## Perturbation theory

Suppose we have a metric $\tilde{g}_{\alpha\beta}(\epsilon)$ that depends on a parameter $\epsilon$. (In our case, $\epsilon = m/M$.) We want to approximate $\tilde{g}_{\alpha\beta}$ for $\epsilon << 1$. e.g., we can write it as a Taylor series

$$\tilde{g}_{\alpha\beta} = \underbrace{\tilde{g}_{\alpha\beta}(0)}_{\equiv g_{\alpha\beta}} + \underbrace{\epsilon \frac{d}{d\epsilon}\tilde{g}_{\alpha\beta}(0)}_{\equiv h^{(1)}_{\alpha\beta}} + \underbrace{\frac{\epsilon^2}{2}\frac{d^2}{d\epsilon^2}\tilde{g}_{\alpha\beta}(0)}_{\equiv h^{(2)}_{\alpha\beta}} + O(\epsilon^3)$$

For short, let's define $h_{\alpha\beta} \equiv \tilde{g}_{\alpha\beta} - g_{\alpha\beta} = \sum_{n\geq 1} \epsilon^n h^{(n)}_{\alpha\beta}$

So $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$.

The exact metric satisfies $G_{\alpha\beta}[\tilde{g}] = 8\pi T_{\alpha\beta}$. What equations do $h^{(n)}_{\alpha\beta}$ satisfy?

To start, let's rewrite the EFE as an exact equation for $h_{\alpha\beta}$. Note that $G_{\alpha\beta}[\tilde{g}]$ is constructed from $R_{\alpha\beta\gamma\delta}[\tilde{g}]$, which is defined from $\tilde{\nabla}_\gamma\tilde{\nabla}_\beta - \tilde{\nabla}_\beta\tilde{\nabla}_\gamma$ $\Rightarrow$ we'll rewrite $\tilde{\nabla}_\gamma$ in terms of $\nabla_\alpha$.

So, define $C^\alpha{}_{\beta\gamma}$ such that

$$\underset{\text{compatible with } \tilde{g}_{\alpha\beta}}{(\tilde{\nabla}_\beta} - \underset{\text{compatible with } g_{\alpha\beta}}{\nabla_\beta)}\, \omega^\alpha = C^\alpha{}_{\beta\gamma}\,\omega^\gamma \qquad \text{for any } \omega^\gamma$$

We can also write $C^\alpha{}_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma}$

$\tilde{\Gamma}$: Christoffel associated with $\tilde{g}_{\alpha\beta}$; $\Gamma$: Christoffel associated with $g_{\alpha\beta}$

Proof:
$$\begin{aligned} C^\alpha{}_{\beta\gamma}\,\omega^\gamma &= \tilde{\nabla}_\beta\,\omega^\alpha - \nabla_\beta\,\omega^\alpha \\ &= \cancel{\partial_\beta\omega^\alpha} + \tilde{\Gamma}^\alpha_{\beta\gamma}\,\omega^\gamma - \cancel{\partial_\beta\omega^\alpha} - \Gamma^\alpha_{\beta\gamma}\,\omega^\gamma \\ &= (\tilde{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma})\,\omega^\gamma \\ \Rightarrow C^\alpha_{\beta\gamma} &= \tilde{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \end{aligned}$$

We'll use this to show $C^\alpha_{\beta\gamma} = \frac{1}{2}\tilde{g}^{\alpha\mu}(\nabla_\beta h_{\gamma\mu} + \nabla_\gamma h_{\beta\mu} - \nabla_\mu h_{\beta\gamma})$

Proof: in a local inertial frame of $g_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma} = 0$

$$\begin{aligned} \Rightarrow C^\alpha_{\beta\gamma} &= \tilde{\Gamma}^\alpha_{\beta\gamma} \qquad \text{in this frame} \\ &= \tfrac{1}{2}\tilde{g}^{\alpha\mu}(\partial_\beta\tilde{g}_{\gamma\mu} + \partial_\gamma\tilde{g}_{\beta\mu} - \partial_\mu\tilde{g}_{\beta\gamma}) \\ &= \tfrac{1}{2}\tilde{g}^{\alpha\mu}(\nabla_\beta\tilde{g}_{\gamma\mu} + \nabla_\gamma\tilde{g}_{\beta\mu} - \nabla_\mu\tilde{g}_{\beta\gamma}) \qquad \text{since } \Gamma^\alpha_{\beta\gamma} = 0 \\ &= \tfrac{1}{2}\tilde{g}^{\alpha\mu}(\nabla_\beta h_{\gamma\mu} + \nabla_\gamma h_{\beta\mu} - \nabla_\mu h_{\beta\gamma}) \qquad \text{since } \nabla_\mu g_{\alpha\beta} = 0 \end{aligned}$$

