The sound of space time
Element of data analysis and effective field theory approach to 2-body gravitational dynamics

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1 GW basics

We know that the GWs $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - 1/2\eta_{\mu\nu}h$ satisfy the wave equation in vacuum

$$\Box \bar{h}_{\mu\nu} = 0 \tag{1}$$

which holds in the gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0\,. \tag{2}$$

However the description in terms of a 10-component tensor is still redundant. We can use the diffeomorphism invariance $x^\mu \to x'^\mu + \xi^\mu$ acting on the metric perturbation as

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} \tag{3}$$

to isolate the physical, radiative degrees of freedom by observing that a diffeomorphism preserving the gauge condition (2) satisfies the *Lorentz condition*

$$\Box \xi^\mu = 0\,. \tag{4}$$

As long as eq. (4) is satisfied we can then perform a gauge transformation to set to 0 some components of $\bar{h}_{\mu\nu}$. Note that this would *not be possible* if $T_{\mu\nu} \neq 0$ (it would appear on the right hand side of eq. (1)), as ξ^μ and $\bar{h}_{\mu\nu}$ would then satisfy different equations.

Let us thus choose a specific ξ^μ to impose 4 conditions on $\bar{h}_{\mu\nu}$. In particular $\bar{h} \to \bar{h}' = \bar{h} + \xi_\mu{}^{;\mu}$ so one can choose ξ_0, say, to set $\bar{h} = 0$, hence $\bar{h}_{\mu\nu} = h_{\mu\nu}$. The 3 ξ^is can be chose to make $h'_{0i} = h_{0i} + \xi_{i,0} + \xi_{0,i} = 0$ hence the 0-component of the Lorentz condition (2) becomes

$$-\dot{h}_{00} + h_{0i}{}^{;i} = 0\,, \tag{5}$$

but since h_{0i} have been made vanish this implies that h_{00} is time-independent, so it can be discarded as it cannot parametrize any radiative degree of freedom. Summarizing we have the 8 conditions

$$h_{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad h_{ij}{}^{;j} = 0 \tag{6}$$

defining the *transverse-traceless (TT)* gauge. The 2 radiative degrees of freedom of the TT gauge can be parametrized by

$$h_{ij}^{TT}(t,x) = \int \frac{d^3k}{(2\pi)^3} A_{ij}(\omega) e^{i(\omega t - \mathbf{kx})} \tag{7}$$

with $\omega = |\mathbf{k}|$, $\mathbf{k} = \omega\hat{\mathbf{n}}$ (we use $c = 1$ units) and the A_{ij} tensor can have only 2 independent non-vanishing components, that for a wave propagating in the z-direction (hence with $e_{iz} = 0$) can be written as

$$A_{ij}(\hat{\mathbf{z}}) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

A convenient way to obtain the TT part of a tensor is to apply the projection tensor

$$\begin{aligned} \Lambda_{ij,kl}(\hat{\mathbf{n}}) &\equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,, \\ P_{ij}(\hat{\mathbf{n}}) &\equiv \delta_{ij} - n_i n_j\,. \end{aligned} \tag{9}$$

A wave propagating along the generic direction $\hat{\mathbf{n}}$ can be obtained from eq. (8) by

$$A_{ij}(\hat{\mathbf{n}}) = R^{(z)}_{ii'}(\phi) R^{(y)}_{i'i''} e_{i''j''}(\hat{\mathbf{z}}) (R^{(y)})^{-1}_{j''j'}(\theta) (R^{(z)})^{-1}_{j'j}(\phi) \tag{10}$$

with

$$R^{(z)}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

$$R^{(y)}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{12}$$

yielding to

$$\begin{aligned} A_{xx}(\hat{\mathbf{n}}) &= h_+\left(\cos^2\theta\cos^2\phi - \sin^2\phi\right) + 2h_\times \cos\theta\sin\phi\cos\phi\,, \\ A_{yy}(\hat{\mathbf{n}}) &= h_+\left(\cos^2\theta\sin^2\phi - \cos^2\phi\right) - 2h_\times \cos\theta\sin\phi\cos\phi\,. \end{aligned} \tag{13}$$

For generic direction propagation it is also convenient to express the tensor e_{ij} in terms of 2 orthogonal unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ spanning the plane orthogonal to the propagation direction, so that $\hat{\mathbf{n}} \times \hat{\mathbf{u}} = \hat{\mathbf{v}}$ and $A_{ij} = e^+_{ij}h_+ + e^\times_{ij}h_\times$ with

$$\begin{aligned} e^+_{ij}(\hat{\mathbf{n}}) &= \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \\ e^\times_{ij}(\hat{\mathbf{n}}) &= \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \end{aligned} \tag{14}$$

We'll see in the next section that this are the only two components we need for an interferometer.

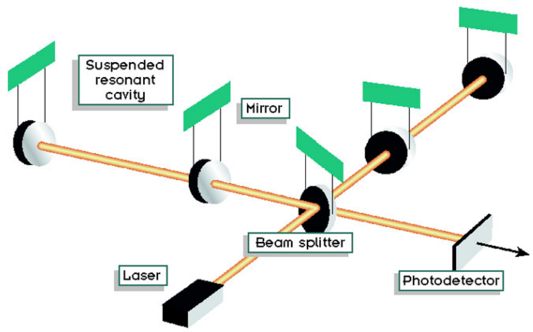

Figure 1: Interferometer scheme. Light emitted from the laser is shared by the two orthogonal arms after going through the beam splitter. After bouncing at the end mirrors it is recombined at the photo-detector.

2 Interaction detector - GWs

The arrival of a gravitational wave (GW) onto a detector will in general alter the state of motion of an observer and the goal of this section is to study the energy exchange of a GW with the laser interferometer components: beam splitter, end mirrors and the laser itself. Let us place them in the $z = 0$ plane at coordinates respectively $(0,0)$, $(L_x, 0)$ and $(0, L_y)$ and consider for simplicity a GW traveling along the z direction. In the Transverse-Traceless (TT) gauge the gravitational perturbation $h_{\mu\nu}$ has components $h_{0\mu} = 0$, $h_{xx} = -h_{yy} = h_+$, $h_{xy} = h_{yx} = h_\times$, and the the metric element restricted to the x-y plane can be written as

$$d\tau^2\big|_{z=0} = dt^2 - (1 + h_+)dx^2 - (1 - h_+)dy^2 - 2(1 + h_\times)dxdy\,. \qquad (15)$$

Interaction between the GW and matter Given two nearby geodesic parametrized by coordinates $x^i(\tau), x'^i(\tau)$, describing the motion of two test masses initially at rest ($dx^i/d\tau|_{\tau=0} = 0 = dx'^i/d\tau|_{\tau=0}$, $dt/d\tau|_{\tau=0} = 1 = dt'/d\tau|_{\tau=0}$), the (space) *coordinate* geodesic deviation $\xi^i \equiv x'^i - x^i$ at initial time satisfies (see sec. 1.3 of [4])

$$\left.\frac{d^2\xi}{d\tau^2}\right|_{\tau=0} = -\dot{h}_{ij}\left.\frac{d\xi^i}{d\tau}\right|_{\tau=0}\,, \qquad (16)$$

as in the TT gauge at linear order $\partial_\mu\Gamma^i_{00} = 0$ and $\Gamma^i_{0j} = \partial_0 h_{ij}/2$, showing that the *coordinate* distance of two particle initially at rest remain constants in the TT gauge under the influence of a GW. However the *proper* distance s between the x-mirror and the beam splitter changes:

$$s = L_x(1 + h_+)^{1/2} \simeq L_x\left(1 + \frac{1}{2}h_+\right)\,, \qquad (17)$$

whose second derivative gives a Newtonian-like equation of motion

$$\ddot{s} \simeq \frac{1}{2}\ddot{h}_+ L_x \simeq \frac{1}{2}\ddot{h}_+ s \qquad (18)$$

as to lowest order in h, $s \simeq L_x$. As the physical distance between two test masses (like the mirror and the beam splitter) is time dependent in the presence of a GW, it is expected that an energy transfer may take place between the GW and the interferometer, as first suggested in [1], by "putting in a spring" between objects in mutual motion.

The mirror and the beam splitter are hung to the ceiling of the laboratory, in a pendulum-like arrangement. The pendulum has a typical restoring period $T \sim \sqrt{l/g} \simeq$ few $\times 10^{-1}$ sec (being l the length of the suspension and g the gravity acceleration), implying that the mirror is approximately in free fall for GW signals whose frequency $f_{GW} \gg Hz$. On longer time scales energy transfer, and eventually dissipation, between the mirror and its suspension will take place.

In a real laboratory, positions are marked by rigid rulers and not by freely falling particles. It is thus instructive to consider the mirror-GW interaction in the *proper detector frame* (PDF). A standard results within General Relativity is that it always possible to set to zero the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ along an entire geodesic by using Fermi normal coordinates in the freely falling frame, see sec. 8.4 of [2]. Considering the relative coordinate distance x^i between an arbitrary space-time point and to the geodesic used to define Fermi normal coordinates, to linear order in x the metric is flat and at second order in x/λ (being λ the curvature scale of the space-time, $\lambda \sim |R_{0i0j}|^{-1/2}$) one has in the proper detector frame

$$d\tau^2_{PDF} \simeq dt^2 \left(1 + R_{0i0j}x^i x^j\right) + 2dtdx^i \left(\frac{2}{3}R_{0jik}x^j x^k\right) - dx^i dx^j \left(\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l\right) . \qquad (19)$$

The laboratory may not be in free fall with respect to earth gravity field, but if we restrict to motion in the $z = 0$ plane and to signals with $f_{GW} \gtrsim 10Hz$ all "environmental" effects can be safely neglected and the coordinate distance between neighboring geodesic results in

$$\ddot{\xi}^i_{PDF} = -R^i_{0j0}\xi^j_{PDF} . \qquad (20)$$

Observing that at $O(x/\lambda)$ the metric is flat and that the Riemann tensor components are not only *covariant* (as common in General Relativity) but actually *invariant* in the linearized theory, so that $R^i_{0j0} = -\ddot{h}_{ij}/2$, being h the TT metric perturbation, we recover eq. (18), which is frame-independent. Since in the proper detector frame coordinates track distances, from eq. (18) we infer that a test particle of mass μ under the influence of a GW is experiencing a time-dependent, Newtonian force $F^i = -\frac{\mu}{2}\ddot{h}^{ij}L^j$, allowing to derive the energy-transfer rate dE/dt due to the force via $dE/dt = F^i dx^i/dt$.

In the presence of the GW only, $F^i dx^i/dt$ is a total derivative and for an oscillating h it averages to 0: after a short transient during which the massive object is set in motion by the GW there is no more energy transfer on average over an oscillation cycle. However the interferometer mirrors are not exactly freely-falling, because of the suspensions hanging them causes dissipation, leading to the actual equation (dropping the proper detector frame subscript)

$$\ddot{\xi}^i + \frac{\omega_0}{Q}\dot{\xi}^i + \omega_0^2 \xi^i = -\frac{1}{2}\ddot{h}_{ij}\xi^j \,, \tag{21}$$

with $\omega_0 = 2\pi/T$ the pendulum proper angular frequency and the ω_0/Q term parametrizing the friction term, for which we assume $Q \gg 1$. Assuming for simplicity a GW of the type $h_+ = h_0 \cos(\omega_{GW} t)$, $h_\times = 0$, we have the solution

$$\xi(t) - L = \left(2Lh_0\omega_{GW}^2/\pi^2\right) \frac{(\omega_{GW}^2 - \omega_0^2)\cos(\omega_{GW}t) - \omega_{GW}\omega_0/Q \sin(\omega_{GW}t)}{\left(\omega_{GW}^2 - \omega_0^2\right)^2 + \omega_{GW}^2\omega_0^2/Q^2} \,, \tag{22}$$

showing that the massive object motion is in phase with the GW, apart for a term proportional to the friction which is responsible for the dissipation

$$\left\langle \frac{dE}{dt} \right\rangle \simeq \left(\mu L^2 h_0^2 \omega_{GW}^8/\pi^4\right) \frac{(\omega_{GW}^2 - \omega_0^2)\omega_0/Q}{\left[\left(\omega_{GW}^2 - \omega_0^2\right)^2 + \omega_{GW}^2\omega_0^2/Q^2\right]^2} \,. \tag{23}$$

