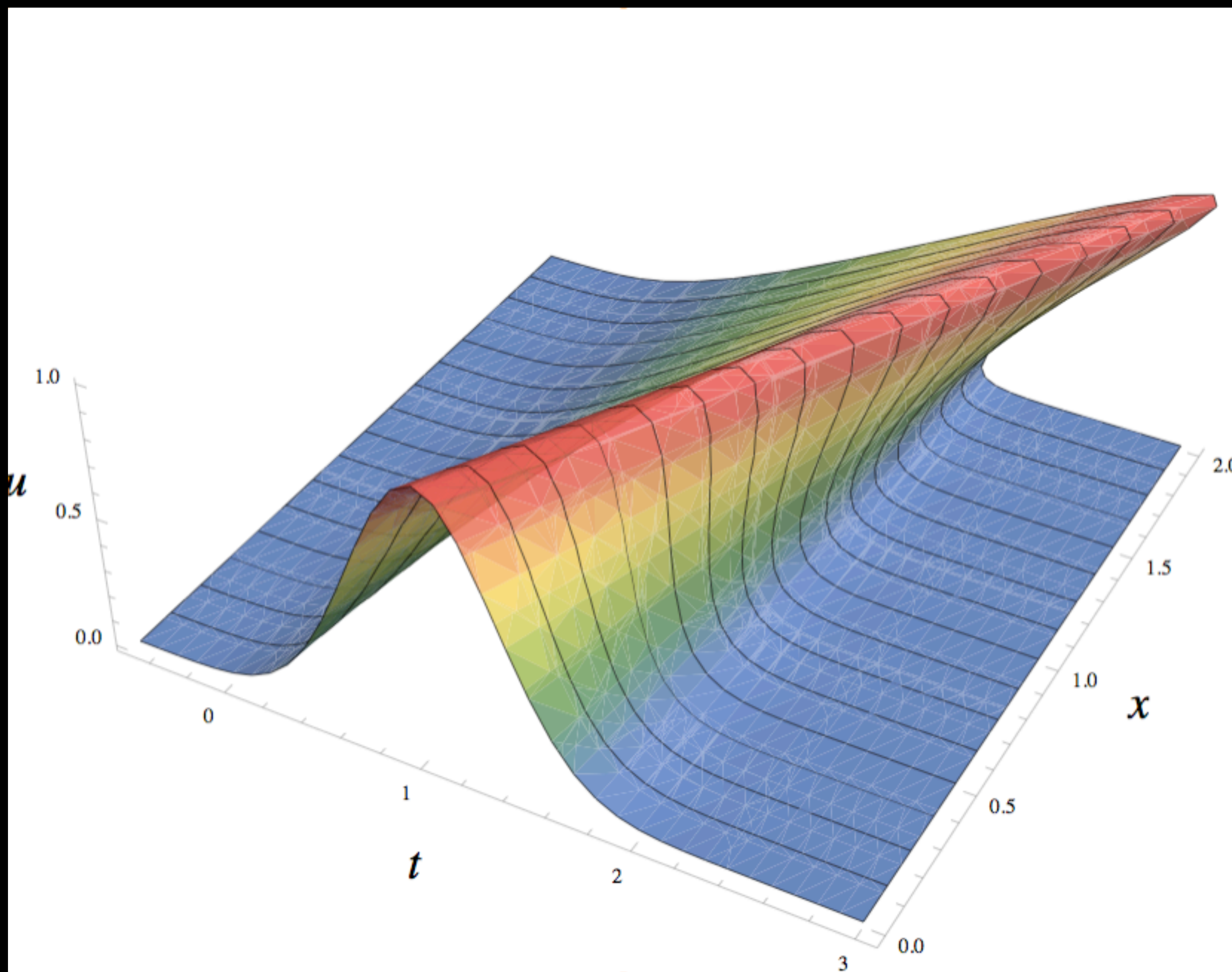
3. Hyperbolic systems of PDEs



Existence of analytic solutions

- For a given PDE, do any solutions exist?
- Consider an initial value problem for the wave equation as an example:
  - wave equation: $$\frac{\partial^2 \phi(t,\vec{x})}{\partial t^2} = \Delta\phi(t,\vec{x})$$
  - set initial data at $t=t_0$: $$\phi(t_0,\vec{x}),\ \frac{\partial}{\partial t}\phi(t,\vec{x})|_{t=t_0}$$
  - Initial data & wave equ. tell us about first and second time derivatives, differentiating the wave equ. in time we can construct all higher time derivatives: $$\frac{\partial^3 \phi(t,\vec{x})}{\partial t^3}|_{t=t_0} = \Delta\frac{\partial\phi(t,\vec{x})}{\partial t}\phi(t,\vec{x})|_{t=t_0}$$
  - Does this formal power series converge? Yes, for analytic initial data! - Theorem of Cauchy-Kowalevskaya!

