Institut d'Astrophysique de Paris
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The Sound of Space-Time
The Dawn of Gravitational Wave Science

INTRODUCTION TO THE POST-NEWTONIAN EXPANSION OF GR
&
ANALYTIC MODELING OF GRAVITATIONAL WAVES

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Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

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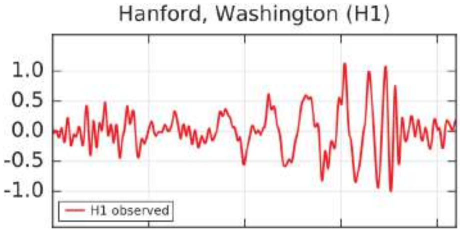
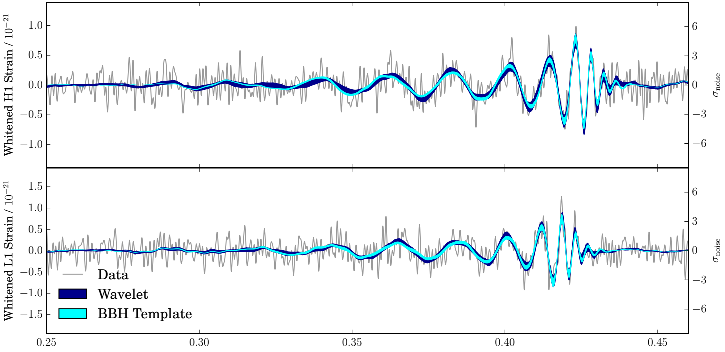
Outline of the lectures

1. Gravitational wave events and gravitational astronomy
2. Methods to compute gravitational wave templates
3. Perturbative methods in general relativity
4. Einstein quadrupole moment formalism
5. Generation of gravitational waves by isolated systems
6. Multipolar post-Minkowskian and matching approach
7. Flux-balance equations for energy, momenta and center of mass
8. Fokker approach to the PN equations of motion
9. Post-Newtonian versus perturbation theory
10. Post-Newtonian versus post-Minkowskian
11. Spin effects in compact binary systems

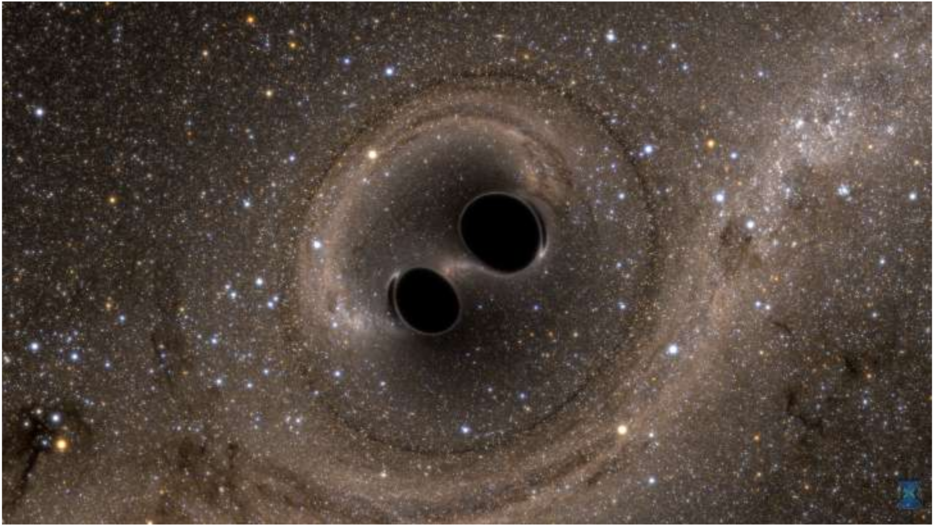
GRAVITATIONAL WAVE EVENTS

Binary black-hole event GW150914 [LIGO/Virgo collaboration 2016]





The Sound of Space-Time



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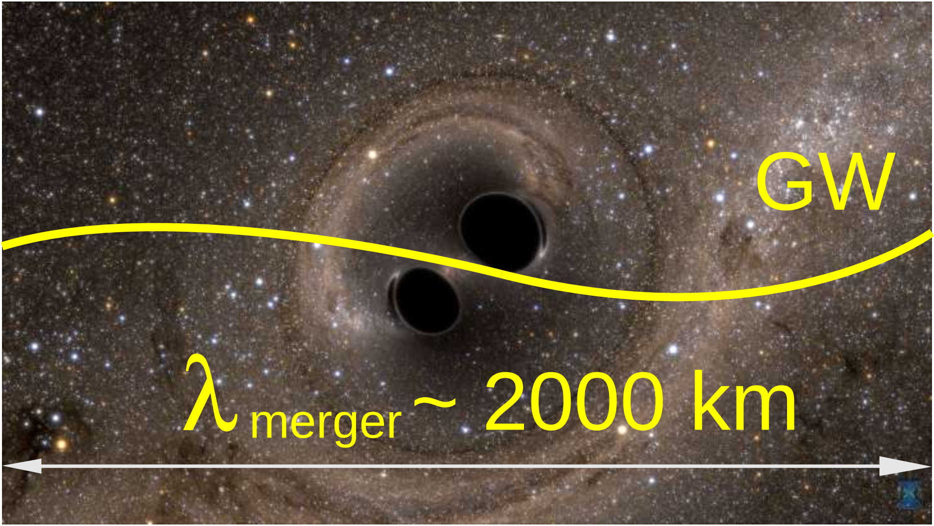
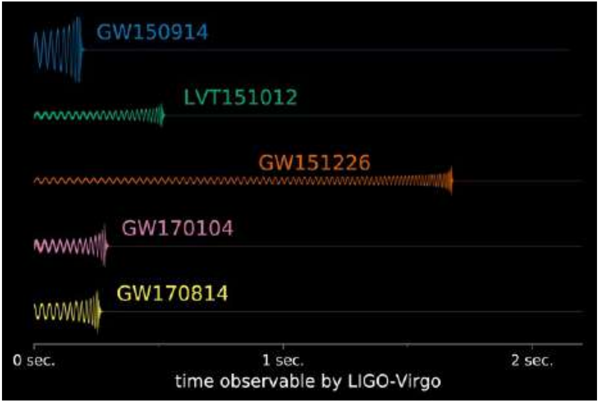
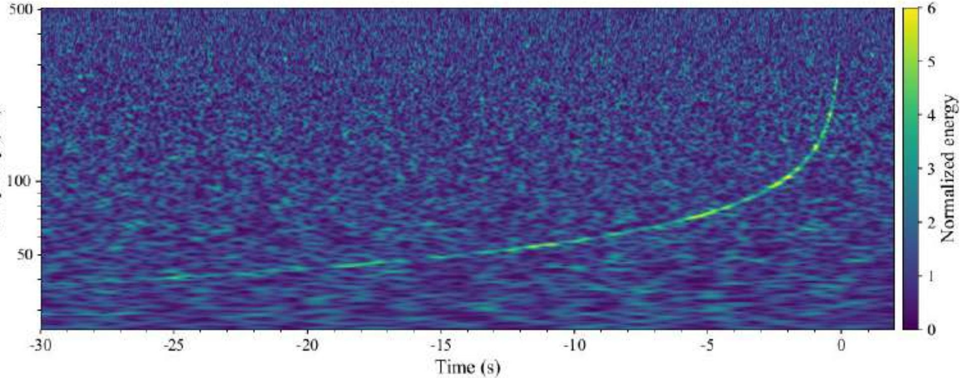


Gravitational wave events [LIGO/Virgo 2016, 2017]



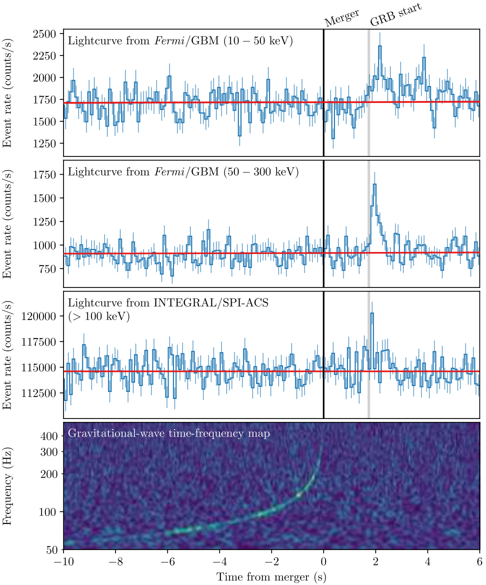
- For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence
- For NS binaries the detectors will be sensitive to the inspiral phase prior the merger and thousands of cycles are observable

Binary neutron star event GW170817 [LIGO/Virgo 2017]



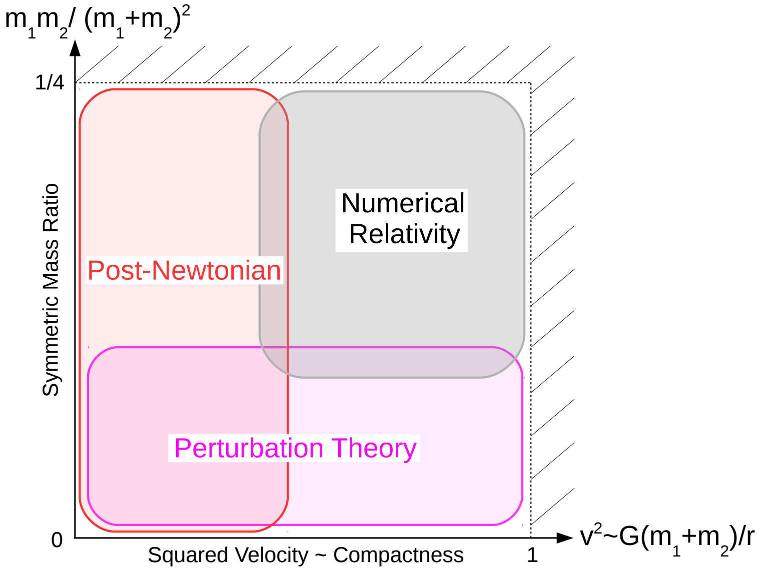
- The signal is observed during $\sim 100\,\mathrm{s}$ and ~ 3000 cycles and is the loudest gravitational-wave signal yet observed with a combined SNR of 32.4
- The chirp mass is accurately measured to $\mathcal{M} = \mu^{3/5} M^{2/5} = 1.98\, M_\odot$
- The distance is measured from the gravitational signal as $R = 40$ Mpc