In the limit $\omega_{GW} \gg \omega_0$ one obtains

$$\begin{aligned} \frac{dE}{dt} \simeq \frac{\mu}{Q\pi^4} L^2 h_0^2 \omega_{GW}^2 \omega_0 \quad &\simeq 2 \times 10^{12} h_0^2 \text{erg/sec} \\ &\times \left(\frac{Q}{10^8}\right)^{-1} \left(\frac{\omega_{GW}}{2\pi kHz}\right)^2 \left(\frac{\omega_0}{2\pi Hz}\right) \left(\frac{\mu}{1kg}\right) \left(\frac{L}{3km}\right)^2 , \end{aligned} \tag{24}$$

showing that the energy absorbed by the system from the GW is proportional to the friction term[1]. This is the energy absorbed by the massive object in order to keep its motion with a constant kinetic energy E_{kin} (averaged over a GW cycle)

$$\langle E_{kin} \rangle \simeq \mu \omega_{GW}^2 L^2 h_0^2/\pi^2 \,. \tag{25}$$

Interaction between the GW and and a Michelson-type interferometer. The laser in an interferometer monitors the distance between mirrors, and its electric field is also affected by the GW. The electric field in the two orthogonal beams in the interferometers "travel" from the beam splitter to the mirrors and back to recombine at the photo-detector at some time t. The phase of the electric field is conserved during free propagation, so at time t the electric fields

[1] In principle one could consider the re-emission by the system made by the beam-splitter and the mirror, which has a time-varying quadrupole $Q_{xx}(t) \simeq \mu\xi^2(t)$. From the standard Einstein quadrupole formula $dE/dt|_{emitted} = G_N \dddot{Q}_{ij}^2/5 \sim G_N \mu^2 L^4 \omega_{GW}^6 h_0^2$, which can be compared to the absorption given from eq. (24) to obtain
$dE/dt|_{emitted} \sim dE/dt|_{absorbed} \simeq 6 \times 10^{-22} \left(\frac{\omega_0}{2\pi Hz}\right)^{-1} \left(\frac{\omega_{GW}}{2\pi kHz}\right)^4 \left(\frac{Q}{10^8}\right) \left(\frac{\mu}{1kg}\right) \left(\frac{L}{3km}\right)^2$,
hence completely negligible.

recombine with the phase they inherit from the times $t_0^{(x)} \neq t_0^{(y)}$ when they left the beam splitter. Denoting by $E^{(x)}$ and $E^{(y)}$ the electric field coming respectively from the x and y arms, once they are recombining at the photo-detector after the beam splitter, we have

$$\begin{aligned} E^{(x)} &= -\frac{1}{2} E_0 e^{-i\omega_l t_0^{(x)}} \,, \\ E^{(y)} &= \frac{1}{2} E_0 e^{-i\omega_l t_0^{(y)}} \,, \end{aligned} \tag{26}$$

with ω_l being the laser angular frequency and the relative minus sign is due to the fact that reflection from opposite sides of the beam splitter brings a π shift in the phase [3]. Using the null geodesic in the metric given by eq. (15) to relate the time t to $t_0^{(x,y)}$, we have (see sec. 9.1 of [4]) at $O(h)$

$$\begin{aligned} t_0^{(x)} &= t - 2L_x - h_+(t - L_x)\sin(\omega_{GW} L_x)/\omega_{GW} \,, \\ t_0^{(y)} &= t - 2L_y + h_+(t - L_y)\sin(\omega_{GW} L_y)/\omega_{GW} \,. \end{aligned} \tag{27}$$

Substituting the above expression for $t_0^{(x,y)}$ in eqs. (26) and expanding at linear order in the GW amplitude one obtains

$$\begin{aligned} E^{(x)}(t) = -\frac{1}{2} E_0 \quad & e^{i(2\omega_l L + \phi_0)} \left[e^{-i\omega_l t} + \frac{i}{2} h_0 \omega_l L \frac{\sin(\omega_{GW} L)}{\omega_{GW} L} \times \right. \\ & \left. \left(e^{-i(\omega_l - \omega_{GW})t} e^{-i\omega_{GW} L} + e^{-i(\omega_l + \omega_{GW})t} e^{i\omega_{GW} L} \right) \right] \end{aligned} \tag{28}$$

where we have introduced $L \equiv (L_x + L_y)/2$ and $\phi_0 \equiv \omega_l \Delta L$, with $\Delta L \equiv L_x - L_y$, and where in $O(h_0)$ terms we have identified $L_x \simeq L_y \simeq L$. This shows that in each arm *sidebands* appear beside the career laser frequency at angular frequencies $\omega_l \pm \omega_{GW}$. The relative amplitude of the sidebands with respect to the career laser signal, for $\omega_{GW} \ll 1/L$, is given approximately by $h_0 L/\lambda_l \gg h_0$, being λ_l the laser wavelength.
Combining eq. (28) with the analogous formula for the y-arm one can determine the total electric field at the photo-detector $E_{pd}(t) = E^{(x)} + E^{(y)}$

$$E_{pd} = -iE_0 e^{-i\omega_l (t-2L)} \sin\left[\phi_0 + h_0 \omega_l L \frac{\sin(\omega_{GW} L)}{\omega_{GW} L} \cos(\omega_{GW}(t - L)) \right] . \tag{29}$$

Detecting a GW from the laser light associated with this electric field is still impractical: in order for the output power be *linear* in h_0 one would be sensitive also to the fluctuations in the laser power at a frequency $\sim \omega_{GW}/(2\pi)$, that would completely hide the GW signal. The solution adopted in actual observatories is to inject sidebands into the laser light so that the input electric field is given by

$$\begin{aligned} E_{in} &= E_0 e^{-i(\omega_l t + \Gamma \sin(\Omega_{mod} t))} \\ &\simeq E_0 \left[e^{-i\omega_l t} + \frac{\Gamma}{2} e^{-i(\omega_l + \Omega_{mod})t} - \frac{\Gamma}{2} e^{-i(\omega_L - \Omega_{mod})t} \right] , \end{aligned} \tag{30}$$

Working with $\phi_0 = 0$, so that $E_{pd} \propto h_0$ as per eq. (29), and combining the effects of the GW with the injected modulating sidebands, one has the output electric field

$$\begin{aligned} E_{out} \simeq \quad & -iE_0 e^{-i(\omega_l t+2L)}\left[\omega_l L \frac{\sin(\omega_{GW}L)}{\omega_{GW}L} h_0 \cos(\omega_{GW}t) \right. \\ & \left. +2\Gamma \sin(\Omega_{mod}\Delta L)\cos(\Omega_{mod}(t-2L))\right], \end{aligned} \tag{31}$$

and the GW signal can be read in the output power from the interference term between the career field and the sidebands oscillating at $\pm\Omega_{mod}$, giving a light power at the photo-detector P_{pd} (for $\omega_{GW}L \ll 1$)

$$P_{pd} = |E_{out}|^2 \simeq 2E_0^2 \Gamma \omega_l L h_0 \cos(\omega_{GW}t)\sin(\Omega_{mod}\Delta L)\sin(\Omega_{mod}(t-2L)) + \ldots \tag{32}$$

where only the term oscillating at $\pm\Omega_{mod} \pm \omega_{GW}$ has been explicitly shown, as it is the only one linear in the GW amplitude.
The output is still sensitive to the power fluctuation (of the sidebands), but now the GW signal has to compete with laser power fluctuation not at $\omega_{GW} \lesssim 10$ kHz, but at $\Omega_{mod} \sim 10$ MHz $\gg \omega_{GW}$ and this is a great advantage as laser power fluctuations generally decrease with frequency [6].

The interferometers actually used as GW observatories contain Fabry-Perot cavities in which the laser beam goes back and forth several times in each arm before recombining. At an effective level, the Fabry-Perot cavity allow to "fold" the light path enhancing its length without changing the region of the laboratory space traveled by the laser. This results in a phase-shift enhanced, in the case $\omega_{GW}L \gg 1$, by a factor $N \equiv 4F/\pi$ (see e.g. sec. 9.2 of [4]) where F is the *finesse* of the cavity related to the *storage time* (i.e. the average time spent by a photon in the cavity) τ_s by $F \simeq \pi\tau_s/L$: the effect of the Fabry-Perot cavity boils down to replace the term $h_0 L$ in the amplitude of the GW sidebands in eq. (28) with

$$h_0 NL \frac{1}{\left[1+(NL\omega_{GW}/2)^2\right]^{1/2}}\,, \qquad \text{for } \omega_{GW}L \ll 1\,. \tag{33}$$

For initial LIGO (Virgo) $N \simeq 60(20)$.

The laser electric fields recombines at the beam splitter to form an output beam directed to the photo-detector and a beam heading back to the laser. We have described how the electric field at the photo-detector depend on the GW in eq. (29). The electric field going back to the laser is $E_l = E^{(x)} - E^{(y)}$ (apart from an irrelevant overall phase), thus we can compute the total laser power

$$|E^{(x)} + E^{(y)}|^2 + |E^{(x)} - E^{(y)}|^2 = E_0^2\,, \tag{34}$$

which is unaffected by the GW, at least at $O(h)$. The appearance of the GW sidebands does not change the total power in the laser beam, but allows to identify a signal at a well-determined frequency and with amplitude highly enhanced with respect to h_0, see the $\omega_l L$ factor in eq. (32).

In order to complete the energy balance of the interferometer interacting with a GW however we still need to consider the radiation pressure exerting a force F_{rp} on the masses set in motion by the GW [5]. The laser power in each arm is approximately $P_{arm} = P_{laser}/2 = E_0^2/2$ and the radiation-pressure induces a force on each end mirror $F_{rp} = 2P_{arm} = P_{laser}$. As the masses are set in motion by the GW with velocity v, the radiation pressure force change because of the Doppler effect to $F_{rp} \simeq 2P_{arm}(1-2v)$, where v can be obtained by deriving eq. (22). The radiation pressure force has thus the effect of a friction term of the kind appearing in eq. (21), with an effective "quality factor" Q_{rp} approximately given by

$$Q_{rp} = \frac{m\omega_0}{4P_{arm}} \simeq 3 \times 10^{15} \left(\frac{P_{laser}}{100W}\right)^{-1} \left(\frac{m}{1kg}\right) \left(\frac{\omega_0}{1Hz}\right) . \tag{35}$$

Summing over the repeated bounces of each photon in the Fabry-Perot cavity, one can derive the dissipation due to radiation pressure [5]

$$\left.\frac{dE}{dt}\right|_{rp} \simeq 4P_{arm}\frac{N^2L^2}{\pi^2}h_0^2\omega_{GW}^2 \tag{36}$$

which can be obtained by substituting Q_{rp} in eq. (24) and replacing L with NL.

3 Detector geometry

We have seen in the previous section that an interferometer responds to the component combination $h_{xx} - h_{yy}$, which can be put into an invariant form by saying that the detector's output $o(t)$ is sensitive to the wave combination $D_{ij}h_{ij}$ with

$$D_{ij} = \frac{1}{2}\left(\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j - \hat{\mathbf{y}}_i\hat{\mathbf{y}}_j\right) . \tag{37}$$

Short-circuiting with the detector tensor D_{ij} with $e_{ij}(\hat{\mathbf{n}})$ computed in sec.1 one obtains the *pattern functions*, which relate the GW amplitude depending only on the source to its actual imprint in the detector.