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$R^\alpha{}_{\beta\gamma\delta}[\tilde g]$ $\qquad$ $R^\alpha{}_{\beta\gamma\delta}[g]$

Next, we show $\tilde R^\alpha{}_{\beta\gamma\delta} = R^\alpha{}_{\beta\gamma\delta} + 2\nabla_{[\gamma}C^\alpha{}_{\delta]\beta} + 2C^\alpha{}_{\mu[\gamma}C^\mu{}_{\delta]\beta}$ where

<u>Proof</u>: By definition, $\tilde R^\alpha{}_{\beta\gamma\delta}\,\omega^\beta = (\tilde\nabla_\gamma\tilde\nabla_\delta - \tilde\nabla_\delta\tilde\nabla_\gamma)\omega^\alpha$

And $$\tilde\nabla_\gamma\tilde\nabla_\delta\omega^\alpha = \nabla_\gamma(\nabla_\delta\omega^\alpha + C^\alpha{}_{\delta\mu}\omega^\mu) - C^\mu{}_{\gamma\delta}(\nabla_\mu\omega^\alpha + C^\alpha{}_{\mu\nu}\omega^\nu) + C^\alpha{}_{\gamma\mu}(\nabla_\delta\omega^\mu + C^\mu{}_{\delta\nu}\omega^\nu)$$

$$\Rightarrow (\tilde\nabla_\gamma\tilde\nabla_\delta - \tilde\nabla_\delta\tilde\nabla_\gamma)\omega^\alpha = 2\tilde\nabla_{[\gamma}\tilde\nabla_{\delta]}\omega^\alpha$$
$$= 2\nabla_{[\gamma}\nabla_{\delta]}\omega^\alpha + 2\nabla_{[\gamma}(C^\alpha{}_{\delta]\mu}\omega^\mu) - 2\cancelto{0}{C^\mu{}_{[\gamma\delta]}}(\nabla_\mu\omega^\alpha + C^\alpha{}_{\mu\nu}\omega^\nu) + 2C^\alpha{}_{\mu[\gamma}(\nabla_{\delta]}\omega^\mu + C^\mu{}_{\delta]\nu}\omega^\nu)$$
$$= R^\alpha{}_{\beta\gamma\delta}\,\omega^\beta + 2(\nabla_{[\gamma}C^\alpha{}_{\delta]\mu})\omega^\mu + \cancel{2C^\alpha{}_{\mu[\delta}\nabla_{\gamma]}\omega^\mu} + \cancel{2C^\alpha{}_{\mu[\gamma}\nabla_{\delta]}\omega^\mu} + 2C^\alpha{}_{\mu[\gamma}C^\mu{}_{\delta]\nu}\omega^\nu$$

$R_{\beta\delta}[\tilde g]$ $\qquad$ $R_{\beta\delta}[g]$

Next, $\tilde R_{\beta\delta} = \tilde R^\alpha{}_{\beta\alpha\delta} = R_{\beta\delta} + 2\nabla_{[\alpha}C^\alpha{}_{\delta]\beta} + 2C^\alpha{}_{\mu[\alpha}C^\mu{}_{\delta]\beta}$

We can write the EFE in terms of this as $\tilde R_{\alpha\beta} - \frac{1}{2}\tilde g_{\alpha\beta}\tilde g^{\mu\nu}\tilde R_{\mu\nu} = 8\pi T_{\alpha\beta}$.

<u>Note</u>: this isn't quite in terms of $h_{\alpha\beta}$ because $C^\alpha{}_{\beta\gamma}$ involves $\tilde g^{\alpha\beta}$. We can write $\tilde g^{\alpha\beta}$ as an expansion in powers of $h_{\alpha\beta}$:

$$\tilde g^{\alpha\beta} = g^{\alpha\beta} - h^{\alpha\beta} + \mathcal{O}(h^2)$$

<u>Convention</u>: we use $g_{\alpha\beta}$ and $g^{\alpha\beta}$ to raise and lower indices, so $h^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu}$