The advent of multi-messenger astronomy



METHODS TO COMPUTE GW TEMPLATES

Methods to compute GW templates



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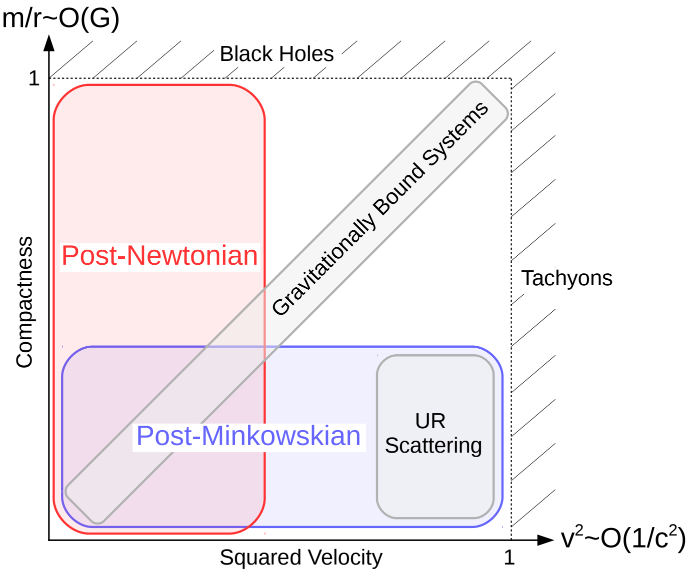
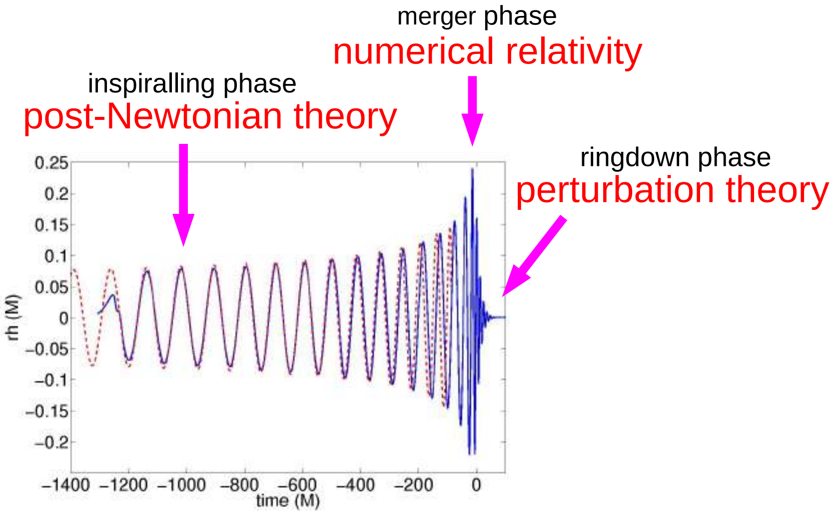


The gravitational chirp of compact binaries



The GW templates of compact binaries

1. In principle, the templates are obtained by matching together:
   - A high-order 3.5PN waveform for the inspiral [Blanchet *et al.* 1998, 2002, 2004]
   - A highly accurate numerical waveform for the merger and ringdown [Pretorius 2005; Baker *et al.* 2006; Campanelli *et al.* 2006]
2. In the practical data analysis, for black hole binaries (such as GW150914), effective methods that interpolate between the PN and NR play a key role:
   - Hybrid inspiral-merger-ringdown (IMR) waveforms [Ajith *et al.* 2011] are constructed by matching the PN and NR waveforms in a time interval through an intermediate phenomenological phase
   - Effective-one-body (EOB) waveforms [Buonanno & Damour 1998] are based on resummation techniques extending the domain of validity of the PN approximation beyond the inspiral phase
3. In the case of neutron star binaries (such as GW170817), the masses are smaller and the templates are entirely based on the 3.5PN waveform

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Methods to compute PN equations of motion

1. ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
2. EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
3. Extended fluid balls [Grishchuk & Kopeikin 1986]
4. Surface-integral approach [Itoh, Futamase & Asada 2000]
5. Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]

- EOM derived in a general frame for arbitrary orbits
- Dimensional regularization is applied for UV divergences[1]
- Radiation-reaction dissipative effects added separately by matching
- Spin effects can be computed within a pole-dipole approximation
- Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

[1] Except in the surface-integral approach

Methods to compute PN radiation field

1. Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, . . . , 1998]
2. Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, . . .]
3. Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]

- Involves a machinery of tails and related non-linear effects
- Uses dimensional regularization to treat point-particle singularities
- Phase evolution relies on balance equations valid in adiabatic approximation
- Spin effects are incorporated within a pole-dipole approximation
- Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

PERTURBATIVE METHODS IN GENERAL RELATIVITY

General problem of linear perturbations

1. Suppose we know a solution $\overline{g}(x)$ of the second-order PDE

$$E\big[\overline{g}(x)\big] = 0$$

2. Assume a one-parameter family of solutions $g(x,\lambda)$ with $g(x,0) = \overline{g}(x)$

$$E\big[g(x,\lambda)\big] = 0$$

3. Defining $h(x) \equiv (\partial g/\partial\lambda)(x,0)$ we obtain the linear second-order PDE

$$\boxed{h\frac{\partial E}{\partial g}\big[\overline{g}\big] + \partial h\frac{\partial E}{\partial(\partial g)}\big[\overline{g}\big] + \partial^2 h\frac{\partial E}{\partial(\partial^2 g)}\big[\overline{g}\big] = 0}$$

4. A good approximation to the exact solution $g(x,\lambda)$ for non-zero but small λ is

$$g_{\text{lin}}(x) = \overline{g}(x) + \lambda\, h(x)$$

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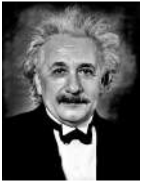
Reliability of the perturbative equations

- To any one-parameter family of solutions $g(x,\lambda)$ corresponds a solution $h(x)$ of the linear perturbative equations
- But the converse is not necessarily true, *i.e.* given a solution $h(x)$ there does not necessarily exist an exact solution such that $h(x) = (\partial g/\partial\lambda)(x,0)$
- More generally, an infinite set of solutions $h_n(x)$ (with $n \in \mathbb{N}$) of the perturbation equations to all non-linear orders n does not necessarily come from the Taylor expansion of some exact solution $g(x,\lambda)$ when $\lambda \to 0$

Knowing if it does is the problem of the reliability of the perturbation equations

Einstein field equations as a "Problème bien posé"

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term



$$S_{\text{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\text{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}$$

- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$)

$$S_{\text{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\left(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\text{m}}$$

- Get a well-posed system of equations [Hadamard 1932; Choquet-Bruhat 1952]

$$\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$

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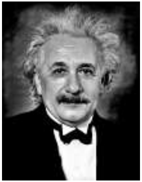

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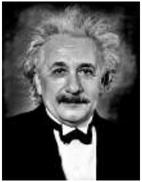

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$$\partial_\mu\mathfrak{g}^{\alpha\mu} = 0$$

Perturbation around Minkowski space-time

Assume space-time slightly differs from Minkowski space-time $\eta_{\alpha\beta}$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} \qquad \text{with} \quad |h| \ll 1$$

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Lambda^{\alpha\beta}[h, \partial h, \partial^2 h]}^{\text{non-linear source term}} \equiv \frac{16\pi G}{c^4} \underbrace{\tau^{\alpha\beta}}_{\text{stress-energy pseudo-tensor}}$$

$$\underbrace{\partial_\mu h^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat d'Alembertian operator

The post-Minkowskian approximation

[Bertotti 1956; Bertotti & Plebanski 1960; Westpfahl *et al.* 1980, 1985; Bel *et al.* 1981; *etc.*]

Appropriate for weakly self-gravitating isolated matter sources

$$\varepsilon_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \qquad \begin{cases} M & \text{mass of source} \\ a & \text{size of source} \end{cases}$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}}_{G \text{ labels the PM expansion}}$$

$$\boxed{\begin{aligned} &\Box h^{\alpha\beta}_{(n)} = \frac{16\pi G}{c^4} |g| T^{\alpha\beta}_{(n)} + \overbrace{\Lambda^{\alpha\beta}_{(n)}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}} \\ &\partial_\mu h^{\alpha\mu}_{(n)} = 0 \end{aligned}}$$

Post-Newtonian expansion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1932; Fock 1959; Chandrasekhar 1965; *etc.*]

Valid for isolated matter sources that are at once slowly moving, weakly stressed and weakly gravitating (so-called post-Newtonian source) in the sense that

$$\boxed{\varepsilon_{\text{PN}} \equiv \max\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \left|\frac{U}{c^2}\right|^{1/2}\right\} \ll 1}$$

- ε_{PN} plays the role of a slow motion estimate $\varepsilon_{\text{PN}} \sim v/c \ll 1$
- For self-gravitating sources the internal motion is due to gravitational forces (*e.g.* a Newtonian binary system) hence $v^2 \sim GM/a$
- Gravitational wavelength $\lambda \sim cP$ where $P \sim a/v$ is the period of motion

$$\boxed{\frac{a}{\lambda} \sim \frac{v}{c} \sim \varepsilon_{\text{PN}}}$$

Post-Newtonian expansion

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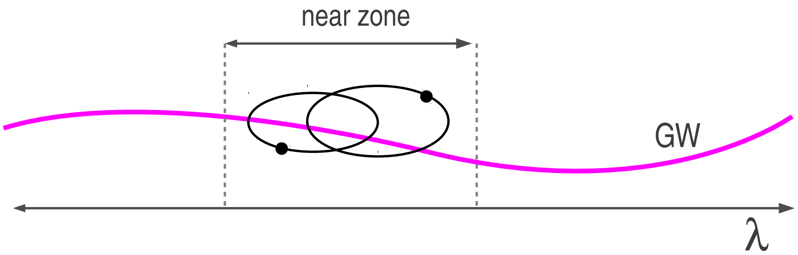

- Near zone defined by $r \ll \lambda$ covers entirely the post-Newtonian source
- General PN expansion inside the source's near zone

$$\boxed{h^{\alpha\beta}_{\text{PN}}(\mathbf{x}, t, c) = \sum_{p\geqslant 2} \frac{1}{c^p} h^{\alpha\beta}_p(\mathbf{x}, t, \ln c)}$$

Multipolar expansion

[*e.g.* Pirani 1964; Geroch 1970; Hansen 1974; Thorne 1980; Simon & Beig 1983; Blanchet 1998]

Valid in the exterior of any possibly strong field isolated source

$$\frac{a}{r} < 1 \qquad \begin{cases} a \text{ size of source} \\ r \text{ distance to source} \\ \lambda \sim cP \text{ wavelength of radiation} \end{cases}$$

$$\underbrace{I_L \sim M a^\ell}_{\text{mass-type multipole moment}} \qquad \underbrace{J_L \sim M a^\ell v}_{\text{current-type multipole moment}} \qquad (L = i_1 \cdots i_\ell)$$

Split space-time into near zone $r \ll \lambda$ and wave zone $r \gg \lambda$

$$\underbrace{h_{\text{NZ}} \sim \frac{G}{c^2} \sum_\ell \left[\frac{I_L}{r^{\ell+1}} + \frac{J_L}{c r^{\ell+1}} \right]}_{r \ll \lambda} \qquad \underbrace{h_{\text{WZ}} \sim \frac{G}{c^2 r} \sum_\ell \left[\frac{I_L^{(\ell)}}{c^\ell} + \frac{J_L^{(\ell)}}{c^{\ell+1}} \right]}_{r \gg \lambda}$$

Multipolar expansion

[*e.g.* Pirani 1964; Geroch 1970; Hansen 1974; Thorne 1980; Simon & Beig 1983; Blanchet 1998]

- The radiative multipolar field in the wave zone

$$h_{\mathrm{WZ}} \sim \frac{G}{c^2 r} \sum_{\ell} \left[\frac{I_L^{(\ell)}}{c^\ell} + \frac{J_L^{(\ell)}}{c^{\ell+1}} \right]$$

is actually a PN expansion in the case of a PN source

$$\boxed{\frac{I_L^{(\ell)}}{c^\ell} \sim \frac{M a^\ell}{\lambda^\ell} \sim M\, \varepsilon_{\mathrm{PN}}^\ell}$$

- The quadrupole moment formalism gives the lowest order PN contribution to the radiation field due to the mass type quadrupole moment ($\ell = 2$)

$$\begin{aligned} I_{ij} &= Q_{ij} + \mathcal{O}(\varepsilon_{\mathrm{PN}}^2) \\ Q_{ij}(t) &= \int_{\text{PN source}} \mathrm{d}^3\mathbf{x}\ \underbrace{\rho_{\mathrm{N}}(\mathbf{x},t)}_{\substack{\text{Newtonian}\\ \text{mass density}}} \left(x_i x_j - \frac{1}{3}\delta_{ij} r^2 \right) \end{aligned}$$