By expressing $h(t)$ as

$$h(t) = D_{ij}h_{ij}^{TT}(t,\mathbf{x}_D) = D_{ij}e_{ij}^A(\hat{\mathbf{n}})h_A(t) \equiv f_+(\theta,\phi)h_+(t) + f_\times(\theta,\phi)h_\times(t) \tag{38}$$

we have introduced the pattern functions F_A, with their explicit dependence on θ, ϕ, the angles determining the arrival direction $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The choice of unit vectors $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ is unique up to a rotation in the plane they span, that is choosing $\hat{\mathbf{u}}' = \hat{\mathbf{u}}\cos\psi - \hat{\mathbf{v}}\sin\psi$ and $\hat{\mathbf{v}}' = \hat{\mathbf{v}}\cos\psi + \hat{\mathbf{u}}\sin\psi$ one obtains different polarization tensors

$$\begin{aligned} e_{ij}^{+'} &= \hat{\mathbf{u}}'_i\hat{\mathbf{u}}'_j - \hat{\mathbf{v}}'_i\hat{\mathbf{v}}'_j = e_{ij}^+\cos 2\psi - e_{ij}^\times \sin 2\psi \,, \\ e_{ij}^{\times'} &= \hat{\mathbf{u}}'_i\hat{\mathbf{v}}'_j + \hat{\mathbf{v}}'_i\hat{\mathbf{u}}'_j = e_{ij}^+\sin 2\psi + e_{ij}^\times \cos 2\psi \,, \end{aligned} \tag{39}$$

hence the patter functions obtained by contracting the polarization tensors with the detector tensor will transform as (with an abuse of notation we write with the same symbol the pattern functions and their transformed under a rotation depending on the extra variable ψ)

$$\begin{array}{l} F_+(\theta,\phi,\psi) = f_+\cos 2\psi - F_\times \sin 2\psi\,, \\ F_\times(\theta,\phi,\psi) = f_+\sin 2\psi + F_\times \cos 2\psi\,. \end{array} \tag{40}$$

When average quantities have to be computed, one has to evaluate terms like

$$\int d\psi d\Omega(\theta,\phi) F_{+,\times} F_{+,\times}\,.$$

However mixed integrals of the type $\int F_+F_\times$ vanish as they involve an odd number of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ vectors and $\hat{\mathbf{u}}$ ($\hat{\mathbf{v}}$) is odd (even) under $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$. Moreover the average of F_+^2 and $F_\times^2$ are equal (since $\int d\psi \cos 2\psi \sin 2\psi = 0$ and $\int d\psi \cos^2 2\psi = \int \sin^2 2\psi = \pi$) For an interferometer the pattern functions are

$$\begin{array}{rcl} f_+(\theta,\phi) &=& \frac{1}{2}\left(1+\cos^2\theta\right)\cos 2\phi \\ f_\times(\theta,\phi) &=& \cos\theta \sin 2\phi \end{array} \tag{41}$$

with average

$$\langle F_+^2 \rangle = \frac{1}{8\pi^2}\int F_+^2(\theta,\phi\psi)\, d\cos\theta d\phi d\psi = \frac{2}{5}\,. \tag{42}$$

4 Source's order of magnitudes

Interferometric detectors are very precise and rapidly responsive ruler, they can detect the change of an arm length down to values of 10^{-15} m, however not at all frequency scales. At very low frequency ($f_{GW} \lesssim 10$ Hz) the noise from seismic activity and generic vibrations degrade the sensitivity of the instrument, whereas at high frequency laser shot noise does not allow to detect signal with frequency larger than few kHz. Considering a binary system in circular orbit, its emission depend only on one parameter according to

$$F(v) = \frac{32\eta^2}{5G_N} v^{10}\left(1 + f_{v^2}(\eta)v^2 + f_{v^3}(\eta)v^3 + \ldots\right)\,. \tag{43}$$

where $\eta \equiv m_1 m_2/M^2$, $M \equiv m_1 + m_2$) and $v \equiv (G_N M \pi f_{GW})^{1/3}$ coincide with Newtonian velocity as it can be derived by using Kepler's law for circular orbits $R^3/T^2 = G_N M/(2\pi)^2$ and remembering that $f_{GW} = 2/T$. What is the typical length, mass, radius scale of the source? Using again third's Kepler law and circular motion we have

$$\begin{array}{rlcl} V = & (G_N M\pi f_{GW})^{1/3} & \simeq & 0.054\left(\frac{M}{M_\odot}\right)^{1/3}\left(\frac{f_{GW}}{10\text{Hz}}\right)^{1/3}, \\ r = & 2G_N M (G_N M \pi f_{GW})^{-2/3} & \simeq & 6.4\text{Km}\left(\frac{M}{M_\odot}\right)^{1/3}\left(\frac{f_{GW}}{10\text{Hz}}\right)^{-2/3}. \end{array} \tag{44}$$

It can also be interesting to estimate how long it will take to for a coalescence to take place. Using the lowest order expression for energy and flux, one has

$$\begin{aligned} -\eta M v \frac{dv}{dt} &= -\frac{32}{5G_N}\eta^2 v^{10} \implies \\ \frac{dv}{v^9} &= \frac{32\eta}{5G_N M} dt \implies \\ \frac{1}{v_i^8} - \frac{1}{v_f^8} &= \frac{256\eta}{5G_N M}\Delta t\,, \end{aligned} \tag{45}$$

for the time Δt taken for the inspiralling system to move from v_i to v_f. If $v_i \ll v_f$ we can estimate

$$\Delta t \simeq \frac{5G_N M}{256\eta} v_i^{-8} \simeq 1.4 \times 10^4 \text{sec} \frac{1}{\eta}\left(\frac{M}{M_\odot}\right)^{-5/3}\left(\frac{f_{iGW}}{10Hz}\right)^{-8/3}. \tag{46}$$

Note that v_f can be comparable to v_i for very massive systems, which enter the detector sensitivity band when $v_i \lesssim 1$. For an estimate of the maximum relative binary velocity during the inspiral, we can take the inner-most stable circular orbit v_{ISCO} of the Schwarzschild case, which gives

$$v_{ISCO} = \frac{1}{\sqrt{6}} \simeq 0.41\,. \tag{47}$$

The number of cycles N the GW spends in the detector sensitivity band can be derived by noting that

$$E = -\frac{1}{2}\eta M \left(G_N M \pi f_{GW}\right)^{2/3} \tag{48}$$

and

$$\begin{aligned} \Delta N(t) &= \int_{t_i}^{t} f_{GW}(t')dt' \implies \\ N(f_{GW}) &\simeq \int_{f_{iGW}}^{f_{GW}} f\frac{dE/df}{dE/dt}df \\ &\simeq \frac{5G_N M}{96\eta}\int_{f_{iGW}}^{f_{GW}} \left(G_N M\pi f\right)^{-8/3} df \\ &= \frac{1}{32\pi\eta}\left(G_N M\pi\right)^{-5/3}\left(\frac{1}{f_{iGW}^{-5/3}} - \frac{1}{f_{GW}^{-5/3}}\right) \\ &\simeq 1.5\times 10^5 \frac{1}{\eta}\left(\frac{M}{M_\odot}\right)^{-5/3}\left(\frac{f_{iGW}}{10\text{Hz}}\right)^{-5/3} \end{aligned} \tag{49}$$

We can finally obtain the time evolution of the GW frequency

$$\dot{f}_{GW} = \frac{96}{5}\pi^{8/3}\eta\left(G_N M\right)^{5/3} f_{GW}^{11/3} \tag{50}$$

which has solution

$$\begin{aligned} f_{GW}(t) &= \frac{1}{\eta^{3/8}\pi}\left(\frac{5}{256}\frac{1}{|t-t_{coa}|}\right)^{3/8} G_N M^{-5/8} \\ &= 151\text{Hz}\frac{1}{\eta^{3/8}}\left(\frac{M}{M_\odot}\right)^{-5/8}\left(\frac{|t|}{1\text{sec}}\right)^{-3/8}, \end{aligned} \tag{51}$$

which can be inverted to give

$$|t - t_{coa}| = \frac{5G_N M}{256\pi\eta}\left(\frac{1}{\pi G_N M f_{GW}}\right)^{5/3}. \tag{52}$$

The typical GW strain at the detector from a binary source is

$$h \sim \frac{2G_N \mu v^2}{d} \sim 10^{-22}\left(\frac{\mu}{M_\odot}\right)\left(\frac{v}{0.1}\right)^2\left(\frac{d}{Mpc}\right)^{-1} \tag{53}$$

5 An example: a specific waveform from binary inspiral

Starting from the GW expression in term of the source quadrupole

$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{2G_N}{d}\Lambda_{ij;kl}(\hat{\mathbf{n}})\ddot{M}_{kl}(t) \tag{54}$$

For $\hat{\mathbf{n}} = \hat{\mathbf{z}}$

$$\Lambda_{ij;kl}M_{kl} = \begin{pmatrix} (M_{xx} - M_{yy})/2 & M_{12} & 0 \\ M_{12} & (M_{yy} - M_{xx})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{55}$$

we have

$$\begin{aligned} h_+ &= G_N\frac{\ddot{M}_{xx} - \ddot{M}_{yy}}{d}, \\ h_+ &= \frac{2G_N\ddot{M}_{xy}}{d}. \end{aligned} \tag{56}$$

Assuming the sources are in circular motion, their relative distance can be parametrized as for

$$\begin{aligned} x(t) &= r\sin(\omega_s t), \\ y(t) &= -r\cos(\omega_s t), \\ z(t) &= 0, \end{aligned} \tag{57}$$

hence yielding to

$$\begin{aligned} \ddot{M}_{xx} &= -\ddot{M}_{yy} = 2\mu r^2\omega_S^2\cos(2\omega_s t) \\ \ddot{M}_{xy} &= 2\mu r^2\omega_S^2\sin(2\omega_s t) \end{aligned} \tag{58}$$

When the orbital planes is inclined by an angle ι with respect to the propagation direction (conventionally kept along the z-axis) one has to compute the rotated projected quadrupole tensor according to

$$M'_{ij} = R^{(y)}(\iota)_{ii'}M_{i'j'}\left(R^{(y)}\right)^{-1}_{j'j}(\iota) \tag{59}$$

and then project it with Λ to obtain

$$\Lambda_{ij,kl}M'_{kl} = \begin{pmatrix} \cos^2\iota M_{xx}/2 - M_{yy}/2 & \cos\iota M_{xy} & 0 \\ \cos\iota M_{xy} & M_{yy}/2 - \cos^2\iota M_{xx}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{60}$$

Now using the explicit expressions eqs. (58) one finds

$$\begin{aligned} h_+ &= \frac{1}{d} 4 G_N \mu \omega_S^2 R^2 \left(\frac{1+\cos^2\iota}{2}\right) \cos(2\omega_s t)\,, \\ h_\times &= \frac{1}{d} 4 G_N \mu \omega_S^2 R^2 \cos\iota \sin(2\omega_s t)\,. \end{aligned} \tag{61}$$

Note that the rotation (59) is not the most generic rotation setting the orbital plane at an angle ι with the view direction $(0,0,1)$: an additional rotation by ψ around the z axis is permitted. However this is not an additional degree of freedom with respect to the polarization angle ψ introduced in eq. (40) as it corresponds to a common rotation of the $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ vector in the orbital plane.

Using $f_{GW} = \omega_s/\pi$ and the explicit expression $f_{GW}(t)$ eq. (51) one has

$$\begin{aligned} h_+(t) &= \frac{1}{d} \left(G_N M_c\right)^{5/4} \left(\frac{5}{\tau}\right)^{1/4} \left(\frac{1+\cos^2\iota}{2}\right) \cos\Phi(\tau) \\ h_\times(t) &= \frac{1}{d} \left(G_N M_c\right)^{5/4} \left(\frac{5}{\tau}\right)^{1/4} (\cos\iota) \sin\Phi(\tau) \end{aligned} \tag{62}$$

For data analysis we actually need this expression in frequency space, so let us compute it for the + polarization. From the leading quadrupole emission derive the waveform formula ($\iota = 0$)

$$\tilde{h}_+(f) = \left(\frac{5}{24}\right)^{1/2} \frac{\pi^{-2/3}}{D} (G_N M_c)^{5/6} f^{-7/6} e^{i(\phi(t_*) - 2\pi t_* f - \pi/4)}\,.$$

Hint: Use the *stationary phase approximation method*:

$$\int_{-\infty}^{\infty} e^{i\phi(t)} dt \simeq e^{i\phi(t_*)} \int_{-\infty}^{\infty} e^{i\ddot{\phi}(t_*)(t-t_*)^2/2} = e^{\phi(t_*)} e^{-i\pi/4} \sqrt{\frac{2\pi}{\ddot{\phi}(t_*)}}$$

and apply to the Fourier transform of $h(t) = 4 G_N M_c^{5/3} (\pi f)^{2/3} \cos(\phi(t))/D$ by expanding around the t_* defined by the relationship $2\pi f = \dot{\phi}(t_*)$. Be careful to express $\cos(\phi(t)) = (e^{i\phi} + e^{-i\phi})/2$ and observe that only one of the two exponentials contribute to the result and finally substitute $\dot{f} = 96/5\pi(\pi M_c f)^{5/3} f^2$.