Let's expand the entire eqn. this way. For a tensor $F[g+h]$ constructed from $g_{\alpha\beta} + h_{\alpha\beta}$, define $\delta^n F[h] = \frac{1}{n!}\frac{d^n}{d\lambda^n}F[g+\lambda h]\big|_{\lambda=0}$, st $F[g+h] = F[g] + \underbrace{\delta F[h]}_{\text{linear in } h} + \underbrace{\delta^2 F[h]}_{\text{quadratic in } h} + \cdots$

e.g., $\tilde g^{\alpha\beta} = g^{\alpha\beta} + \delta g^{\alpha\beta} + \cdots$

where $\delta g^{\alpha\beta} = -h^{\alpha\beta}$

Start with

$$R_{\alpha\beta}[\tilde g] = R_{\alpha\beta}[g] + 2\nabla_{[\mu}C^\mu{}_{\alpha]\beta} + \underbrace{2C^\mu{}_{\nu[\mu}C^\nu{}_{\alpha]\beta}}_{\sim h^2}$$
$$\delta R_{\alpha\beta} = 2\nabla_{[\mu}\delta C^\mu{}_{\alpha]\beta}$$
$$= \frac{1}{2}g^{\mu\nu}(\nabla_\mu\nabla_\alpha h_{\nu\beta} + \nabla_\mu\nabla_\beta h_{\nu\alpha} - \nabla_\mu\nabla_\nu h_{\alpha\beta} - \cancel{\nabla_\alpha\nabla_\mu} h_{\nu\beta} - \nabla_\alpha\nabla_\beta h_{\nu\mu} + \cancel{\nabla_\alpha\nabla_\nu} h_{\mu\beta}$$

$$\Rightarrow \delta R_{\alpha\beta} = -\tfrac{1}{2}\Box h_{\alpha\beta} - \tfrac{1}{2}\nabla_\alpha\nabla_\beta h + \nabla^\mu\nabla_{(\alpha}h_{\beta)\mu}$$ where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$

and $h \equiv g^{\mu\nu}h_{\mu\nu}$

By using the Ricci identity on this, we can rewrite $\delta R_{\alpha\beta}$ as

$$\delta R_{\alpha\beta} = -\tfrac{1}{2}\Box h_{\alpha\beta} - R_\alpha{}^\mu{}_\beta{}^\nu h_{\mu\nu} + R_{(\alpha}{}^\mu h_{\beta)\mu} + \nabla_{(\alpha}\nabla^\mu \bar{h}_{\beta)\mu}$$

where $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}h$ is called the trace-reverse of $h_{\alpha\beta}$

because $g^{\alpha\beta}\bar{h}_{\alpha\beta} = -g^{\alpha\beta}h_{\alpha\beta}$

What we want is $\delta G_{\alpha\beta} = \delta(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}R_{\mu\nu}) = \delta R_{\alpha\beta} - \tfrac{1}{2}h_{\alpha\beta}R - \tfrac{1}{2}g_{\alpha\beta}\,\delta g^{\mu\nu}R_{\mu\nu} - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}\delta R_{\mu\nu}$

Let's specialize to a <u>vacuum background</u>: $R_{\mu\nu}[g] = 0$

Then $\delta G_{\alpha\beta} = \delta R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}\delta R_{\mu\nu}$.

A short calculation yields $\delta G_{\alpha\beta} = -\tfrac{1}{2}\Box\bar{h}_{\alpha\beta} - R_\alpha{}^\mu{}_\beta{}^\nu\bar{h}_{\mu\nu} + \nabla_{(\alpha}\nabla^\mu\bar{h}_{\beta)\mu} - \tfrac{1}{2}g_{\alpha\beta}\nabla_\mu\nabla_\nu\bar{h}^{\mu\nu}$

Getting $\delta^2 G_{\alpha\beta}$ is more laborious, but the idea is the same: just find the quadratic terms in $\tilde{R}_{\alpha\beta}$ to get $\delta^2 R_{\alpha\beta}$, then look at $\delta^2 G_{\alpha\beta} = \delta^2(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}R_{\mu\nu})$

The result is $\delta^2 G_{\alpha\beta} = \delta^2 R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}\delta^2 R_{\mu\nu} - \tfrac{1}{2}h_{\alpha\beta}g^{\mu\nu}\delta R_{\mu\nu} + \tfrac{1}{2}g_{\alpha\beta}h^{\mu\nu}\delta R_{\mu\nu}$

where
$$\delta^2 R_{\alpha\beta} = -\tfrac{1}{2}\nabla_\nu\bar{h}^{\mu\nu}(2\nabla_{(\alpha}h_{\beta)\mu} - \nabla_\mu h_{\alpha\beta}) + \tfrac{1}{4}\nabla_\alpha h^{\mu\nu}\nabla_\beta h_{\mu\nu}$$
$$- \tfrac{1}{2}h^{\mu\nu}(2\nabla_\mu\nabla_{(\alpha}h_{\beta)\nu} - \nabla_\mu\nabla_\nu h_{\alpha\beta} - \nabla_\alpha\nabla_\beta h_{\mu\nu})$$
$$+ \nabla^\nu h^\mu{}_\beta\nabla_{[\nu}h_{\mu]\alpha}$$

