right)^{5/6} \left( \frac{100\,\mathsf{Mpc}}{R} \right) \left( \frac{100\,\mathsf{Hz}}{f_\mathsf{merger}} \right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $R = 400\,\mathsf{Mpc}$ is measured from the signal itself [Schutz 1986]

# The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{f} - t)\right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6}\left(\frac{100\,\mathsf{Mpc}}{R}\right)\left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $R = 400\,\mathsf{Mpc}$ is measured from the signal itself [Schutz 1986]

# The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{f} - t)\right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6}\left(\frac{100\,\mathsf{Mpc}}{R}\right)\left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $R = 400\,\mathsf{Mpc}$ is measured from the signal itself [Schutz 1986]

# Total energy radiated away by GW150914

1. The ADM energy of space-time is constant and reads (at any time $t$)

$$E_{\mathsf{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5} \int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

2. Initially $E_{\mathsf{ADM}} = (m_1 + m_2)c^2$ while finally (at time $t_{\mathsf{f}}$)

$$E_{\mathsf{ADM}} = M_{\mathsf{f}} c^2 + \frac{G}{5c^5} \int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

3. The total energy radiated in GW is

$$\Delta E^{\mathsf{GW}} = (m_1 + m_2 - M_{\mathsf{f}})c^2 = \frac{G}{5c^5} \int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}$$

4. The total power released is

$$\mathcal{P}^{\mathsf{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{49}\,\mathsf{W} \sim 10^{-3} \frac{c^5}{G}$$

# Total energy radiated away by GW150914

1. The ADM energy of space-time is constant and reads (at any time $t$)

$$E_{\mathrm{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

2. Initially $E_{\mathrm{ADM}} = (m_1 + m_2)c^2$ while finally (at time $t_{\mathrm{f}}$)

$$E_{\mathrm{ADM}} = M_{\mathrm{f}}c^2 + \frac{G}{5c^5}\int_{-\infty}^{t_{\mathrm{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

3. The total energy radiated in GW is

$$\Delta E^{\mathrm{GW}} = (m_1 + m_2 - M_{\mathrm{f}})c^2 = \frac{G}{5c^5}\int_{-\infty}^{t_{\mathrm{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t') = \frac{G m_1 m_2}{2 r_{\mathrm{f}}}$$

4. The total power released is

$$\mathcal{P}^{\mathrm{GW}} \sim \frac{3 M_{\odot} c^2}{0.2\,\mathrm{s}} \sim 10^{49}\,\mathrm{W} \sim 10^{-3}\,\frac{c^5}{G}$$

# Total energy radiated away by GW150914

1. The ADM energy of space-time is constant and reads (at any time $t$)

$$E_{\mathsf{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

2. Initially $E_{\mathsf{ADM}} = (m_1 + m_2)c^2$ while finally (at time $t_{\mathsf{f}}$)

$$E_{\mathsf{ADM}} = M_{\mathsf{f}} c^2 + \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

3. The total energy radiated in GW is

$$\boxed{\Delta E^{\mathsf{GW}} = (m_1 + m_2 - M_{\mathsf{f}})c^2 = \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}}$$

4. The total power released is

$$\boxed{\mathcal{P}^{\mathsf{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{49}\,\mathsf{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

# Total energy radiated away by GW150914

1. The ADM energy of space-time is constant and reads (at any time $t$)

$$E_{\mathsf{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

2. Initially $E_{\mathsf{ADM}} = (m_1 + m_2)c^2$ while finally (at time $t_{\mathsf{f}}$)

$$E_{\mathsf{ADM}} = M_{\mathsf{f}}c^2 + \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

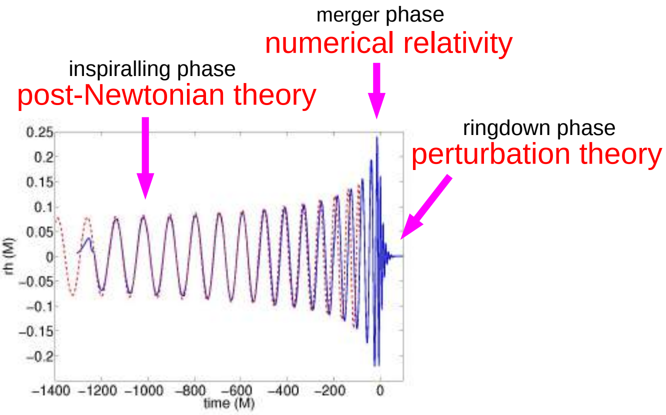
3. The total energy radiated in GW is

$$\boxed{\Delta E^{\mathsf{GW}} = (m_1 + m_2 - M_{\mathsf{f}})c^2 = \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}}$$

4. The total power released is

$$\boxed{\mathcal{P}^{\mathsf{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{49}\,\mathsf{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