The most commonly used approximant is defined in the frequency domain as *TaylorF2*:

$$\tilde{h}_+(f) \propto e^{-i(\psi(f) + \pi/4)}$$

with

$$
\begin{aligned}
\psi(f) &= 2\pi \int^f f' \frac{dt}{df'} df' - 2\pi f t_*(f) \\
&= \frac{2}{G_N M} \int^f \left(v^3(f') - v^3(f) \right) \frac{dt}{df'} df' \\
&= \frac{2}{G_N M} \int^{v(f)} (v'^3 - v^3) \frac{dE/dv}{dE/dt} dv \\
&= \frac{2}{G_N M} \int^f \left(v'^3 - v^3(f) \right) \left[\frac{-\eta M v'}{-32/(G_N 5)\eta^2 v'^{10}} \right] dv' \\
&= \frac{5}{16\eta} \int^f \left(v'^{-6} - \frac{v^3(f)}{v'^9} \right) \left(1 + c_{1PN} v^2 + \ldots \right) dv' \\
&= \frac{5}{16\eta} v(f)^{-5} \left[\left(\frac{1}{8} - \frac{1}{5} \right) + \left(\frac{1}{6} - \frac{1}{3} \right) c_{1PN} + \ldots \right] + \psi_0 \\
&= -\frac{3}{128\eta v^5(f)} \left(1 + \frac{20}{9} c_{1PN} v^2(f) + \ldots \right) + \psi_0 \,.
\end{aligned}
$$

It is useful to introduce

$$
v \equiv (\pi G_N M f_{GW})^{1/3} = (G_N M \omega)^{1/3} \,. \tag{63}
$$

The most commonly used approximant is defined in the frequency domain as *TaylorF2*:

$$
\begin{aligned}
\psi(f) &= 2\pi f t_* + \phi_{ref} - 2 \int^f \omega \frac{dt}{df} df \\
&= \phi_{ref} + \frac{2}{G_N M} \int^f (v_f^3 - v^3(f')) \frac{v(f')}{f'} \frac{dE/dv}{dE/dt} \frac{f'}{v(f')} \frac{dv}{df'} df' \\
&= 2\pi \int^f (v_f^3 v^{-2} - v) \left[\frac{-\eta M v}{-32/(G_N 5)\eta^2 v^{10}} \right] \frac{1}{3} df' \\
&= \frac{5\pi G_N M}{48\eta} \int^f (v_f^3 v^{-11} - v^{-8}) \left(1 + c_{1PN} v^2 + \ldots \right) df' \\
&= \frac{5}{48\eta} (\pi G_N M f)^{-8/3} \left[\left(-\frac{3}{8} + \frac{3}{5} \right) + \left(-\frac{1}{2} + 1 \right) c_{1PN} \ldots \right] \\
&= \frac{3}{128\eta v^5(f)} \left(1 + \frac{20}{9} c_{1PN} v^2(f) + \ldots \right)
\end{aligned} \tag{64}
$$

It may also be useful to have an expression of the phase in time-domain. Let us now relate the phase of the waveform to the dynamics of the sources by defining

$$
\begin{aligned}
\Delta\phi(t) &= 2\pi \int_{t_0}^t f_{GW}(t') dt' = 2 \int_{v(t_0)}^{v(t)} \omega(v) \frac{dE/dv}{dE/dt} dv \\
&= \frac{2}{G_N M} \int_{v(t_0)}^{v(t)} v^3 \frac{dE/dv}{dE/dt} dv \\
&= \frac{5}{16\eta} \int_{v(t_0)}^{v(t)} \frac{1}{v^6} \frac{1 + e_{v^2} v^2 + \ldots}{1 + f_{v^3} v^3 + f_{v^4} v^4 + \ldots} dv \,,
\end{aligned} \tag{65}
$$

where we have inserted the formal Taylor expansion of the energy and flux as functions of v and we have substituted $v = (G_N M\omega)^{1/3}$ (that is valid for circular orbits). We now see that we have different possibilities to compute the phase

- Take the $\dot{\phi} \propto F/(dE/dv)$ expression by re-expanding at the numerator and truncating the v-series at the appropriate order: *TaylorT4*
- Expand the inverse of fraction in the above formula and truncate at some finite v-order $\rightarrow$ *TaylorT2*
- keep numerator and denominator as they are: *TaylorT1.*

6 Elements of data analysis

The output of the detector $o(t)$ is a scalar time domain function, which in general will result of the addition of a part $h(t)$ linear in the impinging GW $h_{ij}(t, \mathbf{x}_D)$ at the location detector $\mathbf{x}_D$ and the instrumental noise $n(t)$. Usually the output of the detector is linearly related to the GW amplitude locally in the frequency space, i.e.

$$\tilde{h}(f) = \tilde{T}_{ij}(f)\tilde{h}_{ij}(f) \tag{66}$$

where $T_{ij}(f)$ is the *transfer function* of the system and

$$\tilde{o}(f) = \tilde{h}(f) + \tilde{n}(f)\,. \tag{67}$$

If the noise is *stationary* then different Fourier components are uncorrelated (see derivation below) and we can write

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \delta(f - f')\frac{1}{2}S_n(f)\,, \tag{68}$$

which defines the *noise correlation function* $S_n(f)$, or *noise power spectral density*, with dimensions Hz^{-1}. Note also that $S_n(-f) = S_n(f)$.

The average in eq. (68) is taken over many noise realizations, but we have only one detectors, so it should be actually replaced over different time span averages:

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \frac{1}{N}\sum_{i=1}^{N}\tilde{n}_i(f)\tilde{n}_i^*(f')\,,$$

where

$$\tilde{n}_i(f) = \int_{t_i - T/2}^{t_i + T/2} n(t)e^{2\pi i f t}dt\,,$$

and $t_i = (\ldots, -2T, -T, 0, T, 2T, \ldots)$. We define the Fourier Transform of a function defined on an interval as

$$\tilde{n}(f) = \int_{t_i - T/2}^{t_i + T/2} n(t)e^{2\pi i f t}\,dt \qquad \text{with } n(t+T) = n(t) \tag{69}$$

with the periodicity requirement implying the Fourier transform $\tilde{n}(f)$ be discrete, with support at $f_n = n/T$, i.e. with frequency resolution $\Delta f = 1/T$. This discreteness condition on the frequency automatically ensure that the inverse Fourier transform returns a periodic function[2]:

$$n(t) = \frac{1}{T}\sum_{n\in\mathbb{N}} \tilde{n}(f)e^{-2\pi i n t/T}\,. \tag{71}$$

A very welcome by-product of the definition (69), restricting the integration domain to a finite interval and imposing periodicity condition on $n(t)$, is

$$\frac{1}{T}\int_{t_i-T/2}^{t_i+T/2} e^{2i\pi f t}dt = \frac{e^{2i\pi f t_i}}{2\pi i f T}\left(e^{2\pi i f T/2} - e^{-2\pi i f T/2}\right) = e^{2\pi i f t_i}\frac{\sin(\pi f T)}{\pi f T} = \delta_{f,0} \tag{72}$$

which is reminiscent of the standard $\int e^{2i\pi f t}dt = \delta(f)$ valid for functions defined on the entire real axis. The only difference between the finite segment and the entire real axis case is that here we deal with a dimension-less, discrete Kronecker delta rather than a delta function with dimension of time, so that the identification

$$\delta(f) \to T\delta_{f,0} \tag{73}$$

can be made.

Let us consider than the definition (68) and exploit the stationarity property of the noise:

$$\begin{aligned}
\langle \tilde{n}(f)\tilde{n}(f')\rangle &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \langle n(t)n(t')\rangle e^{2\pi i(ft+f't')} \\
&= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \langle n(t+T_s)n(t'+T_s)\rangle e^{2\pi i(ft+f't'+T_s(f+f'))} \\
&= \delta_{f+f',0}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \langle n(t)n(t')\rangle e^{2\pi i f(t-t')}
\end{aligned} \tag{74}$$

where we have averaged over a time coordinate T_s and used stationarity to substitute $n(t+T_s) \to n(t)$ inside the average. Short-circuiting the above with

[2] Note that the definition (69) for a function defined on an interval is *not* equivalent to

$$\tilde{n}(f) \neq \int_{-\infty}^{\infty} n(t)e^{2\pi i f t}\theta(t-t_i+T/2)\theta(t_i+T/2-t)dt\,.$$

Had we used this definition one would have dealt with a non-analytic integrand in the time integral, that would have had discontinuities in his derivatives. As the discontinuities in the direct space functions are related to "bad" behaviour at infinity via

$$\frac{d^m n(t)}{dt^m} = \int df(2\pi i f)^m \tilde{n}(f)e^{2\pi i f t} \tag{70}$$

one would expect in general a non-physical high power in $\tilde{n}(f)$ for large $|f|$, or that $f^m\tilde{n}(f)$ must not be integrable, that is $\lim_{f\to\infty}|f^{m+1}n(f)|^2 \neq 0$. It is a general rule that the the large f behaviour of $\tilde{n}(f)$ depends on the continuity property of $n(t)$.

the definition (68) and the correspondence (73) we obtain

$$S(f) = 2\frac{\langle|\tilde{n}(f)|^2\rangle}{T} = 2\langle|\tilde{n}(f)|^2\rangle\Delta f\,. \tag{75}$$

We can make the further connection to the noise auto-correlation function

$$R(\tau) \equiv \langle n(t+\tau)n(t)\rangle\,, \tag{76}$$

white noise corresponding to $R(\tau) \propto \delta(\tau)$. By noting that

$$\begin{aligned} & \langle\int df' \int df\, n(f)n(f')e^{-2\pi i(f(t+\tau)+f't)}\rangle \\ = & \frac{1}{2}\int df S_n(f)e^{-2\pi i f\tau} \end{aligned} \tag{77}$$

we are enabled to interpret the noise spectral density function as the Fourier transform of the noise correlation function

$$\frac{1}{2}S_n(f) = \int d\tau R(\tau)e^{2\pi i f\tau} \tag{78}$$

and hence

$$R(\tau) = \frac{1}{T}\sum_{n\in\mathbb{N}} e^{-2\pi i f\tau}\frac{S_n(f)}{2}\,. \tag{79}$$

The factor 1/2 in the definition is conventionally inserted as

$$\begin{aligned} \langle n^2(t)\rangle &= \frac{1}{T^2}\sum_n\sum_{n'}\langle n(f)n(f')\rangle e^{2\pi i(ft+f't')} \\ &= \frac{1}{2T}\sum_{n\in\mathbb{N}} S_n(f) = \frac{1}{T}\sum_{n\geq 0} S_n(f) \end{aligned} \tag{80}$$

and the factor 1/2 disappears when sum is taken over positive frequencies only (we neglect subtleties about the $n = 0$ mode). The power spectral density of white noise is f-independent.

Actually in practice the time domain noise function will be discrete as well, implying that what we'll be really using are

$$\begin{aligned} \tilde{n}(f = k/T) &= \Delta t \sum_{j=0}^{N-1} e^{2\pi i jk\Delta t/T} \\ n(t = j\Delta t) &= \Delta f \sum_{j=0}^{N-1} e^{2\pi i jk\Delta t/T} \end{aligned} \tag{81}$$

with $\Delta f\Delta t = 1/N$ with $T/\Delta t = N$. With a finite sampling size there has to be a maximum frequency, called the *Nyquist frequency*

$$f_{Nyquist} = \frac{1}{2\Delta t} \tag{82}$$

so that we have N points for $n(t)$ and $N/2+1$ for $\tilde{n}(f)$ if N is even, as we will assume. Note that $\tilde{n}(f=0)$ is real as well as $n(f=N/(2T))$ so that the information stored in the N real numbers of $n(t)$ is fully equivalent to the information stored in the $N/2+1$ numbers making up $\tilde{n}(f)$, 2 of which are real and $N/2-1$ of which are complex.