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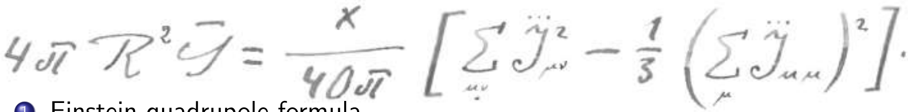
EINSTEIN QUADRUPOLE MOMENT FORMALISM

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]



1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

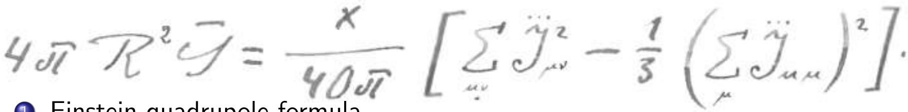
$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]



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3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F^{\mathrm{reac}}_i = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4R}\left\{\frac{\mathrm{d}^2Q_{ij}}{\mathrm{d}t^2}\left(t-\frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

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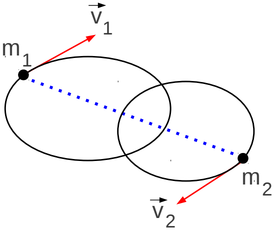
Application to compact binaries [Peters & Mathews 1963; Peters 1964]



$$\begin{cases} a & \text{semi-major axis of relative orbit} \\ e & \text{eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} & \text{orbital frequency} \end{cases}$$

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \qquad 0 < \nu \leqslant \frac{1}{4}}$$

Averaged energy and angular momentum balance equations

$$\langle \frac{\mathrm{d}E}{\mathrm{d}t} \rangle = -\langle \mathcal{F}^{\mathrm{GW}} \rangle \qquad \langle \frac{\mathrm{d}J_i}{\mathrm{d}t} \rangle = -\langle \mathcal{G}_i^{\mathrm{GW}} \rangle$$

are applied to a Keplerian orbit (using Kepler's law $GM = \omega^2 a^3$)

$$\boxed{\begin{aligned} \langle \frac{\mathrm{d}P}{\mathrm{d}t} \rangle &= -\frac{192\pi}{5c^5}\nu \left(\frac{2\pi GM}{P}\right)^{5/3} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}} \\ \langle \frac{\mathrm{d}e}{\mathrm{d}t} \rangle &= -\frac{608\pi}{15c^5}\nu \frac{e}{P} \left(\frac{2\pi GM}{P}\right)^{5/3} \frac{1 + \frac{121}{304}e^2}{(1-e^2)^{5/2}} \end{aligned}}$$

Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. Compact binaries are circularized when they enter the detector's bandwidth

$$E = -\frac{Mc^2}{2}\nu\, x \qquad \mathcal{F}^{\text{GW}} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5$$

where $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ denotes a small PN parameter defined with ω

2. Equating $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\text{GW}}$ gives a differential equation for x

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{64}{5}\frac{c^3\nu}{GM}x^5 \quad \Longleftrightarrow \quad \frac{\dot{\omega}}{\omega^2} = \frac{96\nu}{5}\left(\frac{GM\omega}{c^3}\right)^{5/3}$$

3. This permits to solve for the orbital phase

$$\phi = \int \omega\,\mathrm{d}t = \int \frac{\omega}{\dot{\omega}}\,\mathrm{d}\omega$$

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Orbital phase evolution of compact binaries

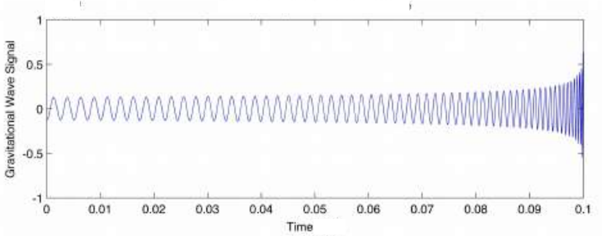
[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. The amplitude and phase evolution follow an adiabatic chirp in time

$$a(t) = \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4}$$

$$\phi(t) = \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8}$$

2. The amplitude and orbital frequency diverge at the instant of coalescence t_c since the approximation breaks down



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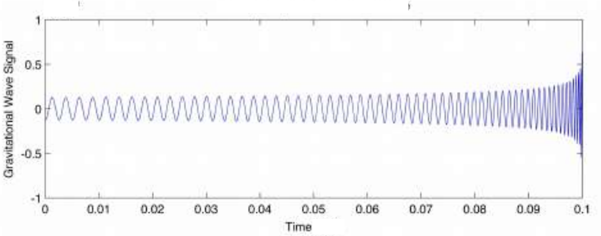
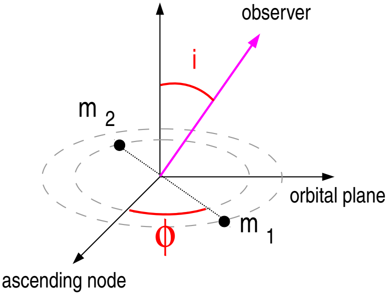


Waveform of inspiralling compact binaries



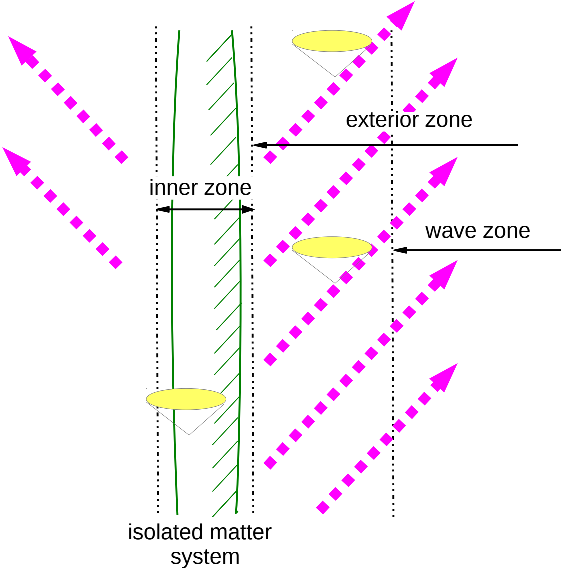
$$h_+ = \frac{2G\mu}{c^2 R}\left(\frac{GM\omega}{c^3}\right)^{2/3}\left(1+\cos^2 i\right)\cos\left(2\phi\right)$$

$$h_\times = \frac{2G\mu}{c^2 R}\left(\frac{GM\omega}{c^3}\right)^{2/3}\left(2\cos i\right)\sin\left(2\phi\right)$$

The distance of the source R is measurable from the GW signal [Schutz 1986]

GENERATION OF GRAVITATIONAL WAVES

Isolated matter system in general relativity



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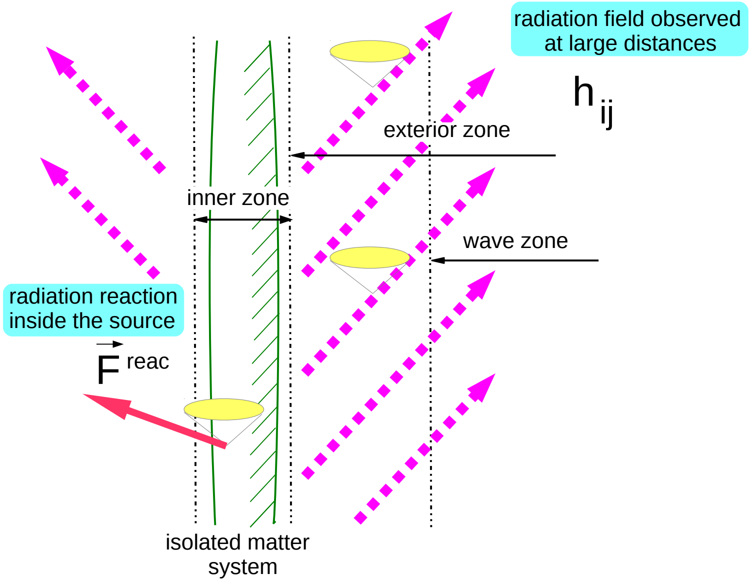

Isolated matter system in general relativity

1. Generation problem
   - What is the gravitational radiation field generated in a detector at large distances from the source?
2. Propagation problem
   - Solve the propagation effects of gravitational waves from the source to the detector, including non-linear effects
3. Motion problem
   - Obtain the equations of motion of the matter source including all conservative non-linear effects
4. Reaction problem
   - Obtain the dissipative radiation reaction forces inside the source in reaction to the emission of gravitational waves

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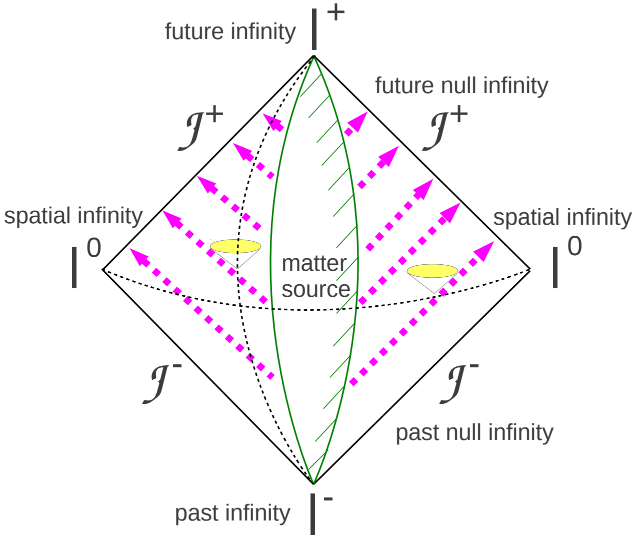
Asymptotic structure of radiating space-time

[Bondi-Sachs formalism 1960s]



Notion of asymptotic flatness [Penrose 1963, 1965]

Definition: [e.g. Geroch & Horowitz 1978]

A space-time $(\mathcal{M}, g_{\alpha\beta})$ is said to be asymptotically simple at null infinity if there exists a C^∞ manifold $\tilde{\mathcal{M}}$ with boundary $\mathcal{J}$ together with a C^∞ Lorentz metric $\tilde{g}_{\alpha\beta}$ and a C^∞ scalar field Ω on $\tilde{\mathcal{M}}$ such that:

1. in the interior $\tilde{\mathcal{M}} \setminus \mathcal{J}$ we have $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$;
2. at the boundary $\mathcal{J}$ we have $\Omega = 0$ and $\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}_\beta\Omega = 0$;
3. $\mathcal{J}$ consists of two parts, $\mathcal{J}^+$ and $\mathcal{J}^-$, each with topology $S^2 \times \mathbb{R}$, with the $\mathbb{R}$'s being complete null generators.