# The gravitational chirp of compact binaries



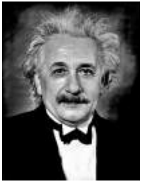

Effective analytical methods interpolate between the PN and NR

- Effective-one-body (EOB) [Buonanno & Damour 1998]
- Hybrid inspiral-merger-ringdown (IMR) [Ajith *et al.* 2011]

# Einstein field equations as a "Problème bien posé"

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term

$$S_{\text{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\text{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}$$



- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$)

$$S_{\text{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\left(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\text{m}}$$

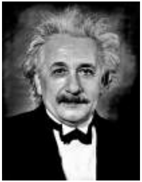
- Get a well-posed system of equations [Hadamard 1932; Choquet-Bruhat 1952]

$$\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$

$$\partial_\mu\mathfrak{g}^{\alpha\mu} = 0$$

# Einstein field equations as a "Problème bien posé"

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term



$$S_{\text{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\text{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}$$

- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$)

$$S_{\text{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\left(\sqrt{-g}\,R \underbrace{- \frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\text{m}}$$

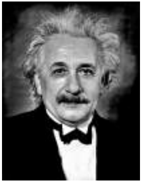
- Get a well-posed system of equations [Hadamard 1932; Choquet-Bruhat 1952]

$$\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$

$$\partial_\mu\mathfrak{g}^{\alpha\mu} = 0$$

# Einstein field equations as a “Problème bien posé”

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term



$$S_{\mathrm{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\mathrm{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}$$

- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$)

$$S_{\mathrm{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\left(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\mathrm{m}}$$

- Get a well-posed system of equations [Hadamard 1932; Choquet-Bruhat 1952]

$$\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$
$$\partial_\mu\mathfrak{g}^{\alpha\mu} = 0$$

# PN approximation scheme in general relativity

1. The dominant radiation reaction effects appears at order $(v/c)^5$ beyond the Newtonian force, where $v$ is the velocity in the source and $c$ the speed of light

$$\boxed{\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \underbrace{\mathbf{F}_{\mathrm{N}}}_{\text{Newtonian acceleration}} + \cdots + \underbrace{\frac{1}{c^5}\mathbf{F}_{\mathrm{RR}}}_{\text{radiation reaction}} + \cdots}$$

At leading order the RR force yields the quadrupole formula for the emission of gravitational radiation

2. During the inspiral phase the dynamics is adiabatic

$$T_{\mathrm{RR}} \gg T_{\mathrm{orbital}}$$

with adiabatic parameter which is small in a post-Newtonian (PN) sense

$$\boxed{\frac{\dot{\omega}}{\omega^2} = \mathcal{O}\left[\left(\frac{v}{c}\right)^5\right]} \qquad \text{(so-called 2.5PN order)}$$

# Orbital phase evolution of compact binaries

1. Obtain the equations of motion of the compact binary system at the $n$PN order beyond Newtonian acceleration. From the conservative part of the equations of motion (deducible from a Lagrangian) we obtain

$$E = \text{binary's center-of-mass energy}$$

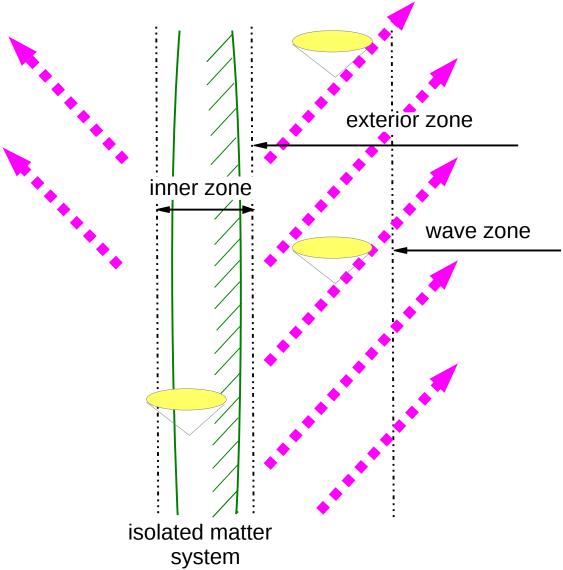
2. Compute the gravitational radiation field (*i.e.* $h^{\mu\nu}$) of the compact binary by means of a GW generation formalism at the same $n$PN order beyond the Einstein quadrupole formula. This yields