The EFE is now $\cancel{G_{\alpha\beta}[g]}^{\,0} + \delta G_{\alpha\beta}[h] + \delta^2 G_{\alpha\beta}[h] + \mathcal{O}(h^3) = 8\pi T_{\alpha\beta}$

Now substitute $h_{\alpha\beta} = \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)$

$$\Rightarrow \epsilon\,\delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2\left(\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}]\right) + \mathcal{O}(\epsilon^3) = 8\pi T_{\alpha\beta}$$

If we suppose that $T_{\alpha\beta} = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)$, then we can equate coefficients of powers of $\epsilon$:

$$\boxed{\begin{aligned}\delta G_{\alpha\beta}[h^{(1)}] &= 8\pi T^{(1)}_{\alpha\beta}\\ \delta G_{\alpha\beta}[h^{(2)}] &= 8\pi T^{(2)}_{\alpha\beta} - \delta^2 G_{\alpha\beta}[h^{(1)}]\end{aligned}}$$

For most of these lectures we'll focus on the first-order equation and just write $h^{(1)}_{\alpha\beta}$ as $h_{\alpha\beta}$

## Gauge freedom

We can alter the form of the field equations using the theory's gauge freedom. The nonperturbative equations are invariant under coordinate transformations; the perturbative equations are invariant under infinitesimal coordinate transformations.

Say $x^\alpha \to x^{\alpha'} = x^\alpha - \epsilon \xi^\alpha + \mathcal{O}(\epsilon^2)$. How does $\tilde{g}_{\alpha\beta}$ transform?

$$\tilde{g}'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \tilde{g}_{\alpha\beta}(x^\gamma)$$

$$= \frac{\partial}{\partial x^{\alpha'}}(x^\mu + \epsilon \xi^\mu) \frac{\partial}{\partial x^{\beta'}}(x^\nu + \epsilon\xi^\nu)\, \tilde{g}_{\mu\nu}(x^{\gamma'} + \epsilon \xi^{\gamma'}) + \mathcal{O}(\epsilon^2)$$

$$= (\delta^\mu_\alpha + \epsilon\, \partial_\alpha \xi^\mu)(\delta^\nu_\beta + \epsilon\, \partial_\beta \xi^\nu)\left[\tilde{g}_{\mu\nu}(x^{\gamma'}) + \epsilon \xi^\gamma \partial_\gamma \tilde{g}_{\mu\nu}(x^{\gamma'})\right] + \mathcal{O}(\epsilon^2)$$

$$= \tilde{g}_{\alpha\beta} + \epsilon\left[\xi^\gamma \partial_\gamma \tilde{g}_{\alpha\beta} + \partial_\alpha \xi^\mu \tilde{g}_{\mu\beta} + \partial_\beta \xi^\mu \tilde{g}_{\alpha\mu}\right] + \mathcal{O}(\epsilon^2)$$

But $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta} + \mathcal{O}(\epsilon^2)$

$$\Rightarrow \tilde{g}'_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta} + \epsilon \underbrace{\left[\xi^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\mu g_{\alpha\mu}\right]} + \mathcal{O}(\epsilon^2)$$

This is $\pounds_\xi g_{\alpha\beta}$. We can also write it as

$$\xi^\gamma \nabla_\gamma g_{\alpha\beta} + \nabla_\alpha \xi^\mu g_{\mu\beta} + \nabla_\beta \xi^\mu g_{\alpha\mu} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$$

So $\tilde{g}'_{\alpha\beta} = g_{\alpha\beta} + \epsilon h'_{\alpha\beta} + \mathcal{O}(\epsilon^2)$

where $h'_{\alpha\beta} = h_{\alpha\beta} + \pounds_\xi g_{\alpha\beta}$
$= h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$