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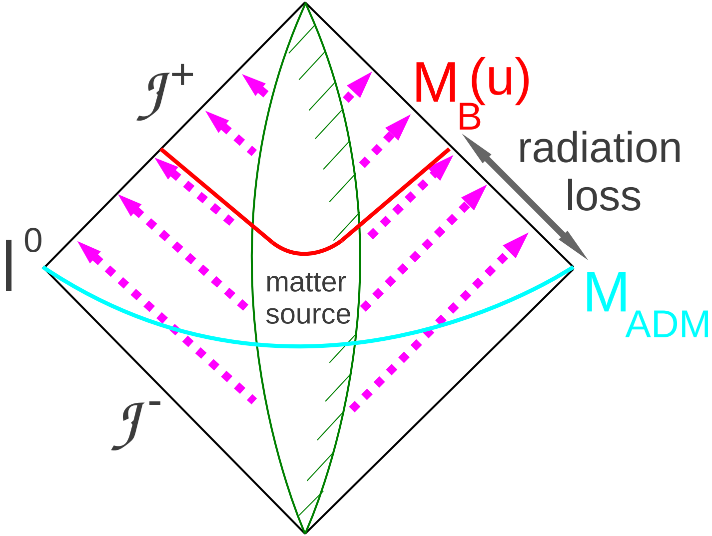
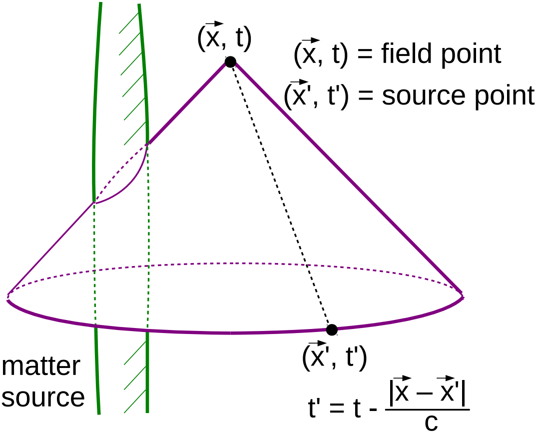
Bondi mass versus ADM mass



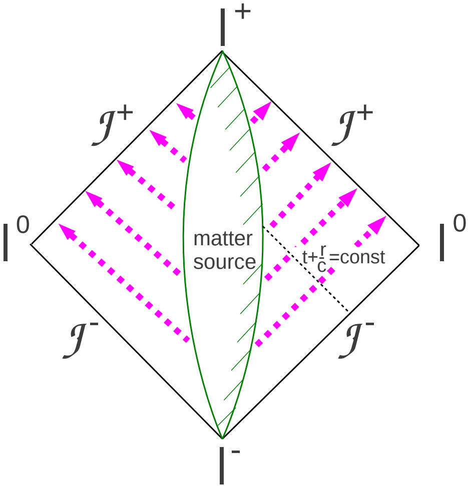
Kirchhoff's formula

For an homogeneous solution of the wave equation $\Box h = 0$

$$h(\mathbf{x},t) = \lim_{|\mathbf{x}'|\to+\infty} \iint \frac{\mathrm{d}\Omega'}{4\pi} \left(\frac{\partial}{\partial r} + \frac{\partial}{c\partial t}\right)(rh)\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

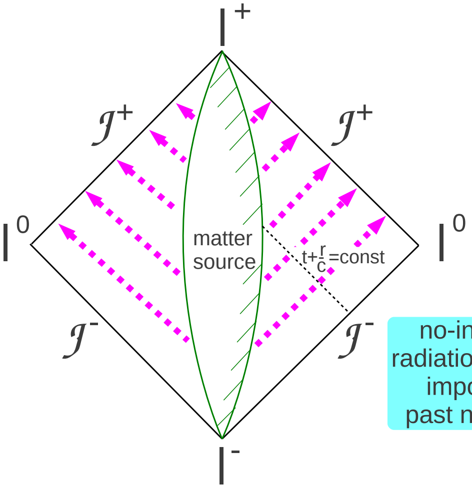


No-incoming radiation condition



$$\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

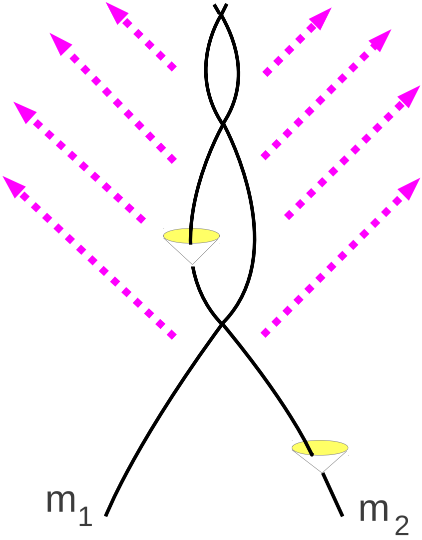
No-incoming radiation condition



no-incoming radiation condition imposed at past null infinity

$$\lim_{\substack{r\to+\infty \\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

Two-body system formed from freely falling particles

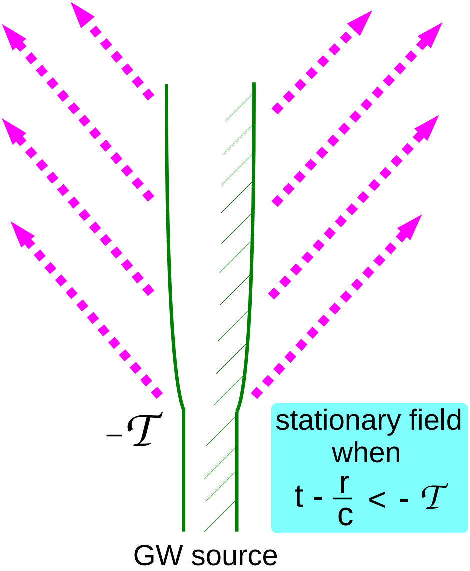


Gravitational motion of initially free particles when $t \to -\infty$ [Eder 1989]

$$\boldsymbol{x}(t) = \boldsymbol{V}\, t + \boldsymbol{W} \ln(-t) + \boldsymbol{X} + o(t^0)$$

where $\boldsymbol{V}$ and $\boldsymbol{X}$ are constant vectors, and $\boldsymbol{W} = GM\boldsymbol{V}/V^3$

Hypothesis of stationarity in the remote past



In practice all GW sources observed in astronomy (*e.g.* a compact binary system) will have been formed and started to emit GWs only from a finite instant in the past $-\mathcal{T}$

MULTIPOLAR POST-MINKOWSKIAN APPROACH

Linearized multipolar vacuum solution [Pirani 1964; Thorne 1980]

Solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}I_L\right) \qquad \boxed{L = i_1 i_2 \cdots i_\ell}$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}\right) + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}J^{(1)}_{j)bL-2}\right)\right\}$$

- multipole moments $I_L(u)$ and $J_L(u)$ are arbitrary functions of $u = t - r/c$
- mass $M \equiv I = \text{const}$, center-of-mass position $G_i \equiv I_i = \text{const}$
  linear momentum $P_i \equiv I^{(1)}_i = 0$, angular momentum $J_i = \text{const}$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

1. The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\text{MPM}} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}$$

where $h^{\alpha\beta}_{(1)}$ is defined from the multipole moments I_L and J_L

2. Hierarchy of perturbation equations is solved by induction over n

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \dots, h_{(n-1)}]$$
$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

3. A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

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[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

1. Multiply source term by r^B where $B \in \mathbb{C}$ and integrate

$$u^{\alpha\beta}_{(n)}(B) = \Box^{-1}_{\text{ret}} \left[r^B \Lambda^{\alpha\beta}_{(n)} \right]$$

2. Consider Laurent expansion when $B \to 0$

$$u^{\alpha\beta}_{(n)}(B) = \sum_{j=j_{\text{min}}}^{+\infty} u^{\alpha\beta}_{j(n)} B^j \quad \text{then} \quad \begin{cases} j \leqslant -1 \quad \Longrightarrow \quad \Box u^{\alpha\beta}_{j(n)} = 0 \\ j \geqslant 0 \quad \Longrightarrow \quad \Box u^{\alpha\beta}_{j(n)} = \frac{(\ln r)^j}{j!} \Lambda^{\alpha\beta}_{(n)} \end{cases}$$

3. Define the finite part (FP) when $B \to 0$ to be the zeroth coefficient $u^{\alpha\beta}_{0(n)}$

$$u^{\alpha\beta}_{(n)} = \text{FP}\Box^{-1}_{\text{ret}} \left[r^B \Lambda^{\alpha\beta}_{(n)} \right] \quad \text{then} \quad \Box u^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}$$

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Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

1. Harmonic gauge condition is not yet satisfied

$$w^{\alpha}_{(n)} = \partial_\mu u^{\alpha\mu}_{(n)} = \mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}\left[B\, r^{B-1} n_i \Lambda^{\alpha i}_{(n)}\right]$$

2. But $\Box w^{\alpha}_{(n)} = 0$ hence we can compute $v^{\alpha\beta}_{(n)}$ such that at once

$$\Box u^{\alpha\beta}_{(n)} = 0 \quad \text{and} \quad \partial_\mu u^{\alpha\mu}_{(n)} = -w^{\alpha}_{(n)}$$

3. Thus we define

$$\boxed{h^{\alpha\beta}_{(n)} = u^{\alpha\beta}_{(n)} + v^{\alpha\beta}_{(n)}}$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

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Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

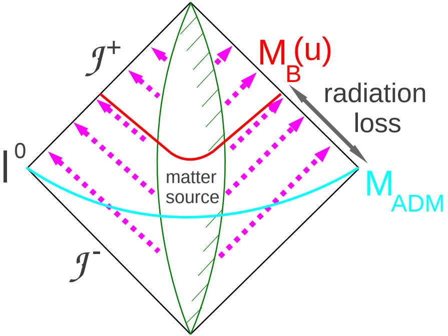
Theorem 2:
The general structure of the PN expansion is

$$h^{\alpha\beta}_{\mathrm{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)$$

Theorem 3:
The MPM solution is asymptotically flat at future null infinity in the sense of Penrose and agrees with the Bondi-Sachs formalism

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]



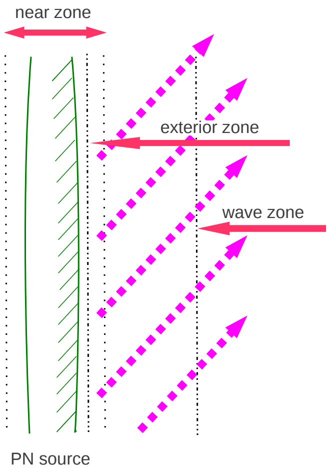
$$M_{\mathsf{B}}(u) \quad = \quad M_{\mathsf{ADM}} - \overbrace{\frac{G}{5c^7}\int_{-\infty}^{u} \mathrm{d}t\, M_{ij}^{(3)}(t) M_{ij}^{(3)}(t)}^{\text{mass-energy emitted in GW}}$$

$$+ \quad \left\{ \begin{array}{l} \text{higher-order multipole moments and} \\ \text{higher-order PM approximations} \\ \text{computable to any order by the MPM algorithm} \end{array} \right.$$

The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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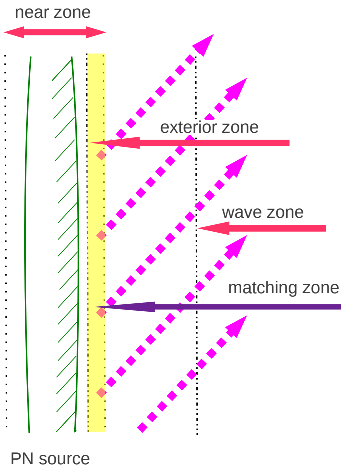

$$\underbrace{\overline{\mathcal{M}(h^{\mu\nu})} = \mathcal{M}(\bar{h}^{\mu\nu})}_{\text{matching equation}}$$

The matching equation

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

1. This is a variant of the theory of matched asymptotic expansions

$$\text{match} \quad \begin{cases} \text{the multipole expansion} \quad \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\text{MPM}} \\ \qquad\qquad \text{with} \\ \text{the PN expansion} \quad \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\text{PN}} \end{cases}$$

$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

- Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
- Right side is the FZ expansion ($r \to +\infty$) of the inner PN field

2. The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
3. It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source