$$\mathcal{F} = \text{binary's gravitational wave flux}$$

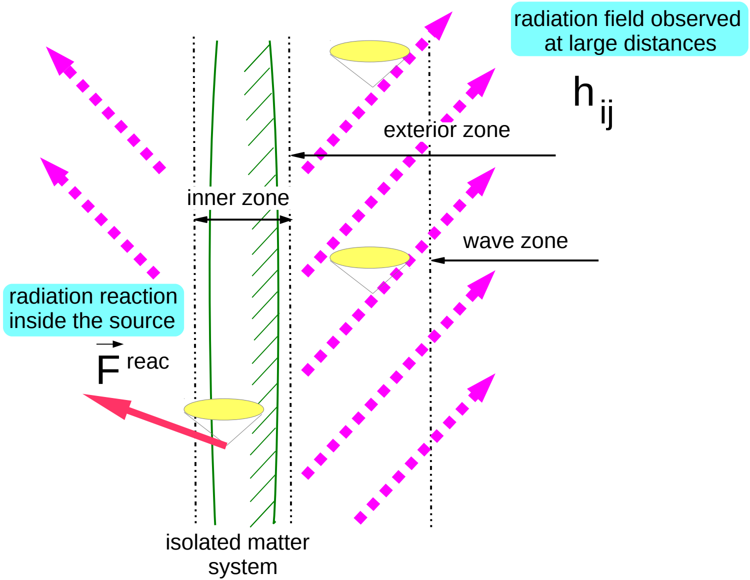
3. The orbital phase $\phi$ (a crucial observable for LIGO/Virgo) follows from the energy balance equation

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \quad \Longrightarrow \quad \phi = \int \omega \, dt = -\int \frac{\omega \, \mathrm{d}E}{\mathcal{F}}}$$

# Isolated matter system in general relativity

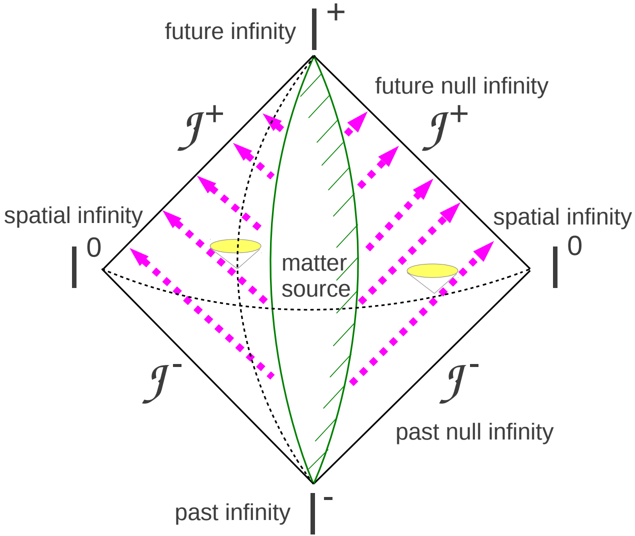


# Isolated matter system in general relativity

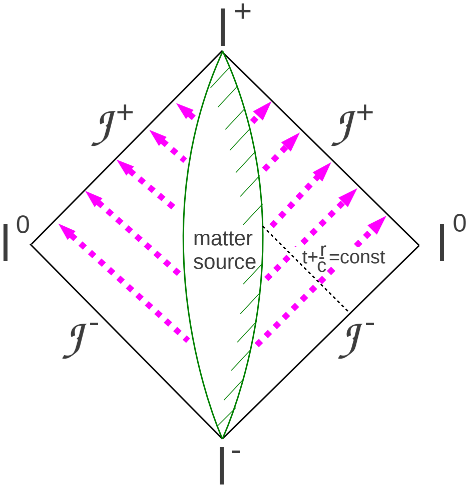


# Asymptotic structure of radiating space-time

[Bondi-Sachs-Penrose formalism 1960s]

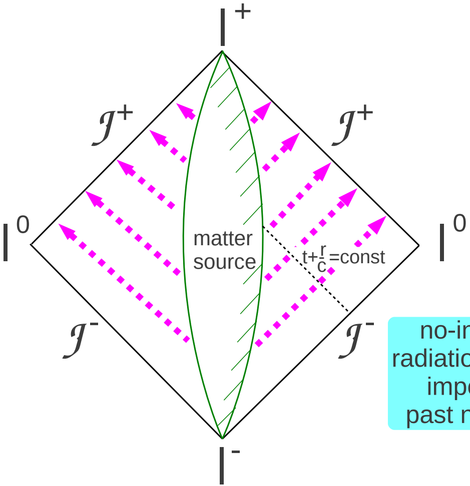



# No-incoming radiation condition




$$\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

# No-incoming radiation condition




no-incoming radiation condition imposed at past null infinity

$$\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

# Linearized multipolar vacuum solution [Pirani 1964; Thorne 1980]

Solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}I_L\right) \qquad \boxed{L = i_1 i_2\cdots i_\ell}$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}\right) + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}J^{(1)}_{j)bL-2}\right)\right\}$$

- multipole moments $I_L(u)$ and $J_L(u)$ are arbitrary functions of $u = t - r/c$
- mass $M \equiv I = \mathrm{const}$, center-of-mass position $G_i \equiv I_i = \mathrm{const}$
  linear momentum $P_i \equiv I^{(1)}_i = 0$, angular momentum $J_i = \mathrm{const}$

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

1. The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\rm MPM} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}$$

where $h^{\alpha\beta}_{(1)}$ is defined from the multipole moments $I_L$ and $J_L$

2. Hierarchy of perturbation equations is solved by induction over $n$

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \ldots, h_{(n-1)}]$$
$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