In particular the Parseval identity has a discrete counterpart

$$\Delta f \sum_{k=0}^{N/2} |\tilde{n}(k/T)|^2 = \Delta t \sum_{j=0}^{N-1} n^2 \left(j\Delta t\right) . \tag{83}$$

6.1 Matched filtering

The signal amplitude is much smaller than the noise, just think of the earth gravitational field that is responsible for $h \sim 10^{-9} \gg 10^{-21}$. However if the signal is known in advance, we can correlate detector's output $o(t)$ with our expectation and dig it out of the noise floor. We thus have to *filter* the detector output to highlight the signal. An important quantity for any experiment is the *signal-to-noise ratio (SNR)* we are going to define now. It must involve a ratio S/N of a quantity linear in the signal h possibly filtered in order to enhance it and a quantity representative of the noise. We want to choose the filter function so to maximize the SNR, i.e. the filter has to *match* the signal. We can assume that by linearly filtering the detector output we can pick only the signal part $h(t)$ in $o(t)$ and we can tentatively define the numerator of the SNR as

$$S = \int dt \langle o(t) \rangle K(t) \, , \tag{84}$$

and since $\langle n(t) \rangle = 0$ we have

$$S = \int dt \langle h(t) \rangle K(t) = \int df \tilde{h}(f) \tilde{K}^*(f) \, , \tag{85}$$

where for simplicity we have moved back to continuum time-frequency space. For the SNR denominator N we want an estimator of the noise. A reasonable guess would be the root mean square of the detector output in absence of the signal, i.e.

$$N^2 \stackrel{?}{=} \langle o^2(t) \rangle - \langle o(t) \rangle^2 |_{h=0} = \langle n(t) \rangle^2 \tag{86}$$

but we want the overall filter scale to drop out of the SNR, so we'd better define

$$\begin{aligned} N^2 &= \int dt\, dt'\, K(t) K(t') \langle n(t) n(t') \rangle \\ &= \int df\, df' \langle n(f) n(f') \rangle e^{2\pi i (ft+ft')} \tilde{K}(f) \tilde{K}(f') \\ &= \frac{1}{2} \int df S_n(f) |\tilde{K}(f)|^2 \, . \end{aligned} \tag{87}$$

We have constructed then our SNR as

$$\frac{S}{N} = \frac{2^{1/2}\int_{-\infty}^{\infty} df \tilde{h}(f)\tilde{K}^*(f)}{\left(\int_{-\infty}^{\infty} df S_n(f)|\tilde{K}(f)|^2\right)^{1/2}} . \tag{88}$$

edm

Matched filtering. By correlating the detector's output with a pre-computed signal we can *filter* out noise and obtain the *signal-to-noise ratio (SNR)*. The filter must *match* the signal and let it be a *linear* operator, so being $o(t) = n(t) + h(t)$ the detector's output and $K(t)$ the filter we can write the *signal* S tentavely as

$$S = \int dt o(t) K(t) ,$$

and assuming $\langle n(t) \rangle = 0$ we have

$$S = \int dt h(t) K(t) = \int df \tilde{h}(f)\tilde{K}^*(f) ,$$

where for simplicity we have moved back to continuum time-frequency space. For the SNR denominator N we want an estimator of the noise. A reasonable guess would be the root mean square of the detector output in absence of the signal, i.e.

$$N^2 \stackrel{?}{=} \langle o^2(t) \rangle - \langle o(t) \rangle^2|_{h=0} = \langle n(t) \rangle^2$$

but we want the overall filter scale to drop out of the SNR, so we'd better define

$$\begin{aligned} N^2 &= \int dt\, dt'\, K(t)K(t')\langle n(t)n(t')\rangle \\ &= \int df\, df' \langle n(f)n(f')\rangle e^{2\pi i(ft+ft')}\tilde{K}(f)\tilde{K}(f') \\ &= \frac{1}{2}\int df S_n(f)|\tilde{K}(f)|^2 , \end{aligned}$$

where in the last passage we have introduced

$$\langle n(f)n(f')\rangle \equiv \frac{1}{2} S_n(f)\delta(f+f') ,$$

being $S_n(f)$ the *noise spectral density* and brackets $\langle \ldots \rangle$ stand for *average over many noise realizations.* Note we have assumed that noise at different frequencies is uncorrelated, i.e. noise at each frequency f correspond to a Gaussian process with variance $S_n(f)$. We have constructed then our SNR as

$$SNR \equiv \frac{S}{N} = \frac{2^{1/2}\int_{-\infty}^{\infty} df \tilde{h}(f)\tilde{K}^*(f)}{\left(\int_{-\infty}^{\infty} df S_n(f)|\tilde{K}(f)|^2\right)^{1/2}} .$$

In order to find the filter function maximizing the SNR we define a positive definite scalar product

$$(A|B) \equiv 2 \int_{-\infty}^{\infty} df \, \frac{A(f)B^*(f)}{S_n(f)} \,,$$

which is real if we assume that $A^*(f) = A(-f)$ and $B^*(f) = B(-f)$, as it is for the Fourier transform of real functions. Verify that the SNR can be re-written as

$$SNR = \frac{(\tilde{u}|\tilde{h})}{(\tilde{u}|\tilde{u})^{1/2}}$$

with $\tilde{u}(f) \equiv 1/2 S_n(f) \tilde{K}(f)$. Show that the vector u that maximizes its scalar product with h must be $\tilde{u} \propto \tilde{h}$, i.e.

$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)}$$

up to an inessential f-independent constant. Verify that one obtains

$$SNR = \left[2 \int_{-\infty}^{\infty} df \, \frac{|\tilde{h}(f)|^2}{S_n(f)} \right]^{1/2} \,.$$

In the previous exercise one has assumed perfect knowledge of the signal. What if we do not know the exact *time location* of the $h(t)$?
Hint:
The Fourier transforms of a function $h(t)$ and of its time-shifted $h_{t_0}(t) \equiv h(t-t_0)$ are in the following relationship

$$\tilde{h}(f) = \tilde{h}_{t_0}(f) \, e^{2\pi i f t_0} \,.$$

If we try to match data $\tilde{h}(f)$ with a *template* signal $\tilde{h}_{t_0}(f)$ allowing for a generic time-shift t_0, we obtain a t_0-dependent SNR given by

$$SNR(t_0) = \sqrt{2} \frac{\int_{-\infty}^{\infty} df \, \frac{\tilde{h}(f)\tilde{h}^*_{t_0}(f)}{S_n(f)} e^{2\pi i f t_0}}{\left(\int_{-\infty}^{\infty} df \, |\tilde{h}(f)|^2 / S_n(f) \right)^{1/2}} \,,$$

where in the denominator we have used that $|\tilde{h}(f)| = |\tilde{h}_{t_0}(f)|$.

Let us consider now the case of a constant phase offset between the template and the signal: consider a signal of the type $h(t) \propto \cos(2\pi f_s t)$, i.e. $\tilde{h}(f) \propto \delta(f - f_s) + \delta(f + f_s)$ and a template of the type $h_{t,\bar{\phi}}(t) \propto \cos(2\pi f_t t + \bar{\phi})$, verify that the SNR would be proportional to $\delta(f_s - f_t) \cos(\bar{\phi})$, i.e. it would not be optimal even for perfect matching of the frequencies.

In view of the result of the previous exercise it is useful to maximize the signal-template correlation over unknown relative phase $\bar{\phi}$.

Show that is possible to maximize analytically over $\bar{\phi}$.
Hint:

Consider the matched-filter integral bewteen a signal and a matching template with a constant phase shift $\bar{\phi}$ (assuming h_t normalized to 1):

$$
\begin{aligned}
SNR^2(t_0 = 0) &= \sqrt{2}\int_{-\infty}^{\infty} S_n^{-1}(f)\tilde{h}(f)\tilde{h}^*_{t,\bar{\phi}}(f)df \\
&= \sqrt{2}\int_{0}^{\infty} S_n^{-1}(f)\left(\tilde{h}(f)\tilde{h}_t(-f)e^{-i\bar{\phi}} + \tilde{h}(-f)\tilde{h}_t(f)e^{i\bar{\phi}}\right) df \\
&= \sqrt{2}\left(\mathcal{R}\cos\bar{\phi} + \mathcal{I}\sin\bar{\phi}\right) ,
\end{aligned}
$$

where the real quantities $\mathcal{R}$ and $\mathcal{I}$ are defined as

$$
\begin{aligned}
\mathcal{R} &\equiv \int_0^{\infty} S_n^{-1}(f)\left(\tilde{h}(f)\tilde{h}^*_t(f) + \tilde{h}^*(f)\tilde{h}_t(f)\right) df &= 2Re\int_0^{\infty}\frac{\tilde{h}(f)\tilde{h}_t(f)}{S_n(f)}df\,, \\
\mathcal{I} &\equiv i\int_0^{\infty} S_n^{-1}(f)\left(h^*(f)h_t(f) - h(f)h^*_t(f)\right) df &= 2Im\int_0^{\infty}\frac{\tilde{h}(f)\tilde{h}^*_t(f)}{S_n(f)}df\,.
\end{aligned}
$$

Maximize the output of the matched filter depends over $\bar{\phi}$ to obtain:

$$
\frac{dSNR}{d\bar{\phi}} = 0 \implies \cos\bar{\phi} = \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2+\mathcal{I}^2}}\,, \qquad \sin\bar{\phi} = \frac{\mathcal{I}}{\sqrt{\mathcal{R}^2+\mathcal{I}^2}}\,,
$$

which give the SNR maximized over $\bar{\phi}$

$$
\left(\overset{Max}{\bar{\phi}}\;SNR\right)^2 = \sqrt{2\left(\mathcal{R}^2+\mathcal{I}^2\right)}\,.
$$

Note that there is an efficient way to compute both the quantities $\mathcal{R}$ and $\mathcal{I}$: it is by computing the *complex inverse Fourier transform*

$$
\rho(t) \equiv \sqrt{2}(\tilde{h}_t|\tilde{h}_t)^{-1/2}\int_0^{\infty}\frac{\tilde{h}(f)\tilde{h}^*_t(f)}{S_n(f)}e^{2\pi i f t}df
$$

and we have

$$
\overset{Max}{\phi_0}\;SNR(t) = 2|\rho(t)|\,.
$$

7 Theoretical modeling

Derive the linearized Einstein equations in vacuum

$$
R_{\mu\nu} = -\frac{1}{2}\left(\Box h_{\mu\nu} + \partial_\mu\partial_\nu h^\alpha{}_\alpha - \partial_\mu\partial_\alpha h^\alpha{}_\nu - \partial_\nu\partial_\alpha h^\alpha{}_\mu\right) = 0
$$

and then simplify them to

$$
\Box\bar{h}_{\mu\nu} = 0\,,
$$

via the *Lorentz* gauge condition $\bar{h}_{\mu\nu},^{\nu} = 0$, having defined

$$
\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha\,.
$$

Use the a coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ to show that

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi_\alpha{}^{,\alpha} \,,$$

and verify that the Lorentz condition is preserved by a coordinate transformation satisfying $\Box\xi_\mu = 0$.

Verify that a coordinate transformation can be chosen so that the D'Alambert equation for gravitational perturbation in vacuum and the Lorentz condition can be used to set $h_{0i} = 0$ (suitably choosing the ξ_i), $h_{00} = 0$ using the time part of the Lorentz gauge condition, $h^\alpha{}_\alpha = 0$ by suitably choosing ξ_0 and $h_{0i}n^i = 0$, being $\vec{n}$ the propagation direction of the wave, using the spatial part of the Lorentz gauge condition.

Derive that gravitational waves have only 2 degrees of freedom that can be isolated by using the projector

$$\Lambda_{ij,kl} \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

where $P_{ij} \equiv (\delta_{ij} - n_i n_j)$.

Gravity in vacuum has only 2 degrees of freedom, but show that this is not the case in presence of a non-vanishing energy-momentum tensor.

Verify that $\Lambda_{ij,kl}$ is a projector.

Verify that the wave equation has the *retarded* Green function

$$G_{ret}(t,\vec{x}) = -\frac{1}{4\pi|\vec{x}|}\delta(t - |\vec{x}|) \,,$$

and verify that

$$\tilde{G}_{ret}(\omega,\vec{k}) = \int dt d^3x G_{ret}(t,\vec{x}) e^{i(\omega t - \vec{k}\cdot\vec{x})} = \frac{1}{(\omega + i\epsilon)^2 - |\vec{k}|^2}$$

Use the identity

$$\theta(s) = i \lim_{\epsilon\to 0^+} \int \frac{d\omega}{2\pi}\frac{e^{-i\omega s}}{\omega + i\epsilon} \,.$$

Verify that the wave equation has the *advanced* Green function

$$G_{adv}(t,\vec{x}) = -\frac{1}{4\pi|\vec{x}|}\delta(t + |\vec{x}|) \,,$$

and verify that

$$\tilde{G}_{adv}(\omega,\vec{k}) = \int dt d^3x G_{adv}(t,\vec{x}) e^{i(\omega t - \vec{k}\cdot\vec{x})} = \frac{1}{(\omega - i\epsilon)^2 - |\vec{k}|^2}$$

Verify that

$$\begin{aligned} G_{ret}(t,\vec{x}) &= -i\theta(t)\left(\Delta_+(t,\vec{x}) - \Delta_-(t,\vec{x})\right) \,, \\ G_{adv}(t,\vec{x}) &= i\theta(-t)\left(\Delta_+(t,\vec{x}) - \Delta_-(t,\vec{x})\right) \,, \end{aligned}$$

where

$$\Delta_\pm(t,\vec{x}) \equiv \int_{\mathbf{k}} \frac{e^{\mp ikt+i\mathbf{k}\cdot\mathbf{x}}}{2k}$$

Show that the Feynman Green function G_F defined by

$$G_F(t,\mathbf{x}) \equiv -i\int_{\mathbf{k}} \frac{d\omega}{2\pi}\frac{e^{-i\omega t+i\mathbf{k}\vec{\mathbf{x}}}}{\omega^2+k^2-i\epsilon}$$

is equivalent to

$$G_F(t,\mathbf{x}) = \theta(t)\Delta_+(t,\mathbf{x})+\theta(-t)\Delta_-(t,\mathbf{x})\,.$$

Derive the relationship

$$G_F(t,\mathbf{x}) = \frac{i}{2}\left(G_{adv}(t,\mathbf{x})+G_{ret}(t,\mathbf{x})\right)+\frac{\Delta_+(t,\mathbf{x})+\Delta_-(t,\mathbf{x})}{2}\,.$$

By integrating over $\mathbf{k}$ the G_F in the $\sim 1/(k^2-\omega^2)$ representation, show that G_F implements boundary conditions giving rise to field h behaving as

$$h(t,\mathbf{x}) \sim \int d\omega e^{-i\omega t+i|\omega|r}\,,$$

corresponding to out-going (in-going) wave for $\omega > (<)0$. Since an $\omega < 0$ solution is equivalent to a $\omega > 0$ solution propagating backward in time, this result can be interpreted by saying that using G_F results into having pure out-going (in-going) wave for $t \to \pm\infty$.