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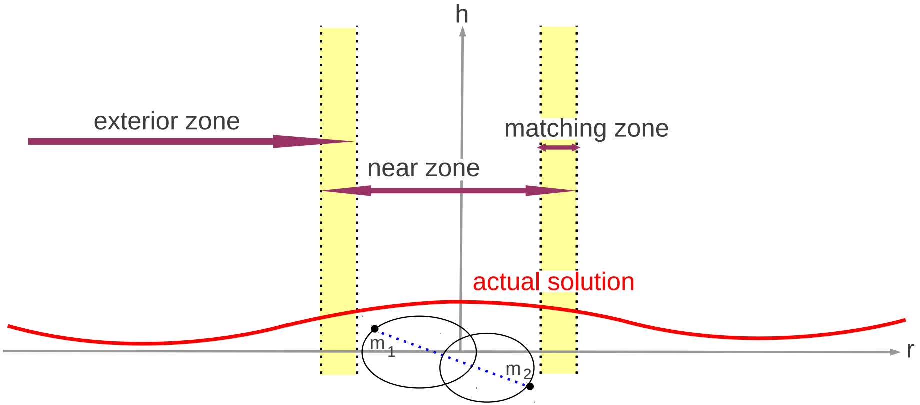

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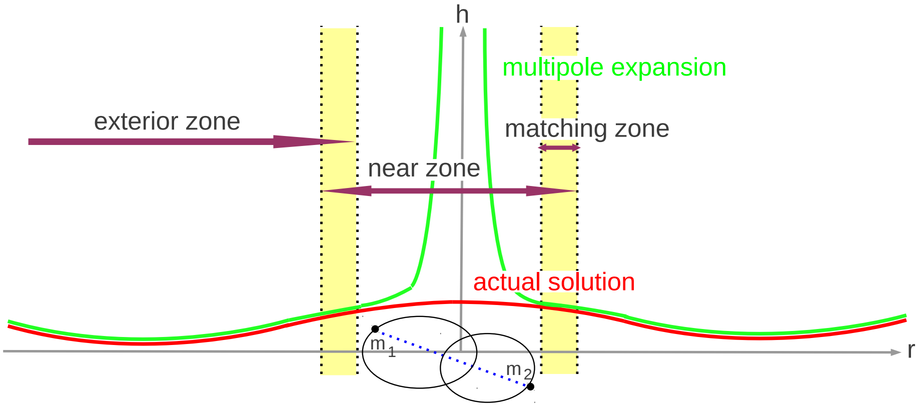

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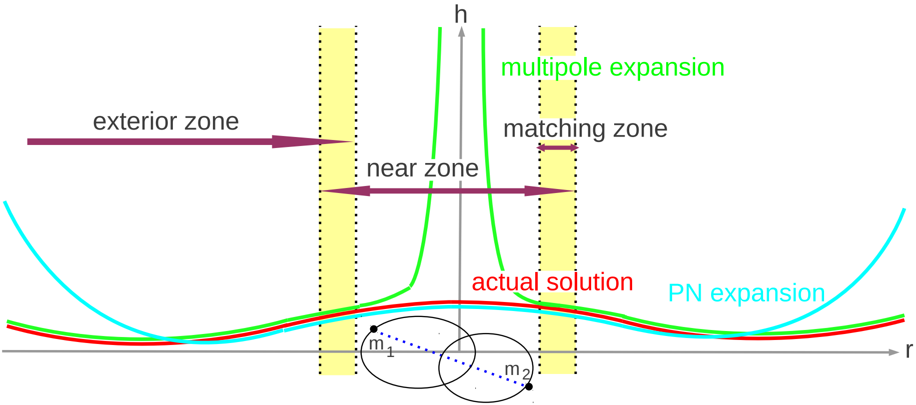

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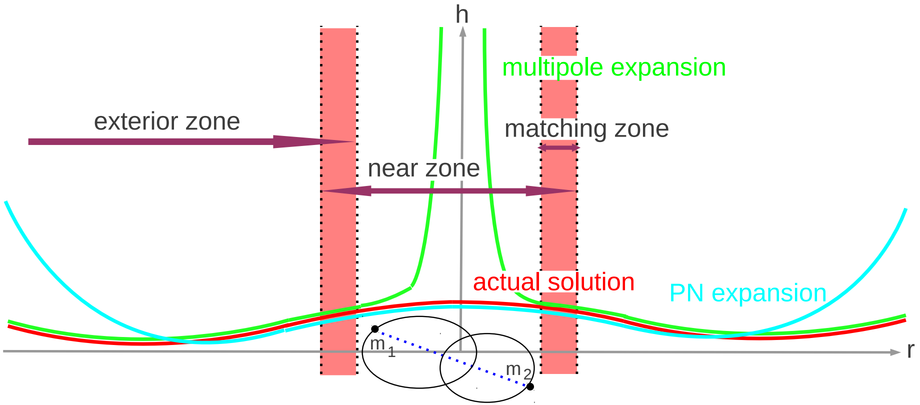

General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \text{FP}\,\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where

$$M_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\;\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem
- This is a formal PN solution *i.e.* a set of rules for generating the PN series regardless of the exact mathematocal nature of this series

General solution for the inner PN field

[Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2004]

$$\bar{h}^{\mu\nu} = \text{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r} \right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \text{FP} \int \mathrm{d}^3\mathbf{x}\, \hat{x}_L \int_1^\infty \mathrm{d}z\, \gamma_\ell(z) \underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Radiative moments at future null infinity

1. Correct for the "tortoise" logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\underbrace{u}_{\text{null coordinate}} \equiv \overbrace{T - \frac{R}{c}}^{\text{radiative coordinates}} = \overbrace{t - \frac{r}{c}}^{\text{harmonic coordinates}} - \overbrace{\frac{2GM}{c^3}\ln\left(\frac{r}{c\tau_0}\right)}^{\text{logarithmic deviation}} + \mathcal{O}\left(\frac{1}{r}\right)$$

2. Asymptotic waveform is parametrized by radiative moments U_L and V_L

$$h_{ij}^{\mathrm{TT}} = \frac{1}{R}\sum_{\ell=2}^{\infty} N_{L-2}\underbrace{U_{ijL-2}(u)}_{\text{mass-type}} + \varepsilon_{ab(i}N_{aL-1}\underbrace{V_{j)bL-2}(u)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

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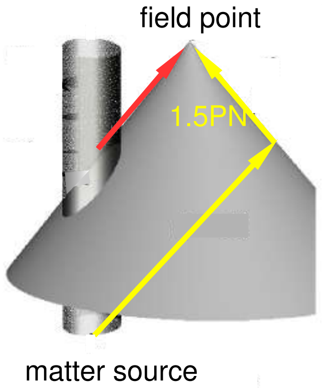
The 4.5PN radiative quadrupole moment

$$
\begin{aligned}
U_{ij}(t) = I^{(2)}_{ij}(t) &+ \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}} \\
&+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\} \\
&+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}} \\
&+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}} \\
&+ \mathcal{O}\left(\frac{1}{c^{10}}\right)
\end{aligned}
$$

Gravitational wave tails

[Bonnor 1959; Bonnor & Rotenberg 1961; Price 1971; Blanchet & Damour 1988, 1992; Blanchet 1993, 1997]

The tails are produced by backscatter of linear GWs generated by the variations of I_{ij} off the curvature induced by the matter source's total mass M

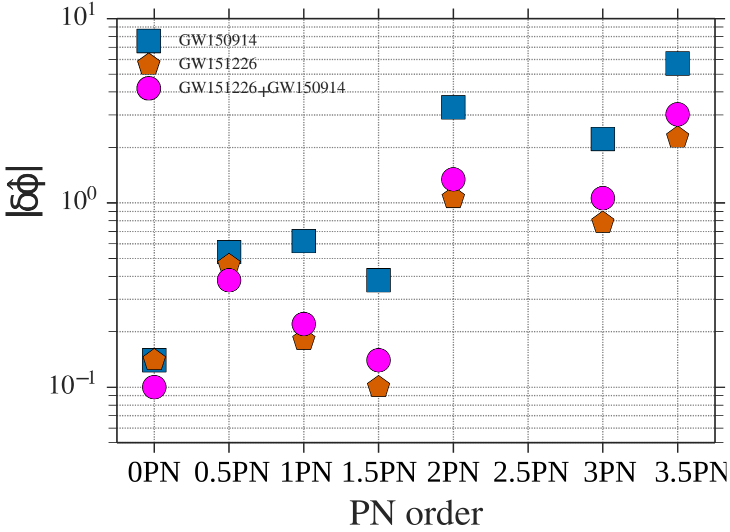


$$\delta h_{ij}^{\text{tail}} = \frac{4G}{c^4 r} \underbrace{\frac{GM}{c^3} \int_{-\infty}^{u} \mathrm{d}t \, I_{ij}^{(4)}(t) \ln\left(\frac{u-t}{\tau_0}\right)}_{\text{The tail is dominantly a 1.5PN effect}} + \cdots$$

3.5PN energy flux of compact binaries

$$\mathcal{F}^{\mathrm{GW}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right)}^{\text{1PN}} x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}}$$
$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}}$$
$$+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_{\mathrm{E}} - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}}$$
$$+ \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3$$
$$+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}$$

Measurement of PN parameters [LIGO/Virgo collaboration 2016]



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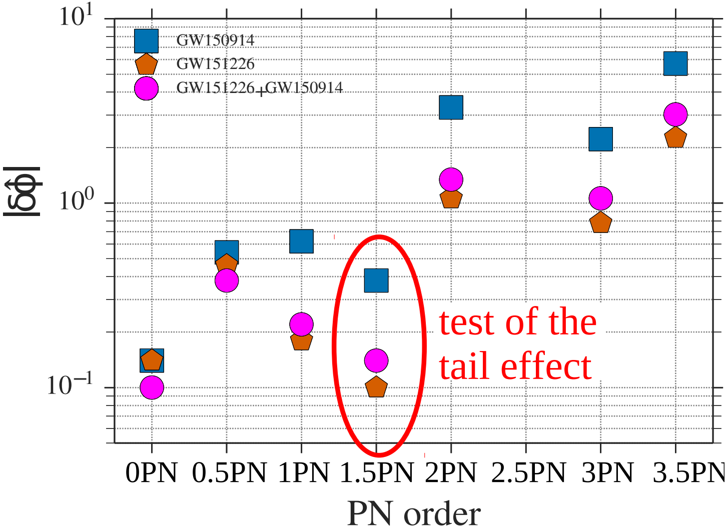
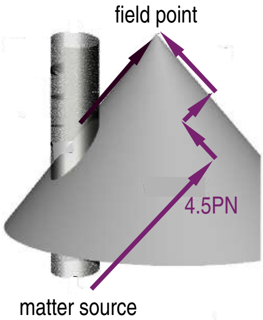


4.5PN coefficient in the GW flux [Marchand, Blanchet, Faye 2017]

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{4.5\mathrm{PN}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ \bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \bigg) \pi x^{9/2} \Bigg\}$$



- The 4.5PN tail effect represents the complete 4.5PN coefficient in the GW energy flux in the case of circular orbits
- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- However the 4PN term in the flux is still in progress

FLUX-BALANCE EQUATIONS FOR ENERGY & MOMENTA

Gravitational radiation reaction to 4PN order

For general matter systems the 4PN radiation reaction derives from radiation reaction potentials valid in a specific extension of the [Burke & Thorne 1971] gauge

$$V^{\text{reac}} = - \underbrace{\frac{G}{5c^5} x^{ij} I^{(5)}_{ij}}_{\text{2.5PN radiation reaction}} + \overbrace{\frac{G}{c^7}\left[\frac{1}{189} x^{ijk} I^{(7)}_{ijk} - \frac{1}{70} r^2 x^{ij} I^{(7)}_{ij}\right]}^{\text{3.5PN scalar correction}}$$
$$\underbrace{- \frac{4G^2 M}{5c^8} x^{ij} \int_0^{+\infty} \mathrm{d}\tau\, I^{(7)}_{ij}(t-\tau) \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right]}_{\text{4PN radiation reaction tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$V_i^{\text{reac}} = \underbrace{\frac{G}{c^5}\left[\frac{1}{21} \hat{x}^{ijk} I^{(6)}_{jk} - \frac{4}{45} \varepsilon_{ijk} x^{jl} J^{(5)}_{kl}\right]}_{\text{3.5PN vector correction}} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Radiation reaction derivation of balance equations