3. A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

1. The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\mathrm{MPM}} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}$$

   where $h^{\alpha\beta}_{(1)}$ is defined from the multipole moments $I_L$ and $J_L$

2. Hierarchy of perturbation equations is solved by induction over $n$

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \dots, h_{(n-1)}]$$
$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

3. A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

1. The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\rm MPM} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}$$

where $h^{\alpha\beta}_{(1)}$ is defined from the multipole moments $I_L$ and $J_L$

2. Hierarchy of perturbation equations is solved by induction over $n$

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \dots, h_{(n-1)}]$$
$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

3. A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

**Theorem 1:**
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

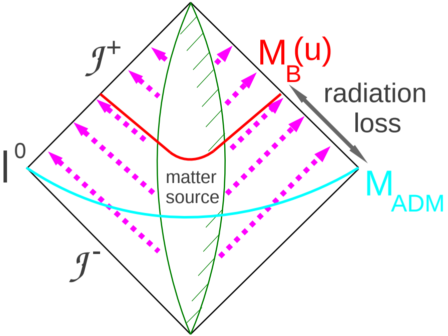
**Theorem 2:**
The general structure of the PN expansion is

$$\boxed{h^{\alpha\beta}_{\mathsf{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)}$$

**Theorem 3:**
The MPM solution is asymptotically flat at future null infinity in the sense of Penrose and agrees with the Bondi-Sachs formalism

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]



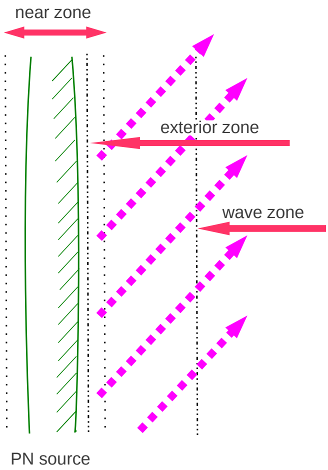

$$M_{\mathrm{B}}(u) = M_{\mathrm{ADM}} - \overbrace{\frac{G}{5c^7}\int_{-\infty}^{u} \mathrm{d}t\, M_{ij}^{(3)}(t) M_{ij}^{(3)}(t)}^{\text{mass-energy emitted in GW}}$$

$$+ \begin{cases} \text{higher-order multipole moments and} \\ \text{higher-order PM approximations} \\ \text{computable to any order by the MPM algorithm} \end{cases}$$

# The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

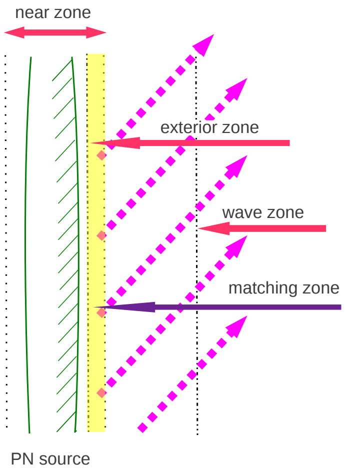
A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



# The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

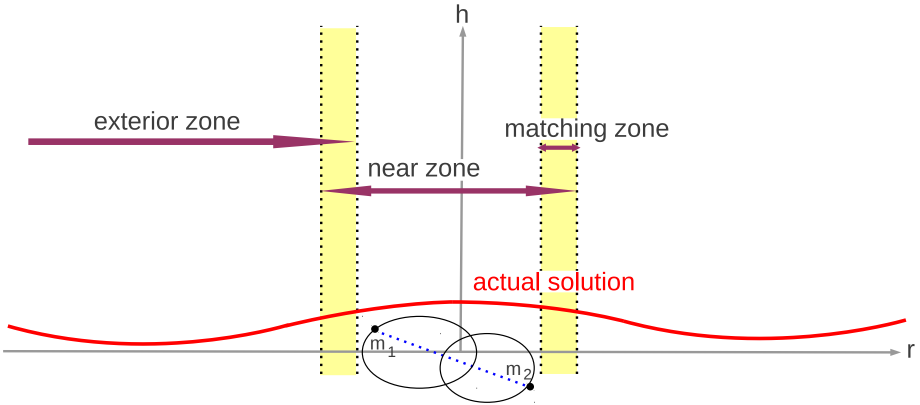
A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone




$$\underbrace{\overline{\mathcal{M}(h^{\mu\nu})} = \mathcal{M}(\bar{h}^{\mu\nu})}_{\text{matching equation}}$$