Derive the linearized Einstein tensor

$$G_{\mu\nu} = -\frac{1}{2}\left(\Box\bar{h}_{\mu\nu}+\eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma}-\partial_\mu\partial_\alpha\bar{h}^\alpha{}_\nu-\partial_\nu\partial_\alpha\bar{h}^\alpha{}_\mu\right) = 8\pi G_N T_{\mu\nu}\,.$$

Use the conservation of the energy momentum tensor to show that

$$\begin{aligned} T^{0k}x^j+T^{0j}x^k &= \left(T^{0l}x^jx^k\right)_{,l}+\dot{T}^{00}x^jx^k\,, \\ \frac{1}{2}\left(T^{ij}x^kx^l\right)_{,ij} &= T^{kl}+\frac{1}{2}\ddot{T}^{00}x^kx^l-\left(\dot{T}^{0k}x^l+\dot{T}^{0l}x^k\right)\,, \end{aligned}$$

Use the result of the above exercises to show that the solution to the linearized Einstein equations is

$$h^{TT}_{ij} = 4G_N\Lambda_{ij;kl}\int d^3x\frac{T_{kl}(t-|x-x'|)}{|x-x'|}\,.$$

Expand

$$\begin{aligned} |\vec{x}-\vec{x}'| &= r-\hat{n}\cdot\vec{x}'+\frac{1}{2r}\left(\delta_{ij}r'^2-x'^ix'^j\right)n^in^j+rO((x'/r)^3)\,, \\ \frac{1}{|\vec{x}-\vec{x}'|} &= \frac{1}{r}\left[1+\hat{n}\cdot\vec{x}'-\frac{1}{2r^2}\left(\delta_{ij}r'^2-3x'^ix'^j\right)n^in^j+rO((x'/r)^3)\right]\,, \end{aligned}$$

to find

$$h_{ij}^{TT} \simeq \frac{4G_N}{r}\Lambda_{ij,kl}\left(\frac{1}{2}\ddot{Q}_{kl}(t-r)+\dot{T}_{kl}(t-r)\vec{x}\cdot\hat{n}+\ldots.\right)$$

What is the expansion parameter?

Let us split in the Einstein equations the part linear in the gravitational perturbation from the rest:

$$R_{\mu\nu}|_{O(h)} = 8\pi G_N\left(T_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}-\frac{1}{8\pi G_N}R_{\mu\nu}|_{O(h^2)}\right)$$

so that we consider the $O(h^2)$ of the Ricci tensor as the energy-momentum tensor of the GWs.

$$R^{(2)}_{\mu\nu} = -\frac{1}{4}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}+\ldots$$

where ... stand for terms which are vanishing when averaged over a wavelength and the TT gauge condition $h_{\mu\nu}{}^{,\nu}$, $h^{\alpha}{}_{\alpha}=0$ as well as the equations of motion are used (and assuming $h\to 0$ sufficiently fast at spatial infinity). One is then lead to consider

$$t^{GW}_{\mu\nu} = \frac{1}{32\pi G_N}\langle\partial_\mu h^{TT}_{\alpha\beta}\partial_\nu h^{TT\alpha\beta}\rangle\,.$$

Verify that $t^{TT}_{\mu\nu}$ is invariant under linear coordinate transformation.
Derive

$$t^{GW}_{00} = \frac{1}{16\pi G_N}\langle\dot{h}_+^2+\dot{h}_\times^2\rangle\,,$$

from which the energy flux can be derived (via energy-momentum conservation)

$$\frac{dE}{dt} = \frac{d}{dt}\int d^3x t^{GW\,00} = -\int d^3x t^{GW\,0i}{}_{,i} = \int dA n^i t^{GW}_{0i}\,.$$

Now observe that for a wave $h(t,|\vec{x}|)=h(t-r)$, hence $t_{0r,r}=\dot{t}_{00}$ to finally obtain

$$\frac{dE}{dAdt} = \frac{1}{16\pi G_N}\langle\dot{h}_+^2+\dot{h}_\times^2\rangle$$

Re-write the energy flux as

$$\frac{dE}{dt} = \frac{r^2}{8G_N}\int\frac{d\Omega}{4\pi}\Lambda_{ij,kl}\langle\dot{h}_{ij}\dot{h}_{kl}\rangle\,,$$

and obtain

$$\frac{dE}{dt} = \frac{G_N}{2}\int\frac{d\Omega}{4\pi}\Lambda_{ij,kl}\langle\dddot{Q}_{ij}\dddot{Q}_{kl}\rangle$$

at leading order. Now use

$$\begin{aligned}\int\frac{d\Omega}{4\pi}n^in^j &= \frac{1}{3}\delta_{ij}\,,\\ \int\frac{d\Omega}{4\pi}n^in^jn^kn^l &= \frac{1}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\,,\\ \int\frac{d\Omega}{4\pi}n^{i_1}\ldots n^{i_n} &= \frac{1}{(2n+1)!!}\left(\delta_{i_1i_2}\ldots\delta_{i_{n-1}i_n}+\ldots\right)\,,\end{aligned}$$

to derive

$$\frac{dE}{dt} = \frac{G_N}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle \left[\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right] = \frac{4G_N}{5}\langle \dddot{Q}^T_{ij}\dddot{Q}^T_{ij}\rangle \,,$$

being $Q^T_{ij} \equiv Q_{ij} - \frac{1}{3}\delta_{ij}\delta_{kl}Q_{kl}$.

Write the GW in the TT gauge beyond quadrupole approximation:

$$h^{TT}_{ij} = \frac{4G_N}{D}\Lambda_{ij;kl}\left(\frac{1}{2}\ddot{Q}_{ij} + \int d^3x'\dot{T}_{kl}(t-r,x')x'^m n_m + \ddot{T}_{kl}\frac{1}{2}x'^r x'^s n_r n_s + \dots\right) \,,$$

show that the NLO term in the expansion can be written as

$$\begin{aligned}\int \mathrm{d}^d x\, T^{ij}x^k = & \ \frac{1}{6}\int \mathrm{d}^d x\, \ddot{T}^{00}x^i x^j x^k \\ & + \frac{1}{3}\int \mathrm{d}^d x\, \left(\dot{T}^{0i}x^j x^k + \dot{T}^{0j}x^i x^k - 2\dot{T}^{0k}x^i x^j\right) \,.\end{aligned}$$

Use that

$$\int d^3x\left[T_{ij}x_k + T_{ki}x_j + T_{jk}x_i\right] = \frac{1}{2}\int d^3x\left[\dot{T}_{i0}x_j x_k + \dot{T}_{k0}x_i x_j + \dot{T}_{k0}x_i x_j\right] \,,$$

and that

$$\int d^3x\left[T_{ij}x_k - T_{ki}x_j\right] = \int d^3x\left[\dot{T}_{j0}x_i x_k - \dot{T}_{k0}x_i x_j\right] \,,$$

to derive that

$$\int d^3x T_{ij}x_k = \frac{1}{6}\int d^3x\ddot{T}_{00}x_i x_j x_k + \frac{1}{3}\left(\dot{T}_{0i}x_j x_k + \dot{T}_{0j}x_i x_k - 2\dot{T}_{0k}x_i x_j\right) \,,$$

where we have an *electric octupole* involving a third order moment of $\ddot{T}_{00}$ and *magnetic quadrupole* incolving second order moment of $\dot{T}_{0i}$. Show that when squaring h^{TT} to compute the energy there are no surviving terms mixing the electric quadrupole with the octupole or the magnetic quadrupole, nor the octupole with the quadrupole. Show that the next-to-leading order emission terms are v^2 suppressed with respect to the leading ones.
Note that to get the NLO emission right, one needs to expand the NNLO in the h^{TT}_{ij} and get its product with the leading h^{TT}_{ij} contribution coming from the mass qaudrupole at leading order. Result can be seen in arXiv:0912.4254.

Show that the electromagnetic vector has 2 propagating degrees of freedom, 1 constrained degree of freedom and 1 pure guauge degree of freedom by breaking the Maxwekk equation $F^{\mu\nu}_{\ \ \ \nu} = 4\pi J^\mu$ into

$$\begin{aligned} -\nabla^2(\dot{a} + A^0) &= 4\pi\rho \,, \\ -\Box\tilde{A}^i &= 4\pi\tilde{J}^i \,, \\ \partial_i\left(\ddot{a} + \dot{A}^0\right. &= 4\pi j) \,, \end{aligned} \tag{89}$$

where the SVT decomposition $A_i = \tilde{A}_i + \partial_i a$ has been adopted.

Show that the metric degrees of freedom are 2 constrained scalar, 1 constrained transver vector and one transverse trace-less symmetric tensor.

Demonstration:

Assuming that $h_{\mu\nu} \to 0$ as $r \to \infty$, $h_{\mu\nu}$ can be decomposed into a number of irreducible quantities ϕ, β_i, b, ψ, σ_i, λ and $h_{ij}^{(\mathrm{TT})}$ via the definitions

$$\begin{aligned} h_{00} &= -2\phi\,, \\ h_{0i} &= \beta_i + \partial_i b\,, \\ h_{ij} &= h_{ij}^{(\mathrm{TT})} - 2\psi\delta_{ij} + \tfrac{1}{2}\left(\partial_i F_j + \partial_j F_i\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right) e\,, \end{aligned}$$

together with the constraints

$$\begin{aligned} \partial_i b_i &= 0 \quad (1\,\text{constraint}) \\ \partial_i \sigma_i &= 0 \quad (1\,\text{constraint}) \\ \partial_i h_{ij}^{(\mathrm{TT})} &= 0 \quad (3\,\text{constraints}) \\ \delta^{ij} h_{ij}^{(\mathrm{TT})} &= 0 \quad (1\,\text{constraint}) \end{aligned}$$

and boundary conditions

$$b \to 0\,, \quad \sigma_i \to 0\,, \quad \lambda \to 0\,, \nabla^2\lambda \to 0$$

as $r \to \infty$. The quantities β_i and $\partial_i b$ are the transverse and longitudinal pieces of h_{0i}. This decomposition is unique (show).

Under coordinate transformations ξ^a with $\xi^a \to 0$ as $r \to \infty$ one has, for the parametrization

$$\xi_\mu = (\alpha, \xi_i + C_{,i})\,,$$

with $\partial_i \xi_i = 0$ and $\alpha, C \to 0$ as $r \to \infty$. Under coordinate transformation one has

$$\begin{aligned} \phi \to \phi' &= \phi - \dot{\alpha}\,, \\ b \to b' &= b + \alpha - \dot{C}\,, \\ \beta_i \to \beta_i' &= \beta_i - \dot{\xi}_i\,, \\ \psi \to \psi' &= \psi + \tfrac{1}{3}\nabla^2 C\,, \\ e \to e' &= e - 2C\,, \\ F_i \to F_i' &= F_i - 2\xi_i\,, \\ h_{ij} \to h_{ij}' &= h_{ij}\,, \end{aligned}$$

The following combinations of these functions are gauge invariant:

$$\begin{aligned} \Phi &\equiv \phi + \dot{b} - \tfrac{1}{2}\ddot{e}\,, \\ \psi^e &\equiv \psi + \tfrac{1}{6}\nabla^2 e\,, \\ V_i &\equiv \beta_i - \tfrac{1}{2}\dot{F}_i\,, \\ h_{ij}^{\mathrm{TT}} & \end{aligned}$$

In the Newtonian limit Φ reduces to the Newtonian potential and $\Psi = \Phi$.