1. Metric accurate to 1PN order for conservative effects and to 3.5PN order for dissipative radiation reaction effects

$$g_{00} = -1 + \frac{2\mathcal{V}}{c^2} - \frac{2\mathcal{V}^2}{c^4} + \frac{1}{c^6}\underset{6}{g_{00}} + \frac{1}{c^8}\underset{8}{g_{00}} + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

$$g_{0i} = -\frac{4\mathcal{V}_i}{c^3} + \frac{1}{c^5}\underset{5}{g_{0i}} + \frac{1}{c^7}\underset{7}{g_{0i}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2\mathcal{V}}{c^2}\right) + \frac{4}{c^4}\Big(W_{ij} - \delta_{ij}W_{kk}\Big) + \frac{1}{c^6}\underset{6}{g_{ij}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

2. Potentials are composed of a conservative part and a dissipative one

$$\mathcal{V}_\mu = V_\mu^{\text{cons}} + \boxed{V_\mu^{\text{reac}}}$$

3. Flux balance equations are obtained by integrating the matter equations of motion $\nabla_\nu T^{\mu\nu} = 0$ over the source

$$\partial_\nu\left(\sqrt{-g}T^\nu_\mu\right) = \frac{1}{2}\sqrt{-g}\,\partial_\mu g_{\rho\sigma}T^{\rho\sigma}$$

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Radiation reaction derivation of balance equations

- Define the matter current and stresses

$$\sigma = \frac{T^{00}+T^{ii}}{c^2} \qquad \sigma_i = \frac{T^{0i}}{c} \qquad \sigma_{ij} = T^{ij}$$

- To conservative 1PN order the invariants of the matter system are given by

$$\boxed{\begin{aligned}
E &= \int \mathrm{d}^3\mathbf{x} \left(\sigma c^2 + \frac{1}{2}\sigma U - \sigma_{ii} + \frac{1}{c^2}\left[-4\sigma W_{ii} + 2\sigma_i U_i + \cdots\right]\right) \\
J_i &= \varepsilon_{ijk} \int \mathrm{d}^3\mathbf{x}\, x_j \left(\sigma_k + \frac{1}{c^2}\left[4\sigma_k U - 4\sigma U_k - \frac{1}{2}\sigma\, \partial_k \partial_t X\right]\right) \\
P_i &= \int \mathrm{d}^3\mathbf{x} \left[\sigma_i - \frac{1}{2c^2}\sigma \partial_i \partial_t X\right] \\
G_i &= \int \mathrm{d}^3\mathbf{x}\, x_i \left(\sigma + \frac{1}{c^2}\left[\frac{\sigma U}{2} - \sigma_{jj}\right]\right)
\end{aligned}}$$

Radiation reaction derivation of balance equations

1. Well known results for the energy and angular momentum

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{G}{c^5}\varepsilon_{ijk}\left(\frac{2}{5}I^{(2)}_{jl}I^{(3)}_{kl} + \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{jlm}I^{(4)}_{klm} + \frac{32}{45}J^{(2)}_{jl}J^{(3)}_{kl}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

2. And for linear momentum (this effect responsible for the recoil of the source)

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{G}{c^7}\left[\frac{2}{63}I^{(4)}_{ijk}I^{(3)}_{jk} + \frac{16}{45}\varepsilon_{ijk}I^{(3)}_{jl}J^{(3)}_{kl}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3. However we find also for the center-of-mass position [Blanchet & Faye 2018]

$$\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \frac{2G}{21c^7}I^{(3)}_{ijk}I^{(3)}_{jk} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

Strangely enough this formula does not appear in the GW litterature

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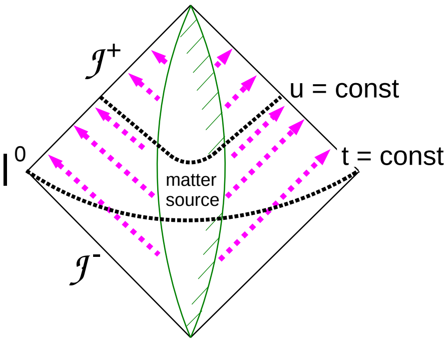
Direct calculation of the GW fluxes at infinity



- Introduce a retarded null coordinate u satisfying

$$\boxed{g^{\mu\nu}\partial_\mu u \partial_\nu u = 0}$$

- For instance choose $u = t - r_*/c$ with the tortoise coordinate

$$r_* = r + \frac{2GM}{c^2}\ln\left(\frac{r}{r_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

Direct calculation of the GW fluxes at infinity

1. Perform a coordinate change $(t, \mathbf{x}) \to (u, \mathbf{x})$ in the conservation law of the pseudo-tensor $\partial_\nu \tau^{\mu\nu} = 0$ to get

$$\frac{\partial}{c\partial u}\Big[\tau^{\mu 0}(\mathbf{x}, u + r_*/c) - n_*^i \tau^{\mu i}(\mathbf{x}, u + r_*/c)\Big] + \partial_i \Big[\tau^{\mu i}(\mathbf{x}, u + r_*/c)\Big] = 0$$

2. Integrating over a volume $\mathcal{V}$ tending to infinity with u =const

$$\frac{\mathrm{d}E}{\mathrm{d}u} = -c \int_{\partial\mathcal{V}} \mathrm{d}S_i \, \tau^{0i}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c)$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}u} = -\varepsilon_{ijk} \int_{\partial\mathcal{V}} \mathrm{d}S_l \, x^j \, \tau^{kl}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c)$$

$$\frac{\mathrm{d}P^i}{\mathrm{d}u} = - \int_{\partial\mathcal{V}} \mathrm{d}S_j \, \tau^{ij}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c)$$

$$\frac{\mathrm{d}G_i}{\mathrm{d}u} = P_i - \frac{1}{c} \int_{\partial\mathcal{V}} \mathrm{d}S_j \left(x^i \, \tau^{0j}_{\mathsf{GW}} - r_* \, \tau^{ij}_{\mathsf{GW}} \right)(\mathbf{x}, u + r_*/c)$$

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\frac{\mathrm{d}E}{\mathrm{d}u} &= -c \int_{\partial\mathcal{V}} \mathrm{d}S_i \, \tau^{0i}_{\text{GW}}(\mathbf{x}, u + r_*/c) \\
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\frac{\mathrm{d}G_i}{\mathrm{d}u} &= P_i - \frac{1}{c} \int_{\partial\mathcal{V}} \mathrm{d}S_j \left(x^i \, \tau^{0j}_{\text{GW}} - r_* \, \tau^{ij}_{\text{GW}} \right) (\mathbf{x}, u + r_*/c)
\end{aligned}$$

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2. Integrating over a volume $\mathcal{V}$ tending to infinity with $u =$const

$$\boxed{\begin{aligned}
E &= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\Big[\tau^{00} - n_*^i\,\tau^{0i}\Big](\mathbf{x}, u + r_*/c)\\
J_i &= \frac{1}{c}\,\varepsilon_{ijk}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,x^j\Big[\tau^{k0} - n_*^l\,\tau^{kl}\Big](\mathbf{x}, u + r_*/c)\\
P_i &= \frac{1}{c}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\Big[\tau^{0i} - n_*^j\,\tau^{ij}\Big](\mathbf{x}, u + r_*/c)\\
G_i &= \frac{1}{c^2}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\Big[x^i\big(\tau^{00} - n_*^j\,\tau^{0j}\big) - r_*\big(\tau^{0i} - n_*^j\,\tau^{ij}\big)\Big](\mathbf{x}, u + r_*/c)
\end{aligned}}$$

Direct calculation of the GW fluxes at infinity

A long calculation to control the leading $1/r^2$ and subleading $1/r^3$ terms in the GW pseudo-tensor when $r \to +\infty$ gives the fluxes as full multipole series parametrized by the multipole moments I_L and J_L up to order $\mathcal{O}(G^2)$

$$\frac{\mathrm{d}E}{\mathrm{d}u} = -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!} \overset{(\ell+1)}{I}_L \overset{(\ell+1)}{I}_L + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \overset{(\ell+1)}{J}_L \overset{(\ell+1)}{J}_L \right\}$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}u} = -\varepsilon_{ijk} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \overset{(\ell)}{I}_{jL-1} \overset{(\ell+1)}{I}_{kL-1} + \frac{4\ell^2(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \overset{(\ell)}{J}_{jL-1} \overset{(\ell+1)}{J}_{kL-1} \right\}$$

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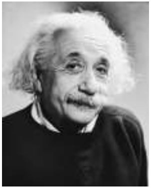
$$\frac{\mathrm{d}G_i}{\mathrm{d}u} = P_i \underbrace{- \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+3}} \Bigg\{ \frac{2(\ell+2)(\ell+3)}{\ell\,\ell!(2\ell+3)!!} \overset{(\ell+1)}{I}_{iL} \overset{(\ell+1)}{I}_{L} + \frac{8(\ell+3)}{c^2\ell!(2\ell+3)!!} \overset{(\ell+1)}{J}_{iL} \overset{(\ell+1)}{J}_{L} \Bigg\}}_{\text{[Blanchet \& Faye 2018]}}$$

FOKKER APPROACH TO THE PN EOM

The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]

$$\boxed{\begin{aligned}\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} &= -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\left[1-4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}}-\sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1-\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right)\right.\\ &\qquad\left.+\frac{1}{c^2}\left(\boldsymbol{v}_A^2+2\boldsymbol{v}_B^2-4\boldsymbol{v}_A\cdot\boldsymbol{v}_B-\frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\right]\\ &\quad+\sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B-4\boldsymbol{v}_A)]-\frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}\end{aligned}}$$

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\text{2.5PN radiation reaction}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\text{3.5PN radiation reaction}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\text{4PN conservative \& radiation tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc]	Fokker Lagrangian
	[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

The Fokker Lagrangian approach to the 4PN EOM

Based on collaborations with





Laura Bernard, **Alejandro Bohé**, **Guillaume Faye**,
Tanguy Marchand & **Sylvain Marsat**

[PRD **93**, 084037 (2016); **95**, 044026 (2017); **96**, 104043 (2017); **97**, 044023 (2018); PRD **97**, 044037 (2018)]

Fokker action of N particles [Fokker 1929]

1. Gauge-fixed Einstein-Hilbert action for N point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} \Big[R \underbrace{- \frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu}_{\text{Gauge-fixing term}} \Big]$$
$$- \sum_A \underbrace{m_A c^2 \int \mathrm{d}t \sqrt{-(g_{\mu\nu})_A \, v_A^\mu v_A^\nu / c^2}}_{N \text{ point particles}}$$

2. Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

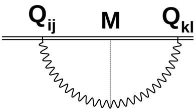
$$g_{\mu\nu}(\mathbf{x}, t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x}; \boldsymbol{x}_B(t), \boldsymbol{v}_B(t), \cdots)$$

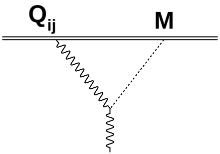
3. The PN equations of motion of the N particles (self-gravitating system) are