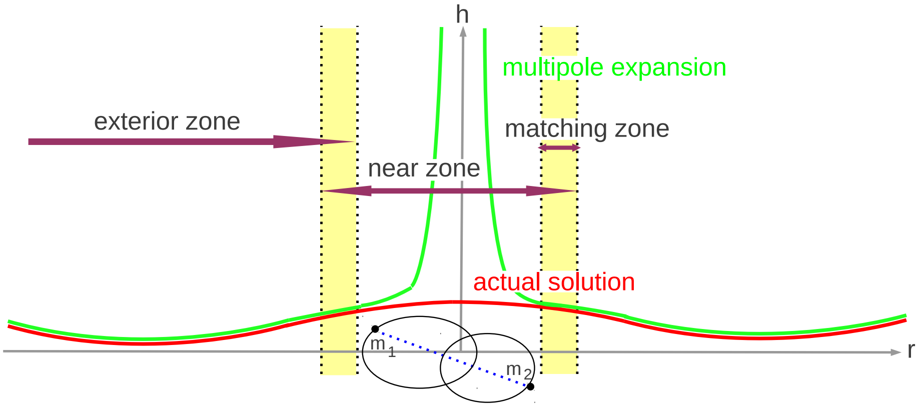
# The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]



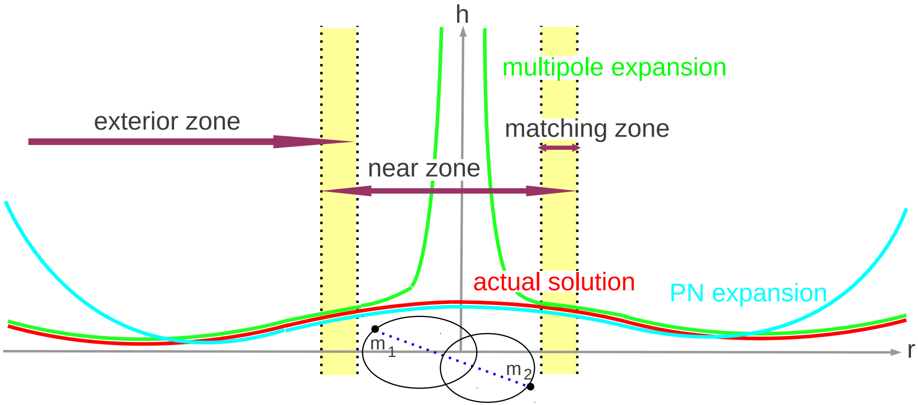
# The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]



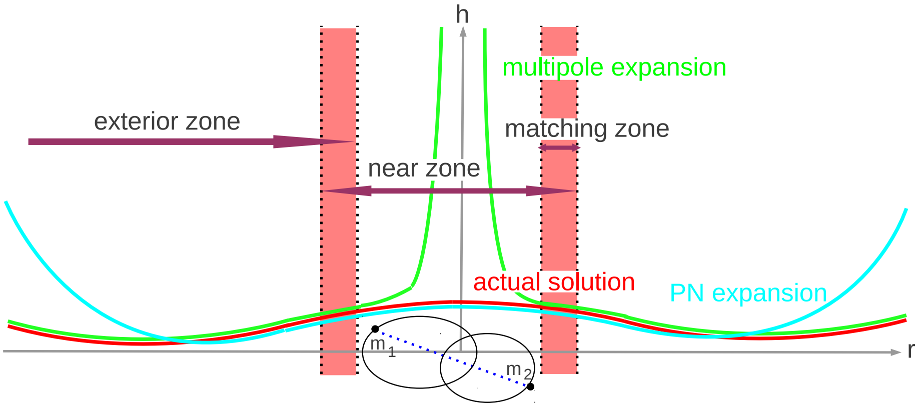
# The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]



# The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]



# Radiative moments at future null infinity

1. Correct for the "tortoise" logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\overbrace{u}^{\text{null coordinate}} \equiv \overbrace{T - \frac{R}{c}}^{\text{radiative coordinates}} = \overbrace{t - \frac{r}{c}}^{\text{harmonic coordinates}} - \overbrace{\frac{2GM}{c^3} \ln\left(\frac{r}{c\tau_0}\right)}^{\text{logarithmic deviation}} + \mathcal{O}\left(\frac{1}{r}\right)$$

2. Asymptotic waveform is parametrized by radiative moments $U_L$ and $V_L$

$$h_{ij}^{\text{TT}} = \frac{1}{R} \sum_{\ell=2}^{\infty} N_{L-2} \underbrace{U_{ijL-2}(u)}_{\text{mass-type}} + \varepsilon_{ab(i} N_{aL-1} \underbrace{V_{j)bL-2}(u)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

3. The radiative moments $U_L$ and $V_L$ are the observables of the radiation field at future null infinity

# Radiative moments at future null infinity

1. Correct for the "tortoise" logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\overbrace{u}^{\text{null coordinate}} \equiv \overbrace{T - \frac{R}{c}}^{\text{radiative coordinates}} = \overbrace{t - \frac{r}{c}}^{\text{harmonic coordinates}} - \overbrace{\frac{2GM}{c^3} \ln\left(\frac{r}{c\tau_0}\right)}^{\text{logarithmic deviation}} + \mathcal{O}\left(\frac{1}{r}\right)$$