In general, for a tensor $T^{\alpha\ldots}{}_{\beta\ldots} = T_0{}^{\alpha\ldots}{}_{\beta\ldots} + \epsilon T_1{}^{\alpha\ldots}{}_{\beta\ldots} + \mathcal{O}(\epsilon^2)$,
$\Delta T_1{}^{\alpha\ldots}{}_{\beta\ldots} = \pounds_\xi T_0{}^{\alpha\ldots}{}_{\beta\ldots}$
So if $T_0{}^{\alpha\ldots}{}_{\beta\ldots} = 0$, then $T_1{}^{\alpha\ldots}{}_{\beta\ldots}$ is gauge invariant

We can check that $\delta G_{\alpha\beta}[\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu] = 0$ if $G_{\alpha\beta}[g] = 0$

$\Rightarrow$ $h'_{\alpha\beta}$ also satisfies $\delta G_{\alpha\beta}[h'_{\alpha\beta}] = 8\pi T^{(1)}_{\alpha\beta}$.

$\Rightarrow$ if $G_{\alpha\beta}[g]=0$, then $\delta G_{\alpha\beta}[h]$ is gauge-invariant.
$\therefore\ \delta G_{\alpha\beta}[h'] = \delta G_{\alpha\beta}[h]$

We can use this freedom to simplify the equation.

For example, we can impose the Lorenz gauge condition $\nabla_\beta \bar{h}^{\alpha\beta} = 0$.

With this condition we have

$$\delta G_{\alpha\beta}[\bar{h}] = -\tfrac{1}{2}\left(\Box \bar{h}_{\alpha\beta} + 2R_\alpha{}^\mu{}_\beta{}^\nu \bar{h}_{\mu\nu}\right)$$

$$\Rightarrow \boxed{\Box \bar{h}_{\alpha\beta} + 2R_\alpha{}^\mu{}_\beta{}^\nu \bar{h}_{\mu\nu} = -16\pi T^{(1)}_{\alpha\beta}}$$

or $E_{\alpha\beta}[\bar{h}] = -16\pi T^{(1)}_{\alpha\beta}$

How do we know we can impose $\nabla_\beta \bar{h}^{\alpha\beta} = 0$?

Say $h^{old}_{\alpha\beta}$ is in an unspecified gauge, and we want $h^{new}_{\alpha\beta}$ to be in the Lorenz gauge.

Then $\nabla_\beta(\bar{h}^{\alpha\beta}_{new}) = 0$

$$\Leftrightarrow \nabla_\beta\left(h^{\alpha\beta}_{new} - \tfrac{1}{2} g^{\alpha\beta} g_{\mu\nu} h^{\mu\nu}_{new}\right) = 0$$

$$\Leftrightarrow \nabla_\beta h^{\alpha\beta}_{new} - \tfrac{1}{2}\nabla^\alpha (g_{\mu\nu} h^{\mu\nu}_{new}) = 0$$

$$\Leftrightarrow \nabla_\beta h^{\alpha\beta}_{old} + \nabla_\beta\left(\nabla^\alpha \xi^\beta + \nabla^\beta \xi^\alpha\right) - \tfrac{1}{2}\nabla^\alpha h_{old} - \tfrac{1}{2}\nabla^\alpha\left(2\nabla_\mu \xi^\mu\right) = 0$$

$$\Leftrightarrow \nabla_\beta \bar{h}^{\alpha\beta}_{old} + \underbrace{\nabla^\beta \nabla^\alpha \xi_\beta}_{\nabla^\alpha \nabla^\beta \xi_\beta + R_\beta{}^{\gamma\beta\alpha}\xi_\gamma} + \Box \xi^\alpha - \nabla^\alpha \nabla_\mu \xi^\mu = 0$$

$0$

$$\Leftrightarrow \Box \xi^\alpha = -\nabla_\beta \bar{h}^{\alpha\beta}_{old}$$

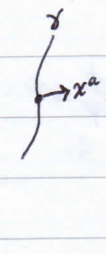
This is a hyperbolic equation that we can always solve $\Rightarrow$ we can always impose the Lorenz gauge condition.