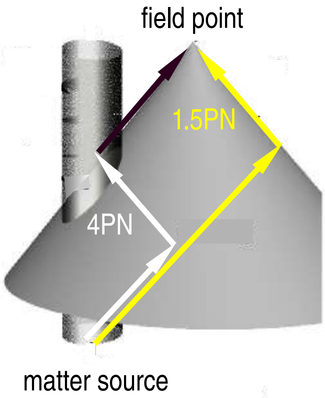
$$\boxed{\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{x}_A} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{x}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

The gravitational wave tail effect

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley, Leibovich, Porto *et al.* 2016]







- In the near zone (4PN effect)

$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t)\, I^{(3)}_{ij}(t')$$

- In the far zone (1.5PN effect)

$$h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{t} \mathrm{d}t'\, I^{(4)}_{ij}(t') \ln\left(\frac{t-t'}{\tau_0}\right)$$

Problem of the UV divergences

[t'Hooft & Veltman 1972; Bollini & Giambiagi 1972; Breitenlohner & Maison 1977]

1. Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

2. For two point-particles $\rho = m_1 \delta_{(d)}(\mathbf{x} - \boldsymbol{x}_1) + m_2 \delta_{(d)}(\mathbf{x} - \boldsymbol{x}_2)$ we get

$$\boxed{U(\mathbf{x},t) = \frac{2(d-2)k}{d-1} \left(\frac{G m_1}{|\mathbf{x} - \boldsymbol{x}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x} - \boldsymbol{x}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}}$$

3. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
4. Dimensional regularization is then followed by a renormalization of the worldline of the particles so as to absorb the poles $\propto (d-3)^{-1}$

Problem of the IR divergences

1. The tail effect implies the appearance of IR divergences in the Fokker action at the 4PN order
2. Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (FP procedure when $B \to 0$)
3. However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
4. The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\text{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\Big(\delta_1 (n_{12} v_{12})^2 + \delta_2 v_{12}^2\Big)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

5. Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

Conserved energy for a non-local Hamiltonian

1. Because of the tail effect at 4PN order the Lagrangian or Hamiltonian becomes non-local in time

$$H[\mathbf{x},\mathbf{p}] = H_0(\mathbf{x},\mathbf{p}) + \underbrace{H_{\text{tail}}[\mathbf{x},\mathbf{p}]}_{\text{non-local piece at 4PN}}$$

2. Hamilton's equations involve functional derivatives

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\delta H}{\delta p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\delta H}{\delta x^i}$$

3. The conserved energy is not given by the Hamiltonian on-shell but $E = H + \Delta H^{\text{AC}} + \Delta H^{\text{DC}}$ where the AC term averages to zero and

$$\boxed{\Delta H^{\text{DC}} = -\frac{2GM}{c^3}\mathcal{F}^{\text{GW}} = -\frac{2G^2M}{5c^5}\langle\left(I_{ij}^{(3)}\right)^2\rangle}$$

4. On the other hand [DJS] perform a non-local shift to transform the Hamiltonian into a local one, and both procedure are equivalent

Conserved energy for circular orbits at 4PN order

- The 4PN energy for circular orbits in the small mass ratio limit is known from GSF of the redshift variable [Le Tiec, Blanchet & Whiting 2012; Bini & Damour 2013]
- This permits to fix the ambiguity parameter α and to complete the 4PN equations of motion

$$
\begin{aligned}
E^{\mathsf{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{&1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 \\
&+ \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3 \\
&+ \Bigg(-\frac{3969}{128} + \left[-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_{\mathsf{E}} + \frac{448}{15}\ln(16x)\right]\nu \\
&\qquad + \left[-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right]\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\Bigg)x^4\Bigg\}
\end{aligned}
$$

Periastron advance for circular orbits at 4PN order

The periastron advanced (or relativistic precession) constitutes a second invariant which is also known in the limit of circular orbits from GSF calculations

$$
\begin{aligned}
K^{\mathsf{4PN}} = 1 + 3x + \left(\frac{27}{2} - 7\nu\right) x^2 \\
+ \left(\frac{135}{2} + \left[-\frac{649}{4} + \frac{123}{32}\pi^2\right]\nu + 7\nu^2\right) x^3 \\
+ \left(\frac{2835}{8} + \left[-\frac{275941}{360} + \frac{48007}{3072}\pi^2 - \frac{1256}{15}\ln x \right.\right. \\
\left. - \frac{592}{15}\ln 2 - \frac{1458}{5}\ln 3 - \frac{2512}{15}\gamma_{\mathsf{E}}\right]\nu \\
\left. + \left[\frac{5861}{12} - \frac{451}{32}\pi^2\right]\nu^2 - \frac{98}{27}\nu^3\right) x^4
\end{aligned}
$$

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I_{\mathcal{R}}^{\mathsf{HR}} = \underset{B=0}{\mathrm{FP}} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I_{\mathcal{R}}^{\mathsf{DR}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} \, F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d-3$)

$$\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon)$$

Ambiguity-free completion of the 4PN EOM

[Marchand, Bernard, Blanchet & Faye 2017]

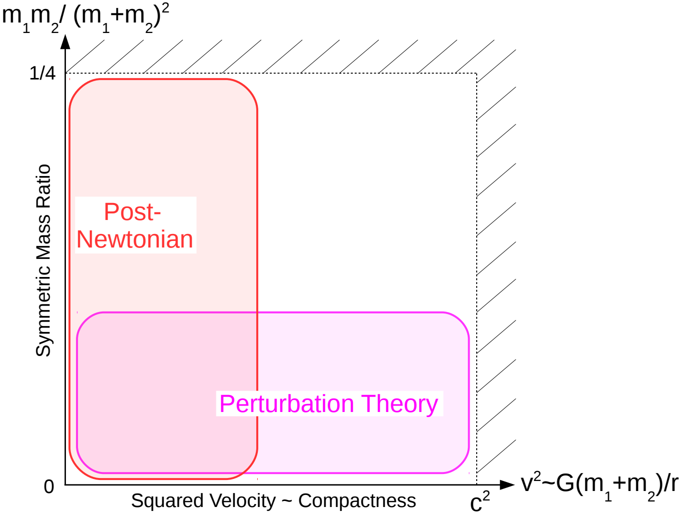
1. The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

$$g_{00}^{\text{tail}} = -\frac{8G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\frac{41}{60}\right]I_{ij}^{(7)}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

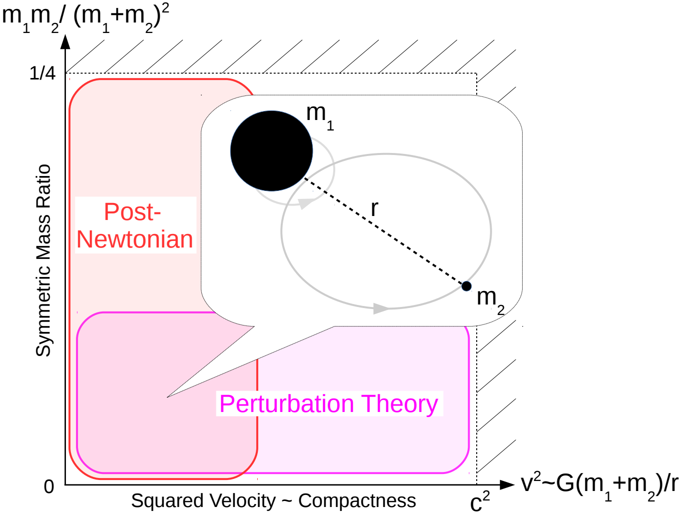
2. Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters δ_1 and δ_2
3. It is likely that the EFT formalism will also succeed in deriving the full EOM without ambiguities [Porto & Rothstein 2017]
4. The lack of a consistent matching between the near zone and the far zone in the ADM Hamiltonian formalism [DJS] forces this formalism to be still plagued by one ambiguity parameter

PN VERSUS PERTURBATION THEORY

Post-Newtonian versus perturbation theory



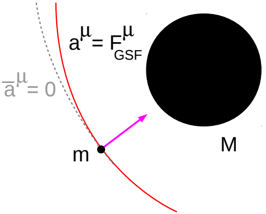
Post-Newtonian versus perturbation theory



Problem of the gravitational self-force (GSF)

[Mino, Sasaki & Tanaka 1997; Quinn & Wald 1997; Detweiler & Whiting 2003]

- A particle is moving on a background space-time of a massive black hole
- Its stress-energy tensor modifies the background gravitational field
- Because of the back-reaction the motion of the particle deviates from a background geodesic hence the gravitational self force

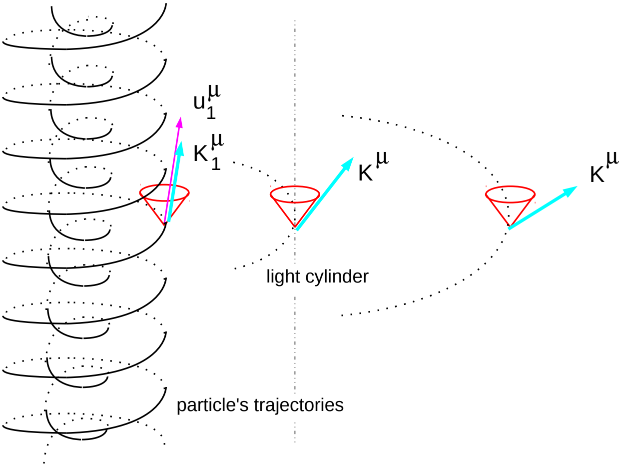


$$\bar{a}^\mu = F^\mu_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)$$

The GSF is computed to high accuracy by

- numerical methods [Sago, Barack & Detweiler 2008; Shah, Friedmann & Whiting 2014]
- analytical ones [Mano, Susuki & Takasugi 1996ab; Bini & Damour 2013, 2014]

Looking at the conservative part of the dynamics



Space-time for exact circular orbits admits a Helical Killing Vector (HKV) K^μ

Choice of a gauge-invariant observable [Detweiler 2008]

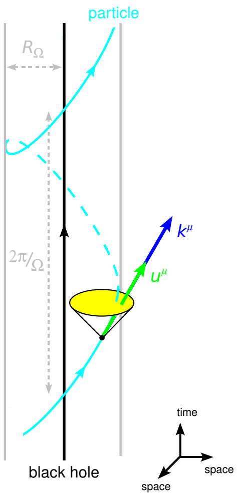
1. For exactly circular orbits the geometry admits a helical Killing vector with

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi \quad \text{(asymptotically)}$$

2. The four-velocity of the particle is necessarily tangent to the Killing vector hence

$$K_1^\mu = z_1\, u_1^\mu$$

3. This z_1 is the Killing energy of the particle associated with the HKV and is also a redshift
4. The relation $z_1(\Omega)$ is well-defined in both PN and GSF approaches and is gauge-invariant

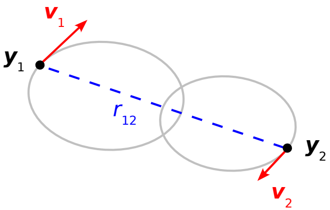


Post-Newtonian calculation of the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011; Blanchet, Faye & Whiting 2014, 2015]

In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \omega\, \partial_\varphi$ we have

$$z_1 = \frac{1}{u_1^t} = \left(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \right)^{1/2}$$



One needs a self-field regularization

- Hadamard "partie finie" regularization is extremely useful in practical calculations but yields (UV and IR) ambiguity parameters at high PN orders
- Dimensional regularization is an extremely powerful regularization which seems to be free of ambiguities at any PN order

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\text{SF}} = & -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
& + \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_\text{E} - \frac{64}{5}\ln(16y)\right) y^5 \\
& - \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
& + \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
& - \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
& + \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