2. Asymptotic waveform is parametrized by radiative moments $U_L$ and $V_L$

$$h_{ij}^{\mathsf{TT}} = \frac{1}{R} \sum_{\ell=2}^{\infty} N_{L-2} \underbrace{U_{ijL-2}(u)}_{\text{mass-type}} + \varepsilon_{ab(i} N_{aL-1} \underbrace{V_{j)bL-2}(u)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

3. The radiative moments $U_L$ and $V_L$ are the observables of the radiation field at future null infinity

# Radiative moments at future null infinity

1. Correct for the "tortoise" logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\overbrace{u}^{\text{null coordinate}} \equiv \overbrace{T-\frac{R}{c}}^{\text{radiative coordinates}} = \overbrace{t-\frac{r}{c}}^{\text{harmonic coordinates}} - \overbrace{\frac{2GM}{c^3}\ln\left(\frac{r}{c\tau_0}\right)}^{\text{logarithmic deviation}} + \mathcal{O}\left(\frac{1}{r}\right)$$

2. Asymptotic waveform is parametrized by radiative moments $U_L$ and $V_L$

$$h_{ij}^{\mathrm{TT}} = \frac{1}{R}\sum_{\ell=2}^{\infty} N_{L-2}\underbrace{U_{ijL-2}(u)}_{\text{mass-type}} + \varepsilon_{ab(i}N_{aL-1}\underbrace{V_{j)bL-2}(u)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

3. The radiative moments $U_L$ and $V_L$ are the observables of the radiation field at future null infinity

# The 4.5PN radiative quadrupole moment

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty}\mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty}\mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

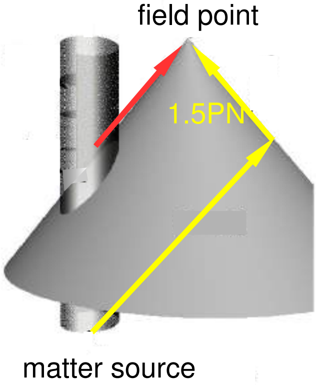
$$+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty}\mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+ \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

# Gravitational wave tails

[Bonnor 1959; Bonnor & Rotenberg 1961; Price 1971; Blanchet & Damour 1988, 1992; Blanchet 1993, 1997]

The tails are produced by backscatter of linear GWs generated by the variations of $I_{ij}$ off the curvature induced by the matter source's total mass $M$




$$\delta h_{ij}^{\text{tail}} = \frac{4G}{c^4 r} \underbrace{\frac{GM}{c^3} \int_{-\infty}^{u} \mathrm{d}t\, I_{ij}^{(4)}(t) \ln\left(\frac{u-t}{\tau_0}\right)}_{\text{The tail is dominantly a 1.5PN effect}} + \cdots$$

# 3.5PN energy flux of compact binaries

$$\mathcal{F}^{\mathrm{GW}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right)}^{\text{1PN}} x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}}$$

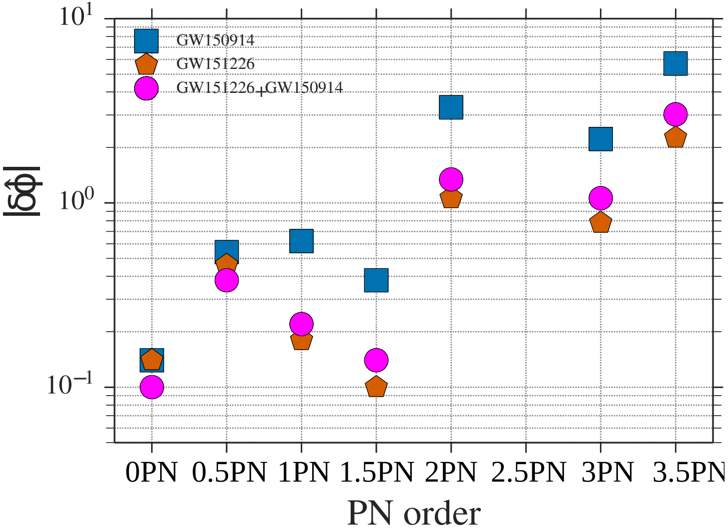
$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}}$$

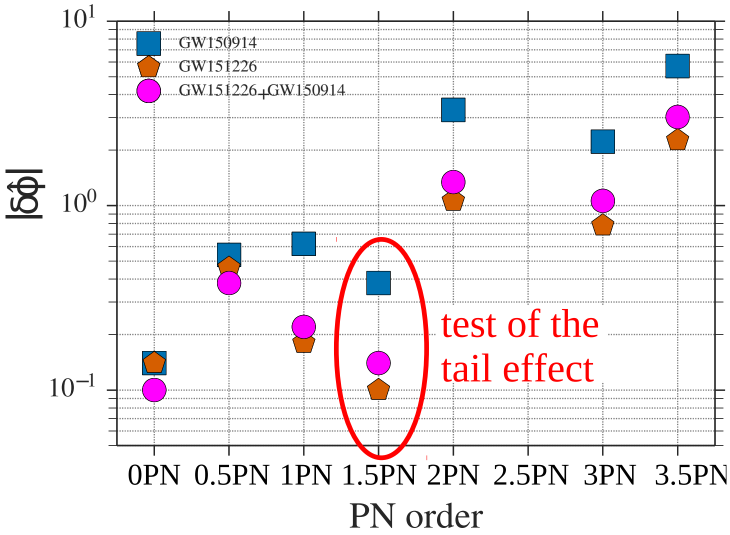
$$+ \Bigg[ \frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}}$$