## Point particle approximation

What we've said about perturbation theory so far is very generic. Now let's assume that at leading order, the small object in an EMRI can be approximated as a point mass. That is, $T^{(1)}_{\alpha\beta}$ is a delta function supported on the worldline $\gamma$:

$$T^{(1)}_{\alpha\beta} = m\int_\gamma u_\alpha u_\beta \frac{\delta^4(x^\mu - z^\mu(\tau))}{\sqrt{-g}}\, d\tau$$

determinant of $g_{\alpha\beta}$



e.g. in local coordinates $(\tau, x^a)$ comoving with $\gamma$, $T^{(1)}_{\alpha\beta} = m\int_\gamma \delta^\tau_\alpha \delta^\tau_\beta\, \delta(\tau-\tau')\,\delta^3(x^a)\, d\tau'$

$$= m\,\delta^\tau_\alpha \delta^\tau_\beta\, \delta^3(x^a)$$

We'll return to whether this approximation makes sense. For now we'll assume it does.

$$\cancel{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = r\sin\theta\cos\phi\left(\frac{\partial r}{\partial y}\partial_r + \frac{\partial\theta^A}{\partial y}\right)}$$

$$\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} = \cos\phi\left(\frac{\partial x}{\partial\theta}\partial_x + \frac{\partial y}{\partial\theta}\partial_y + \frac{\partial z}{\partial\theta}\partial_z\right) - \cot\theta\sin\phi\left(\frac{\partial x}{\partial\phi}\partial_x + \frac{\partial y}{\partial\phi}\partial_y\right)$$

$$= \cos\phi\left(\cos\theta\cos\phi\,\partial_x + \cos\theta\sin\phi\,\partial_y - \sin\theta\,\partial_z\right) r$$

$$-r\cot\theta\sin\phi\left(-\sin\theta\sin\phi\,\partial_x + \sin\theta\cos\phi\,\partial_y\right)$$

say we need errors << 1 rad ⇒ second order

$$= r\cos\theta\left(\cos^2\phi + \sin^2\phi\right)\partial_x + \cancel{r\cos\theta\sin\phi\cos\phi\,(1-1)\,\partial_y}$$

$$-r\sin\theta\cos\phi\,\partial_z$$

$$= z\,\partial_x - x\,\partial_z$$

$$\cancel{\partial\phi = x\,\partial_y}$$

$$\partial\phi = x\,\partial_y - y\,\partial_x$$

$$T_r = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{E^2 - V}}$$

$$\sqrt{\frac{p}{p-6-2e\cos v}}$$

$$\Lambda_r = 2\int_{r_{min}}^{r_{max}} \frac{d\lambda}{dr}\,dr$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{V_r}}$$

$$T_r$$

$$\frac{d\phi}{d\psi} = \frac{d\phi}{d\tau}\frac{d\tau}{dr}\frac{dr}{d\psi}$$

$$= \frac{L_z/r^2}{\sqrt{E^2 - V(r(\psi))}}$$

$x = p-6-2e\cos\psi < p$

$$\Rightarrow \cancel{\frac{p}{x}}$$

$$\cancel{\frac{d\psi}{d\lambda} = d}$$

$$\frac{d\psi_r}{d\lambda} =$$

$x < p$

$$\Rightarrow \frac{p}{x} > 1$$

$$\cancel{\frac{d\lambda}{d\psi} = \frac{d\lambda}{d\tau}\frac{dr}{d\psi}\frac{d\tau}{dr}}$$

$$\cancel{=}$$

$$r = \frac{pM}{1+e\cos\psi} \Rightarrow \frac{dr}{d\psi} = \frac{pM}{(1+e\cos\psi)^2}(-e\sin\psi)$$

$$= \frac{-er\sin\psi}{1+e\cos\psi}$$

$$\frac{d\tau}{d\psi} = \frac{d\tau}{dr}\frac{dr}{d\psi}$$

$$= \frac{1}{\sqrt{E^2 - V(r(\psi))}}$$

$$\cancel{\Lambda_\phi = \int_0^{2\pi} \frac{d\lambda}{d\phi}\frac{d\phi}{d}}$$

$$\psi(\phi) = \int \frac{d\psi}{d\phi}\,d\phi$$

$$= \int \frac{\dot\psi}{\dot\phi}\,d\phi$$

$$T_\phi = \int_0^{2\pi} \frac{d\tau}{d\phi}\,d\phi =$$

$$= \int_0^{2\pi} \frac{d\phi}{L_z/r^2}$$

$$\Rightarrow \tau(\lambda) = \int \frac{d\tau}{d\psi}\,d\psi$$

$$\phi(\psi) = \int \frac{d\phi}{d\psi}\,d\psi$$

$$= \int$$

$$= \int \cancel{\frac{d\phi}{d\psi}}\frac{d\tau}{dr}\frac{dr}{d\psi}\,d\psi$$

$$= \int \frac{1}{\sqrt{E^2 - V(r(\psi))}}\,\frac{-er\sin\psi}{1+e\cos\psi}\,d\psi$$