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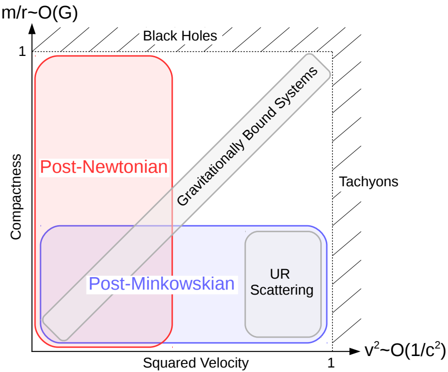
POST-NEWTONIAN VERSUS POST-MINKOWSKIAN

The post-Minkowskian approximation



- The ultra relativistic gravitational scattering of two particles has been solved up to the 2PM order [Westpfahl *et al.* 1980, 1985; Portilla 1980]
- A closed-form expression for the Hamiltonian of N particles at the 1PM order has been found [Ledvinka, Schäfer & Bičak 2008]

Comparing 4PN with 1PM [Blanchet & Fokas 2018]

1. The 1PM field equations of N particles in harmonic coordinates read

$$\Box h^{\mu\nu} = \frac{16\pi}{c^2} \sum_{a=1}^{N} G m_a \int_{-\infty}^{+\infty} \mathrm{d}\tau_a \, u_a^\mu u_a^\nu \delta^{(4)}(x - y_a)$$

2. The Lienard-Wiechert solution is

$$h^{\mu\nu}(x) = -\frac{4}{c^2} \sum_a \frac{G m_a \, u_a^\mu u_a^\nu}{r_a^{\text{ret}} \, (ku)_a^{\text{ret}}}$$

where $r_a^{\text{ret}} = |\boldsymbol{x} - \boldsymbol{x}_a^{\text{ret}}|$ and $(ku)_a^{\text{ret}}$ is the redshift factor

3. In small 1PM terms trajectories are straight lines hence the retardations can be explicitly performed

$$h^{\mu\nu}(\boldsymbol{x}, t) = -\frac{4}{c^2} \sum_a \frac{G m_a \, u_a^\mu u_a^\nu}{r_a \sqrt{1 + (n_a u_a)^2}}$$

Comparing 4PN with 1PM [Blanchet & Fokas 2018]

1. This yields the 1PM equations of motion but in PN like form[2]

$$\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} = -\gamma_a^{-2} \sum_{b\neq a} \frac{G m_b}{r_{ab}^2\, y_{ab}^{3/2}} \Bigg[(2\epsilon_{ab}^2 - 1)\boldsymbol{n}_{ab} + \gamma_b \Big(-4\epsilon_{ab}\gamma_a (n_{ab}v_a) + (2\epsilon_{ab}^2 + 1)\gamma_b (n_{ab}v_b) \Big) \frac{\boldsymbol{v}_{ab}}{c^2} \Bigg]$$

2. These equations of motion are conservative and admit a conserved energy

$$E = \sum_a m_a c^2 \gamma_a + \sum_a \sum_{b\neq a} \frac{G m_a m_b}{r_{ab}\, y_{ab}^{1/2}} \Bigg\{ \gamma_a \Big(2\epsilon_{ab}^2 + 1 - 4\frac{\gamma_b}{\gamma_a}\epsilon_{ab} \Big) + \frac{\gamma_b^2}{\gamma_a} (2\epsilon_{ab}^2 - 1) \frac{\dot{r}_{ab}(n_{ab}v_b) - (v_{ab}v_b)}{(v_{ab}^2 - \dot{r}_{ab}^2) y_{ab} + \frac{\gamma_b^2}{c^2}\big(\dot{r}_{ab}(n_{ab}v_b) - (v_{ab}v_b)\big)^2} \Bigg\}$$

[2] $y_{ab} = 1 + (n_{ab}u_a)^2$ and $\epsilon_{ab} = -(u_a u_b)$

Comparing 4PN with 1PM [Blanchet & Fokas 2018]

1. The 1PM Lagrangian in harmonic coordinates is a generalized one

$$L = \sum_a -\frac{m_a c^2}{\gamma_a} + \lambda + \underbrace{\sum_a q_a^i a_a^i}_{\text{accelerations}}$$

2. The 1PM Lagrangian can be computed up to any PN order from the terms of order G in the conserved energy say $E = \sum_a m_a c^2 \gamma_a + \varepsilon$

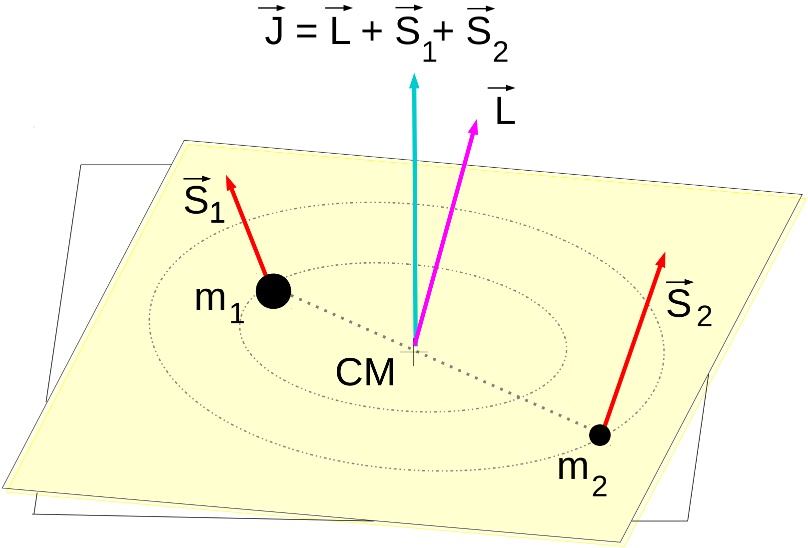
$$\boxed{\lambda = \text{FP} \int_c^{+\infty} \frac{\mathrm{d}c'}{c}\, \varepsilon\Big(\boldsymbol{x}_a, \frac{\boldsymbol{v}_a}{c'}\Big)}$$

3. We checked in a particular case that the Hamiltonian differs by a canonical transformation from the closed-form expression of the 1PM Hamiltonian in ADM coordinates [Ledvinka, Schäfer & Bičak 2008]

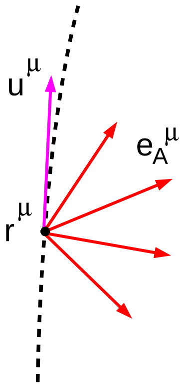
4. All the results reproduce the terms linear in G in the 4PN harmonic coordinates equations of motion and Lagrangian [BBBFMM]

SPIN EFFECTS IN COMPACT BINARIES

Black hole binary system with spins



Spinning particles in a pole-dipole approximation



1. The spin degrees of freedom are described by an orthonormal moving tetrad along the worldline

$$g_{\mu\nu}\, e_A{}^\mu e_B{}^\nu = \eta_{AB}$$

2. The rotation tensor of the tetrad is defined as

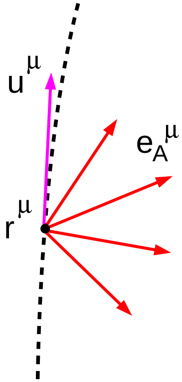
$$\frac{\mathrm{D}e_A{}^\mu}{\mathrm{d}\tau} = -\Omega^{\mu\nu} e_{A\nu}$$

3. Because of the orthonormality condition the rotation tensor is antisymmetric

$$\Omega^{\mu\nu} = -\Omega^{\nu\mu}$$

4. The dynamical degrees of freedom of the particle are the particle's position and the moving tetrad and the internal structure of the particle is neglected

Spinning particles in a pole-dipole approximation



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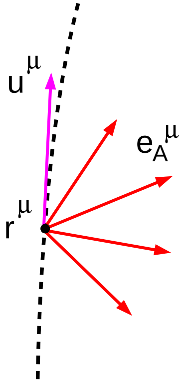

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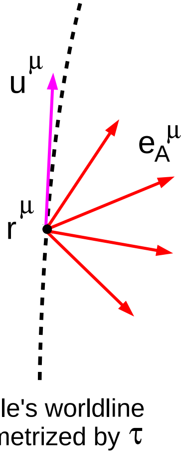

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Action for a system of spinning point particles

[Hanson & Regge 1974; Bailey & Israel 1975]

1. Following effective field theories we define a general action principle

$$S\left[r^\mu, e_A{}^\mu\right] = \sum_{\text{particles}} \int_{-\infty}^{+\infty} \mathrm{d}\tau\, L\left(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}\right)$$

2. The particle's linear momentum and spin tensor are the conjugate momenta

$$p_\mu = \frac{\partial L}{\partial u^\mu} \qquad S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$$

3. We just impose that the action obeys basic symmetry principles:
   - It should be a Lorentz scalar
   - It should be a covariant scalar

$$2\frac{\partial L}{\partial g_{\mu\nu}} = p^\mu u^\nu + S^\mu{}_\rho \Omega^{\nu\rho}$$

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Equations of motion and of spin precession

1. Varying the action with respect to the tetrad ${e_A}^\mu$ (holding the metric $g_{\mu\nu}$ fixed) gives the spin precession equation

$$\boxed{\frac{\mathrm{D}S_{\mu\nu}}{\mathrm{d}\tau} = p_\mu u_\nu - p_\nu u_\mu}$$

2. Varying with respect to the position r^μ gives the famous Mathisson-Papapetrou [Mathisson 1937; Papapetrou 1951] equation of motion

$$\boxed{\frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} = -\frac{1}{2}u^\nu R_{\mu\nu\rho\sigma}S^{\rho\sigma}}$$

3. Varying with respect to the metric $g_{\mu\nu}$ (keeping $e_{A[\mu}\delta {e^A}_{\nu]} = 0$) gives the stress-energy tensor of the spinning particles [Trautman 1958; Dixon 1979]

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Spin suplementary condition (SSC)

1. To correctly account for the number of degrees of freedom associated with the spin we impose a suplementary condition [Tulczyjew 1957, 1959]

$$\boxed{S^{\mu\nu}p_\nu = 0}$$

2. With the latter choice for the SSC, the particle's mass $m^2 = -g^{\mu\nu}p_\mu p_\nu$ and the four-dimensional spin magnitude $s^2 = S^{\mu\nu}S_{\mu\nu}$ are constant

$$\frac{\mathrm{D}m}{\mathrm{d}\tau} = 0 \qquad \frac{\mathrm{D}s}{\mathrm{d}\tau} = 0$$

3. The link between the four velocity u^μ and the four linear momentum p^μ is entirely specified, hence the Lagrangian is specified. At linear order in the spins we have

$$p^\mu = m\,u^\mu + \mathcal{O}(S^2)$$

4. The equation for the spin reduces to the equation of parallel transport

$$\frac{\mathrm{D}S_{\mu\nu}}{\mathrm{d}\tau} = \mathcal{O}(S^2)$$

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$$p^\mu = m\,u^\mu + \mathcal{O}(S^2)$$

4. The equation for the spin reduces to the equation of parallel transport

$$\frac{\mathrm{D}S_{\mu\nu}}{\mathrm{d}\tau} = \mathcal{O}(S^2)$$

Spin suplementary condition (SSC)

1. To correctly account for the number of degrees of freedom associated with the spin we impose a suplementary condition [Tulczyjew 1957, 1959]

$$\boxed{S^{\mu\nu} p_\nu = 0}$$

2. With the latter choice for the SSC, the particle's mass $m^2 = -g^{\mu\nu} p_\mu p_\nu$ and the four-dimensional spin magnitude $s^2 = S^{\mu\nu} S_{\mu\nu}$ are constant

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