$$+ \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \Bigg] x^3$$

$$+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}$$

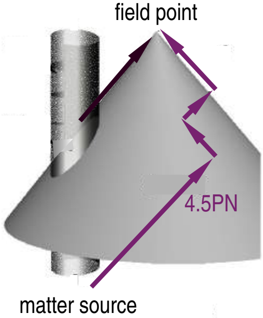
# Measurement of PN parameters [LIGO/Virgo collaboration 2016]




GW150914
GW151226
GW151226+GW150914
$10^1$
$10^0$
$10^{-1}$
$|\delta\hat{\varphi}|$
0PN
0.5PN
1PN
1.5PN
2PN
2.5PN
3PN
3.5PN
PN order
test of the
tail effect

# 4.5PN coefficient in the GW flux [Marchand, Blanchet, Faye 2017]

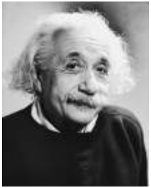
$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{4.5PN}} = \frac{32c^5}{5G}\nu^2 x^5 \left\{ \left( \frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\text{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \right) \pi x^{9/2} \right\}$$




- The 4.5PN tail effect represents the complete 4.5PN coefficient in the GW energy flux in the case of circular orbits
- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- However the 4PN term in the flux is still in progress

# The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]

$$\begin{aligned}\frac{\mathrm{d}^2 \boldsymbol{r}_A}{\mathrm{d}t^2} = &-\sum_{B\neq A} \frac{G m_B}{r_{AB}^2} \boldsymbol{n}_{AB} \Bigg[1 - 4\sum_{C\neq A} \frac{G m_C}{c^2 r_{AC}} - \sum_{D\neq B} \frac{G m_D}{c^2 r_{BD}} \left(1 - \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right) \\ &\qquad + \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg] \\ &+ \sum_{B\neq A} \frac{G m_B}{c^2 r_{AB}^2} \boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B} \frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3} \boldsymbol{n}_{BD}\end{aligned}$$

# 4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

| | | |
|---|---|---|
| 3PN | [Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab] | ADM Hamiltonian |
| | [Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002] | Harmonic EOM |
| | [Itoh & Futamase 2003; Itoh 2004] | Surface integral method |
| | [Foffa & Sturani 2011] | Effective field theory |
| 4PN | [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014] | ADM Hamiltonian |
| | [Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc] | Fokker Lagrangian |
| | [Foffa & Sturani 2012, 2013] (partial results) | Effective field theory |

# The Fokker Lagrangian approach to the 4PN EOM

*Based on collaborations with*





**Laura Bernard**, **Alejandro Bohé**, **Guillaume Faye**,
**Tanguy Marchand** & **Sylvain Marsat**

[PRD **93**, 084037 (2016); **95**, 044026 (2017); **96**, 104043 (2017); **97**, 044023 (2018); PRD **97**, 044037 (2018)]

# Fokker action of $N$ particles [Fokker 1929]

1. Gauge-fixed Einstein-Hilbert action for $N$ point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{Gauge-fixing term}}\Big]$$

$$-\sum_A \underbrace{m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\, v_A^\mu v_A^\nu/c^2}}_{N\text{ point particles}}$$

2. Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

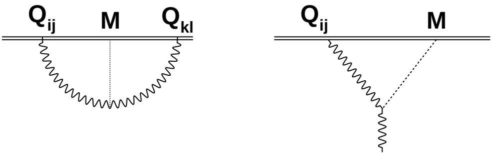
$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{x}_B(t),\boldsymbol{v}_B(t),\cdots)$$

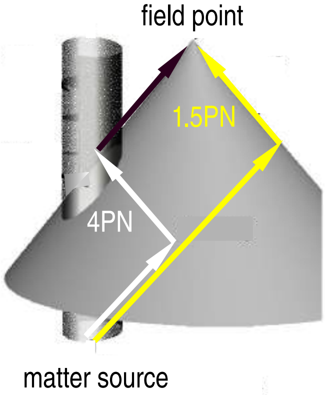
3. The PN equations of motion of the $N$ particles (self-gravitating system) are

$$\boxed{\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{x}_A} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{x}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

# The gravitational wave tail effect

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley, Leibovich, Porto *et al.* 2016]




- In the near zone (4PN effect)

$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t)\, I^{(3)}_{ij}(t')$$

- In the far zone (1.5PN effect)

$$h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{t} \mathrm{d}t'\, I^{(4)}_{ij}(t') \ln\left(\frac{t-t'}{\tau_0}\right)$$