Decomposing the energy-momentum tensor we have

$$\begin{aligned}
T_{00} &= \rho\,, \\
T_{0i} &= S_i + \partial_i S\,, \\
T_{ij} &= P\delta_{ij} + \sigma_{ij} + \tfrac{1}{2}\left(\partial_i\sigma_j + \partial_j\sigma_i\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\sigma,
\end{aligned}$$

together with the constraints

$$\begin{aligned}
\partial_i S_i &= 0\,, \\
\partial_i \sigma_i &= 0\,, \\
\partial_i \sigma_{ij} &= 0\,, \\
\delta^{ij}\sigma_{ij} &= 0,\,,
\end{aligned}$$

and boundary conditions

$$S \to 0, \quad \sigma_i \to 0, \quad \sigma \to 0 \quad \nabla^2\sigma \to 0$$

as $r \to \infty$. These quantities are not all independent. The variables ρ, P, S_i and σ_{ij} can be specified arbitrarily, energy-momentum conservation then determines the remaining variables S, σ, and σ_i via

$$\begin{aligned}
\nabla^2 S &= \dot{\rho}\,, \\
\nabla^2 \sigma &= -\tfrac{3}{2}P + \tfrac{3}{2}\dot{S}\,, \\
\nabla^2 \sigma_i &= 2\dot{S}_i\,.
\end{aligned}$$

The Einsten tensor at linear order is given by

$$\begin{aligned}
G_{00} &= 2\nabla^2\psi^e\,, \\
G_{0i} &= -\tfrac{1}{2}\nabla^2 V_i + 2\psi^e_{,i}\,, \\
G_{ij} &= \delta_{ij}\left[2\ddot{\psi}^e + \nabla^2(\Phi - \psi^e)\right] + (\psi^e - \Phi)_{,ij}
\end{aligned}$$
$$-\tfrac{1}{2}\left(\dot{V}_{i,j} + \dot{V}_{j,i}\right) - \tfrac{1}{2}\Box h^{(\mathrm{TT})}_{ij}$$

and the Einstein's equations read:

$$\begin{aligned}
\nabla^2\psi^e &= 4\pi G_N \rho\,, \\
\nabla^2 V_i &= -16\pi G_N S_i\,, \\
\nabla^2 \Phi &= 4\pi G_N(\rho + 3P - 3\dot{S})\,, \\
\Box h^{(\mathrm{TT})}_{ij} &= -16\pi\sigma_{ij}\,.
\end{aligned}$$

Consider a binary system in circular motion in the $x-y$ plane, hence the quadrupole components are

$$\begin{aligned}
Q_{xx} &= \mu x^2 = \mu r^2\cos^2(\omega t) = \tfrac{1}{2}\mu r^2\cos(2\omega t) + const \\
,Q_{xy} &= \mu xy = \mu r^2\cos(\omega t)\sin(\omega t) = \tfrac{1}{2}\mu r^2\sin(2\omega t) + const\,, \\
Q_{yy} &= \mu y^2 = \mu r^2\sin^2(\omega t) = -\tfrac{1}{2}\mu r^2\cos(2\omega t) + const\,, \\
Q_{iz} &= 0\,.
\end{aligned}$$

Substitute in the formula for dE/dt in terms of the quadupole and find

$$\frac{dE}{dt} = \frac{32 G_N \eta^2 M^2 R^4 \omega^6}{5} = \frac{32\eta^2}{G_N 5}v^{10}\,,$$

where we have substituded $v^2 = (\omega r)^2 = G_N M/r$, $\eta \equiv \mu/M$.

Use the energy of circular orbit $E = -\frac{1}{2}\eta M v^2$ to derive the time to coalescence Δt_c

$$\Delta t_c = \int \frac{dE}{dv}\frac{1}{\dot{E}}dv \simeq \frac{5}{256\eta} G_N M v^{-8}\,,$$

and the equation for the GW frequency $f_{GW} = \omega/\pi = v^3/(\pi G_N M)$ evolution

$$\frac{df_{GW}}{dt} = \eta\frac{96\pi}{5}(\pi G_N M)^{5/3} f_{GW}^{11/3}\,.$$

Suppose gravitational radiation is quantized, use the quadrupole formula to estimate how many gravitons are emitted per orbital period for different systems:

- Hydrogen atom
- Earth-sun system

Result:
The number of gravitons at frequency f_{GW} can be estimated as

$$\frac{1}{hf_{GW}}\frac{2\pi}{\omega}\frac{dE}{dt} = \frac{2}{hf_{gw}^2}\frac{dE}{dt}\,,$$

with $dE/dt = 32/5 G_N \eta^2 M^2 R^4 \omega^6$ for a generic system (not necessarily gravitationally bound) . Substitue the values for an hydrogen atom, having $\eta \simeq 2\times 10^{-3}$, $M \simeq 1$GeV, $R \simeq a_0 = 5\times 10^{-10}$m, $\omega = 5.1\times 10^{16}$Hz (angular velocity of circular orbits), to obtain a flux $G_N \times de/dt \simeq 2\times 10^{-101}$, giving a number of gravitons per orbit of 7.4×10^{-48}.

In case the system emitting gravitational waves is bound together by gravitational interaction, use the quadrupole formula to estimate how many gravitational quanta would be emitted per orbital period.
Result: For the system earth-sun $\eta = 3\times 10^{-7}$, $M = 2\times 10^{33}$ gr, $R = 1.5\times 10^8$km, $\omega = 2\pi/(\pi\times 10^7\text{sec})$ one finds a number of emitted gravitons per orbit equal to 1.6×10^{48}.

Find the numer of emitted gravitons for a lab system experiment (masses $\sim$ kg, frequency $\sim$ Hz, size $\sim$ m). Can one envisage a system to test the eventual quantum nature of gravity?

Consider a massless scalar field ϕ interacting with non-relativistic sources $J_{1,2}$ and derive the non-relativistic potential between J_1 and J_2 mediated by ϕ.
Hint: Fromt the action

$$\begin{aligned} S &= \int dt d^d x \left[-\frac{1}{2}\left(\partial\Phi(t,\mathbf{x})\right)^2 + (J_1(t,\mathbf{x}) + J_2(t,\mathbf{x}))\Phi(t,\mathbf{x})\right] \\ &= \int_{\mathbf{k}} \frac{dk_0}{2\pi}\left[\Phi(k_0,\mathbf{k})\Phi^*(k_0,\mathbf{k})\left(k_0^2 - k^2\right) + \frac{1}{2}\left((J_1(k_0,\mathbf{k}) + J_2(k_0,\mathbf{k}))\Phi^*(k_0,\mathbf{k}) + c.c.\right)\right] \end{aligned}$$

consider the partition function $Z_0[J]$ obtained by performing the Gaussian integral over each of the (continously) infinite $\phi(k_0,\mathbf{k})$ fields, considered independent for different $(k_0, \mathbf{k})$.

$$
\begin{aligned}
Z_0[J] \equiv & \int \mathcal{D}\Phi \exp\left\{ i \int_{\mathbf{k}} \frac{dk_0}{2\pi} \left[\frac{1}{2}\left(k_0^2 - k^2\right) \Phi(k_0, \mathbf{k})\Phi(-k_0, -\mathbf{k}) + \right.\right. \\
& \left.\left. J(k_0, \mathbf{k})\Phi(-k_0, -\mathbf{k}) + i\epsilon|\Phi(k_0, \mathbf{k})|^2 \right] \right\} ,
\end{aligned}
$$

where the ϵ has been added term ensure convergence for $|\Phi| \to \infty$. Perform the Gaussian integral by using the new variable

$$
\Phi'(k_0, \mathbf{k}) = \Phi(k_0, \mathbf{k}) + (k_0^2 - k^2 + i\epsilon) J(k_0, \mathbf{k}) ,
$$

enabling to rewrite

$$
\begin{aligned}
Z_0[J] = & \exp\left[-\frac{i}{2} \int_{\mathbf{k}} \frac{dk_0}{2\pi} \frac{J(k_0, \mathbf{k}) J^*(k_0, \mathbf{k})}{k_0^2 - k^2 + i\epsilon} \right] \\
& \times \int \mathcal{D}\Phi' \exp\left\{ i \int_{\mathbf{k}} \frac{dk_0}{2\pi} \left[\frac{1}{2}\left(k_0^2 - k^2\right) \Phi'(k_0, \mathbf{k}) \Phi^{*'}(k_0, \mathbf{k}) + i\epsilon|\Phi'|^2 \right] \right\} .
\end{aligned}
$$

The integral over Φ' gives an uninteresting normalization factor $\mathcal{N}$, thus we can write the result of the functional integration as

$$
\begin{aligned}
Z_0[J] &= \mathcal{N} \exp\left[-\frac{i}{2} \int_{\mathbf{k}} \frac{dk_0}{2\pi} \frac{J_1(k_0, \mathbf{k}) J_2(-k_0, -\mathbf{k}) + J_1(-k_0, -\mathbf{k}) J_2(k_0, \mathbf{k})}{k_0^2 - k^2 + i\epsilon} \right] \\
&= \mathcal{N} \exp\left[- \int dt d^3x \, G_F(t - t', \mathbf{x} - \mathbf{x}') J(t, \mathbf{x}) J(t', \mathbf{x}') \right] ,
\end{aligned}
$$

where terms J_1^2 and J_2^2 have been neglected.
Explain why it is correct to neglect them when computing the effective potential between sources 1 and 2. Verify the result for the effective potential after plugging in the Feynman Green function.

Consider the same problem for a vector field A_μ coupled to a source J_μ. Is it possible to invert the quadratic term of A_μ in the Lagrangean?
Hint:

$$
\begin{aligned}
S_A &= \int dt d^dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu \right) \\
&\to \int dt d^dx \left(-\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \partial^\mu A^\nu - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 + J_\mu A^\mu \right)
\end{aligned}
$$

show that it is not possible to perform the Gaussian integral unless the term proportional to $1/\xi$ is added.

Find the *propagator*, i.e. the inverse of the quadratic term of the Lagrangean by writing the quadratic Lagrangean in Fourier space as

$$
-\frac{1}{2} \tilde{A}_\mu(k) \tilde{A}^*_\nu(k) \left(k^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \right) \equiv -\frac{1}{2} \tilde{A}_\mu(k) \tilde{A}^*_\nu(k) B^{\mu\nu}(k)
$$

and looking for a tensor B^{-1} such that $(B^{-1})_{\mu\rho} B^{\rho\nu} = \delta_\mu^\nu$.

Derive the potential between 2 non-relativistic sources mediated by the vector field A_μ.

For the gravitational field the quadratic Lagrangean is

$$S_{EH} = -\frac{1}{64\pi G_N}\int dt d^3x \left[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h \partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu\right] .$$

Schematically the the Lagrangean density in Fourier space up to quadratic order is of the type

$$\mathcal{L} = -\frac{1}{64\pi G_N}\left(A^{\mu\nu\rho\sigma}(k_\alpha)h_{\mu\nu}(k)h_{\rho\sigma}(-k)\right) - \frac{1}{2}h_{\mu\nu}(k)T^{\mu\nu}(-k)$$

with

$$\begin{aligned} A_{\mu\nu\rho\sigma} \quad = \quad & \frac{1}{2}k^2\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right) \\ & -k^2\eta_{\mu\nu}\eta_{\rho\sigma} \\ & +k_\mu k_\nu \eta_{\rho\sigma} + \eta_{\mu\nu}k_\rho k_\sigma \\ & -\frac{1}{2}\left(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}\right) , \end{aligned}$$

so that it leads to the equation of motions

$$A^{\mu\nu\rho\sigma}(k)h_{\rho\sigma} = 16\pi G_N T^{\mu\nu} ,$$

Note however that analogously to the electromagnetic case it is impossible to solve the equation

$$A^{\mu\nu\rho\sigma}X_{\alpha\nu\beta\sigma} = \delta^\mu_\alpha \delta^\rho_\beta ,$$

for the unknown X since $A^{\mu\nu\rho\sigma}$ has no inverse. To solve the problem one can add a gauge fixing term. Find the propagator corresponding to the gauge fixing term $\frac{1}{2}\Gamma^\mu_{\alpha\mu}\Gamma^\nu_{\beta\nu}g^{\mu\nu}$.

Consider the source in (space) Fourier domain

$$\tilde{J}_A(k) = m\int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\mathbf{x}}\delta^{(3)}(\mathbf{x}-\mathbf{x}_A) ,$$

and the quasi static limit of the Green function

$$\frac{1}{k^2-\omega^2+i\epsilon} \to \frac{1}{k^2}\left(1+\frac{\omega^2}{k^2}+\ldots\right) ,$$

and derive the leading behaviour of the effective potential.

Result:

$$\begin{aligned} & -i\int dt_A dt_B \int_{\mathbf{k}} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t_A-t_B)+i\mathbf{k}(\mathbf{x}_A-\mathbf{x}_B)}}{k^2-\omega^2+i\epsilon} \\ \simeq \quad & -i\int dt_A dt_B \int_{\mathbf{k}} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t_A-t_B)+i\mathbf{k}(\mathbf{x}_A-\mathbf{x}_B)}}{k^2}\left(1+\frac{\omega^2}{k^2}+\ldots\right) \\ \simeq \quad & -i\int dt_A dt_B \delta(t_A-t_B)\int_{\mathbf{k}}\frac{e^{i\mathbf{k}(\mathbf{x}_A-\mathbf{x}_B)}}{\mathbf{k}^2}\left(1+\frac{\partial_{t_1}\partial_{t_2}}{\mathbf{k}^2}\right) \\ = \quad & -i\int dt\left[\frac{1}{4\pi|\mathbf{x}|}+\frac{O(v^2)}{|\mathbf{x}|}\right] \end{aligned}$$

one recover the instantaneous $1/r$, Newtonian interaction (plus $O(v^2)$ corrections). Note that we have implemented the substitution $\omega = -i\partial_{t_1} = i\partial_{t_2}$ in order to work out the systematic expansion in v.

Consider the neglected terms of the type $J^2_{1,2}$, involving $G_F(0,0)$, which is divergent. Show that it can be absorbed by shifting (of an infinite amount) the value of $m_{1,2}$.
This is an example of regularization of a computation giving an un-physical infinite result via the addition of a *local* counter term, i.e. a counter-term involving a non-negative power of the momentum k. This is OK as long as we do not aim at *predicting* the parameters of the theory, but rather take them as input to compute other quantities like interaction potential.

If there are interaction terms which cannot be written with terms linear or quadratic in the field, the Gaussian integral cannot be done analytically, so the rule to follow is to separate the quadratic action $S_{quad}[\Phi]$ of the field (its kinetic term) and Taylor expand all the rest: for an action $S = S_{quad} + (S - S_{quad})$ one would write

$$\begin{aligned} Z[J] &= \int \mathcal{D}\Phi e^{iS} \\ &= \int \mathcal{D}\Phi e^{iS_{quad}} \left[1 + i\left(S - S_{quad}\right) - \frac{\left(S - S_{quad}\right)^2}{2} + \ldots\right], \end{aligned}$$

As long as the $S - S_{quad}$ contains only polynomials of the field which is integrated over, the integral can be *perturbatively* performed analytically, inheriting the rules from Gaussian integrals

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + Jx} dx &= \left(\frac{2\pi}{a}\right)^{1/2} \exp\left(\frac{J^2}{2a}\right), \\ \int e^{-\frac{1}{2}x^i A_{ij} x^j + J^i x_i} dx_1 \ldots dx_n &= \frac{(2\pi)^{n/2}}{(detA)^{1/2}} \exp\left(\frac{1}{2} J^t A^{-1} J\right), \\ \int x^k e^{-\frac{1}{2}ax^2 + Jx} dx &= \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{d}{dJ}\right)^k \exp\left(\frac{J^2}{2a}\right) \end{aligned}$$

from which it follows that

$$\begin{aligned} \int x^{2n} e^{-\frac{1}{2}ax^2} dx &= \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{d}{dJ}\right)^{2n} \exp\left(\frac{J^2}{2a}\right)\Bigg|_{J=0} \\ &= \frac{(2n-1)!!}{a^n} \left(\frac{2\pi}{a}\right)^{1/2}. \end{aligned}$$

Re-derive the previous of the Newtonian potential by making the 2nd order Taylor expansion described above.
Hint:

$$\begin{aligned} e^{iS_{eff}} &= Z[J, x_A]|_{J=0} = \int \mathcal{D}\Phi e^{iS_{quad}} \times \{1 \\ &\quad -\frac{1}{2}\left[\sum_A m_A \int dt_A \Phi\left(t_A, \mathbf{x}_A(t_A)\right)\right]\left[\sum_B m_B \int dt_B \Phi\left(t_B, \mathbf{x}_B(t_B)\right)\right] + \ldots\Big\} \end{aligned}$$

If following Green function's lines all the vertices can be connected the diagram is said *connected*, otherwise it is said *disconnected*: only connected diagrams contribute to the effective action. We will not demonstrate this last statement, but its proof relies on the following argument. Taking the logarithm of we get

$$S_{eff} = -i \log Z_0[0] - i \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} Z_0^{-1}[0] \left(Z[0] - Z_0[0] \right)^n \ .$$

All terms with $n > 1$ describe disconnected diagrams, and some disconnected diagrams can also be present in the $n = 1$ term. However the $n = 1$ disconnected contribution is precisely canceled by the $n = 2$ terms. Verify that the 4th order expansion involving a double ϕ exchange is canceled by the square of the 2nd order term in $(Z - Z_0)^2$.

Derive the 1PN potential.

Result:

Considering that the gravitational field can be parametrized as

$$g_{\mu\nu} = e^{2\phi/\Lambda} \left(\begin{array}{cc} -1 & A_i/\Lambda \\ A_j/\Lambda & e^{-4\phi/\Lambda}\gamma_{ij}/\Lambda + A_i A_j/\Lambda^2 \end{array} \right) ,$$

with $\gamma_{ij} \equiv \delta_{ij} + \gamma_{ij}$, to obtain a gauge-fixed Einstein Hilbert-Lagrangean of the type

$$S_{E-H+GF} = \int dt d^3x \sqrt{\gamma} \left(-4\partial\phi^2 + 4\dot{\phi}^2 + \frac{1}{2} F_{ij}^2 + \vec{\nabla} \cdot \vec{A} + \ldots \right) ,$$

where terms non-relevant for the present computation has been omitted. Using the couping to world-line

$$S_{pp} = -\frac{m}{\Lambda} \int dt \left(\phi \left[1 + \frac{3}{2} v^2 \right] + \frac{\phi^2}{2\Lambda} - A_i v_i + \ldots \right) ,$$

computes the three contributions to the effective action at 1PN order

1. one ϕ exchange:

$$\begin{aligned} iS_{eff}|_\phi &= -i^3 \frac{m_1 m_2}{8\Lambda^2} \int dt_1\, dt_2 \delta(t_1 - t_2) \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2))}}{k^2} \left[1 + \frac{3}{2} \left(v_1^2 + v_2^2 \right) \right] \left(1 + \frac{\partial_{t_1} \partial_{t_2}}{k^2} \right) \\ &= i \frac{m_1 m_2}{8\Lambda^2} \int dt \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1(t) - \mathbf{x}_2(t))}}{k^2} \left[1 + \frac{3}{2} \left(v_1^2 + v_2^2 \right) \right] \left(1 + \frac{v_1^i v_2^j k_i k_j}{k^2} \right) \\ &\simeq i \frac{G_N m_1 m_2}{r} \left[1 + \frac{3}{2} \left(v_1^2 + v_2^2 \right) + \frac{1}{2} \left(v_1 v_2 - (v_1 \hat{r})(v_2 \hat{r}) \right) \right] \end{aligned}$$

where in the last pasage we have equated $\Lambda^2 = (32\pi G_N)^{-1}$ to recover the standard Newtonian potential,

2. one A_i exchange

$$iS_{eff}|_{A_i} = i^3 \frac{m_1 m_2}{2\Lambda^2} \int dt \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}}{k^2} \delta_{ij} v_1^i v_2^j = -i \frac{4G_N m_1 m_2}{r} v_1 v_2 \,,$$

3. two ϕ exchange

$$iS_{eff}|_{\phi^2} = -i^5 \frac{m_1^2 m_2}{128\Lambda^4} \int dt \left(\int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}}{k^2} \right)^2 = i \frac{G_N^2 m_1^2 m_2}{2r^2} \,,$$

where the formulae

$$\begin{aligned} \int_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{1}{k^{2\alpha}} &= \frac{\Gamma(d/2-\alpha)}{(4\pi)^{d/2}\Gamma(\alpha)} \left(\frac{r}{2}\right)^{2\alpha-d} \,, \\ \int_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{k^i k^j}{k^{2\alpha}} &= \left(\frac{1}{2}\delta^{ij} - \left(\frac{d}{2} - \alpha + 1\right) \hat{r}^i \hat{r}^j \right) \frac{\Gamma(d/2-\alpha+1)}{(4\pi)^{d/2}\Gamma(\alpha)} \left(\frac{r}{2}\right)^{2\alpha-d-2} \,, \end{aligned}$$

have been used.

The three contributions add up together to give the Einstein-Infeld-Hoffman potential

$$V_{EIH} = -\frac{G_N m_1 m_2}{2r} \left[3\left(v_1^2 + v_2^2\right) - 7 v_1 v_2 - (v_1 \hat{r})(v_2 \hat{r}) \right] + \frac{G_N^2 m_1 m_2 (m_1 + m_2)}{2r^2} \,.$$

(the Lagrangean is minus the potential!).

Derive the time to coalescence

$$t_c - t(f) = \frac{5}{256\pi f} \left(\pi M_c f\right)^{-5/3} \,.$$

Derive the time evolution of he instantaneous frequency

$$f(t) = \left(\frac{5}{256\pi(t_c - t)} \right)^{3/8} \left(\pi G_N M_c\right)^{-5/8}$$

After introducing

$$v \equiv \left(\pi G_N M f_{GW}\right)^{1/3} = \left(G_N M \omega\right)^{1/3} \,,$$

derive the leading order expression for

$$\phi(t) = 2\pi \int_{t_0}^{t} f dt' = 2\pi \int_{f_0}^{f} \frac{f'}{\dot{f}'} df' = \frac{1}{16\eta v^5}$$

Derive the emitted energy spectrum of gravitational waves

$$\begin{aligned} \frac{dE}{df} &= \frac{\pi}{2G_N} f^2 r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\iota \, |h|^2 \left[\left(\frac{1+\cos^2\iota}{2} \right)^2 + \cos^2\iota \right] d\iota \\ &= \frac{\pi^{2/3}}{3G_N} \left(G_N M_c\right)^{5/3} f^{-1/3} \,, \end{aligned}$$

obtained after substituting the expression $\tilde{h}(f)$ from the SPA approximation, performing the angular integral over ι

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\iota \left(\frac{\left(1+\cos^2\iota\right)^2}{4} + \cos^2\iota \right) = \frac{16\pi}{5} \,,$$

and using that

$$\frac{dE}{dA} = \frac{1}{16\pi G_N} \int dt \left(\dot{h}_+^2 + \dot{h}_\times^2 \right) \,,$$

from which it follows

$$\Delta E = \frac{2 \times 4\pi^2 f^2 r^2}{16\pi G_N} \int_0^f df' \int d\Omega \left(|h_+(f')|^2 + |h_\times(f')|^2 \right) \,.$$

Compute approximate SNR of a waveform assuming the noise is almost constant and introducing the angular function

$$Q(\iota,\theta,\phi,\psi) = F_+^2(\theta,\phi,\psi) \left(\frac{1+\cos^2\iota}{2} \right)^2 + F_\times(\theta,\phi,\psi)\cos^2\iota$$

whose average is $\bar{Q} \equiv \langle Q \rangle = 4/25$:

1. in time domain

$$\begin{aligned} SNR &\simeq \sqrt{\frac{2}{S_N}} \left(\int h^2(t) dt \right)^{1/2} \\ &= \sqrt{\frac{2}{S_n}} \frac{2 G_N M_c^{5/3} \pi^{2/3}}{D} \left(\int f^{4/3}(t) dt \right)^{1/2} \bar{Q}^{1/2} \\ &= \frac{2}{5^{3/4}} \frac{S_n^{-1/2} M_c^{5/4}}{D} \left[(t_c - t_i)^{1/2} - (t_c - t)^{1/2} \right]^{1/2} \,, \end{aligned}$$

2. in f-domain

$$\begin{aligned} SNR &\simeq \sqrt{\frac{2}{S_N}} \left(\int h^2(f) df \right)^{1/2} \\ &= \sqrt{\frac{5}{24}} \frac{\pi^{-2/3} M_c^{5/6}}{D} \left(\int f^{-7/3} df \right)^{1/2} \bar{Q} \\ &= \frac{1}{2\pi^{2/3} 5^{1/2}} \frac{S_n^{-1/2} M_c^{5/6}}{D} \left(f_i^{-4/3} - f^{-4/3} \right)^{1/2} \,. \end{aligned}$$

Use the explicit formula for $f(t)$ to verify that the two expressions agree.

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