# Problem of the UV divergences

[t'Hooft & Veltman 1972; Bollini & Giambiagi 1972; Breitenlohner & Maison 1977]

1. Einstein's field equations are solved in $d$ spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

2. For two point-particles $\rho = m_1 \delta_{(d)}(\mathbf{x} - \boldsymbol{x}_1) + m_2 \delta_{(d)}(\mathbf{x} - \boldsymbol{x}_2)$ we get

$$\boxed{U(\mathbf{x},t) = \frac{2(d-2)k}{d-1} \left( \frac{G m_1}{|\mathbf{x} - \boldsymbol{x}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x} - \boldsymbol{x}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}}$$

3. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
4. Dimensional regularization is then followed by a renormalization of the worldline of the particles so as to absorb the poles $\propto (d-3)^{-1}$

# Problem of the IR divergences

1. The tail effect implies the appearance of IR divergences in the Fokker action at the 4PN order
2. Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (FP procedure when $B \to 0$)
3. However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
4. The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\text{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\Big(\delta_1 (n_{12} v_{12})^2 + \delta_2 v_{12}^2\Big)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

5. Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

# Conserved energy for a non-local Hamiltonian

1. Because of the tail effect at 4PN order the Lagrangian or Hamiltonian becomes non-local in time

$$H\left[\mathbf{x},\mathbf{p}\right] = H_0\left(\mathbf{x},\mathbf{p}\right) + \underbrace{H_{\text{tail}}\left[\mathbf{x},\mathbf{p}\right]}_{\text{non-local piece at 4PN}}$$

2. Hamilton's equations involve functional derivatives

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\delta H}{\delta p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\delta H}{\delta x^i}$$

3. The conserved energy is not given by the Hamiltonian on-shell but $E = H + \Delta H^{\text{AC}} + \Delta H^{\text{DC}}$ where the AC term averages to zero and

$$\boxed{\Delta H^{\text{DC}} = -\frac{2GM}{c^3}\mathcal{F}^{\text{GW}} = -\frac{2G^2M}{5c^5}\left\langle\left(I_{ij}^{(3)}\right)^2\right\rangle}$$

4. On the other hand [DJS] perform a non-local shift to transform the Hamiltonian into a local one, and both procedure are equivalent

# Conserved energy for circular orbits at 4PN order

- The 4PN energy for circular orbits in the small mass ratio limit is known from GSF of the redshift variable [Le Tiec, Blanchet & Whiting 2012; Bini & Damour 2013]
- This permits to fix the ambiguity parameter $\alpha$ and to complete the 4PN equations of motion

$$
\begin{aligned}
E^{\text{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{ & 1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right) x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right) x^2 \\
& + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right) x^3 \\
& + \Bigg(-\frac{3969}{128} + \left[-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_E + \frac{448}{15}\ln(16x)\right]\nu \\
& \quad + \left[-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right]\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\Bigg) x^4 \Bigg\}
\end{aligned}
$$

# Periastron advance for circular orbits at 4PN order

The periastron advanced (or relativistic precession) constitutes a second invariant which is also known in the limit of circular orbits from GSF calculations

$$\begin{aligned} K^{\mathsf{4PN}} = 1 + 3x &+ \left(\frac{27}{2} - 7\nu\right) x^2 \\ &+ \left(\frac{135}{2} + \left[-\frac{649}{4} + \frac{123}{32}\pi^2\right]\nu + 7\nu^2\right) x^3 \\ &+ \left(\frac{2835}{8} + \left[-\frac{275941}{360} + \frac{48007}{3072}\pi^2 - \frac{1256}{15}\ln x \right.\right. \\ &\left. - \frac{592}{15}\ln 2 - \frac{1458}{5}\ln 3 - \frac{2512}{15}\gamma_\mathsf{E}\right]\nu \\ &\left. + \left[\frac{5861}{12} - \frac{451}{32}\pi^2\right]\nu^2 - \frac{98}{27}\nu^3\right) x^4 \end{aligned}$$

# Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I^{\mathrm{HR}}_{\mathcal{R}} = \mathop{\mathrm{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I^{\mathrm{DR}}_{\mathcal{R}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d-3$)

$$\boxed{\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi^{(\varepsilon)}_{3,q}(\mathbf{n}) + \mathcal{O}(\varepsilon)}$$

# Ambiguity-free completion of the 4PN EOM

[Marchand, Bernard, Blanchet & Faye 2017]

1. The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

$$g_{00}^{\text{tail}} = -\frac{8G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\frac{41}{60}\right]I_{ij}^{(7)}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

2. Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters $\delta_1$ and $\delta_2$
3. It is likely that the EFT formalism will also succeed in deriving the full EOM without ambiguities [Porto & Rothstein 2017]
4. The lack of a consistent matching between the near zone and the far zone in the ADM Hamiltonian formalism [DJS] forces this formalism to be still plagued by one ambiguity parameter