Theorem of Cauchy-Kowalevskaya

- Let t, x_1, ..., x_{n-1} be coordinates of R^n .

  Consider a system of m PDEs for m unknowns $\Phi_i(t, x_\mu)$, i=1,...,m, where each RHS function F_i is an analytic function of its variables:

$$\frac{\partial^2 \phi_i(t,\vec{x})}{\partial t^2} = F_i(t,\vec{x},\phi_j,\frac{\partial \phi_j}{\partial t},\frac{\partial \phi_j}{\partial x^\mu},\frac{\partial^2 \phi_j}{\partial t \partial x^\mu},\frac{\partial^2 \phi_j}{\partial x^\mu \partial x^\nu})$$

  - Let $f_i(x_\mu)$ and $g_i(x_\mu)$ be analytic functions.

  - => ∃ open neighbourhood O of the hypersurface $t=t_0$:
    within O ∃! analytic solution of the PDE system with initial data $\Phi_i(t_0, x_j) = f_i$, $\partial_t\Phi_i(t_0, x_j) = g_i$.

- **CK-theorem shows that:**
  - **the wave equation and similar equations have an initial value formulation for analytic initial data.**
  - **There is a large class of solutions (as many as there are pairs of analytic functions of the spatial coordinates x_μ).**

Non-analytic equations: example of Lewy

Even linear PDEs with non-analytic coefficients do not in general have solutions!

On $\mathbb{R}\times\mathbb{C}$, suppose that $u(t,z)$ is a function satisfying, in a neighborhood of the origin,

$$\frac{\partial u}{\partial \bar{z}} - iz\frac{\partial u}{\partial t} = \varphi'(t)$$

for some C^1 function ϕ. Then ϕ must be real-analytic in a (possibly smaller) neighborhood of the origin.

http://en.wikipedia.org/wiki/Lewy's_example

Analytic solutions are not enough!

- For analytic solutions, any finite neighbourhood determines the whole solution - makes no sense for relativistic theories, where information propagates at finite speed.

- We can only require C^k, or C^∞ (smooth is sufficient for us).

- C-K does not distinguish between wave and Laplace equations:

  - Let's see the difference between wave and Laplace equations in an example ...

Example (Hadamard)

$$U_n(t,x) = \frac{1}{n^2}\sin(nt)\sin(nx), \quad V_n(t,x) = \frac{1}{n^2}\sinh(nt)\sin(nx)$$

- Functions U_n satisfy wave equation, V_n satisfy Laplace eq.:

$$\ddot{U} = U_n^{''}, \quad \ddot{U} + U_n^{''} = 0$$

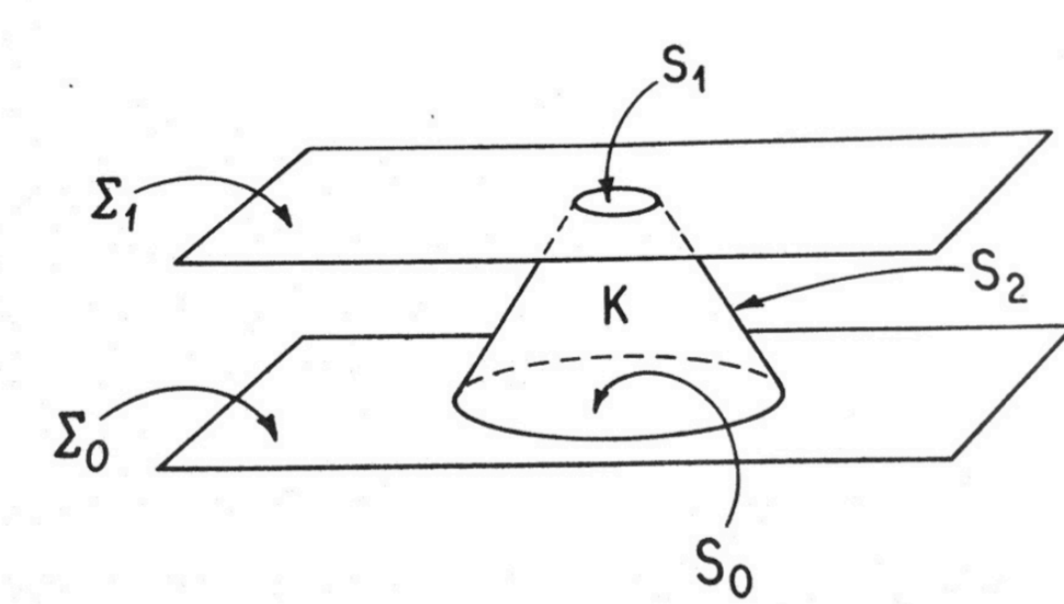
- At t=0 we have

$$U_n(0,x) = V_n(0,x) = 0, \quad \partial_t U_n(0,x) = \frac{1}{n}\sin(nx)$$

- The Cauchy data converge to 0 as n-> ∞. For wave eq., solutions converge to 0, For the Laplace eq. the V_n blow up for any t>0.

- Key idea of ‘hyperbolic’ eqs: have stable solutions for the initial value problem.

Outline of well-posedness proof for KG (Wald)



- Klein-Gordon equation in flat spacetime:

$$\partial_a \partial^a \phi = -\frac{\partial^2 \phi}{\partial t^2} + \Delta \phi = m^2 \phi$$

  - energy momentum tensor divergence free:

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} \eta_{ab} \left(\partial_c \phi \partial^c \phi + m^2 \phi^2 \right) \qquad \partial^a T_{ab} = 0$$

- **satisfies dominant energy condition: if v^a is a future directed timelike vector, then $-T^a{}_b v^b$ is a future directed timelike or null vector (mass energy can not be observed to flow faster than light)**
- **Using the Gauss law we can rewrite as:**

$$\int_{S_1} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + |\nabla \phi|^2 + m^2 \phi^2 \right] \leq \int_{S_0} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + |\nabla \phi|^2 + m^2 \phi^2 \right]$$

Well-posedness proof for KG - II

$$\int_{S_1}\left[\left(\frac{\partial\phi}{\partial t}\right)^2+|\nabla\phi|^2+m^2\phi^2\right]\leq\int_{S_0}\left[\left(\frac{\partial\phi}{\partial t}\right)^2+|\nabla\phi|^2+m^2\phi^2\right]$$

- There can at most be 1 solution in $D^+(S_0)$ with given initial data $(\Phi, \partial_t\Phi)$ on S_0:
  If Φ_1, Φ_2 are both C^2 solutions with the same initial data, then
  $\psi = \Phi_1 - \Phi_2$ would be a solution with vanishing initial data (using linearity!).
  The RHS of the above inequality thus vanishes, implying $\psi=0$ at S_1, S_1 was arbitrary, so ψ vanishes on $D^+(S_0)$ and $D^-(S_0)$.

- -> A variation of the initial data outside of S_0 can not affect the solution within $D^+(S_0)$ and $D^-(S_0)$.

- Solutions depend continuously on initial data in the above “energy norm”.

- Other norms (Sobolev) can be constructed to bound the solution and its partial derivatives directly, see e.g. Wald, GR, p. 249).

Well-posedness proof for KG - III

Outline of existence proof for smooth solutions Φ for arbitrary initial data $\Phi_i(t_0, x_j)$, $\partial_t\Phi_i(t_0, x_j)$ on Σ_0

- Smooth functions can be approximated (with uniform convergence) by analytical functions.
- By C-K theorem, these give rise to analytical solutions of the KG equation. Using the energy norm (and derived Sobolev norms) one can show that these analytical solutions have to converge to a solution of KG.
- As seen before, the limiting solution has to be unique.
- Unlike C-K, this proof uses specific properties of the wave equation: linearity, conserved T_{ab}, dominant energy condition, “wave equation character” - proof would not work for Laplace equation!
- Can we obtain a proof of well-posedness for a general class of equations?

Nonlinear PDEs

- Non-linear PDEs in general have to be discussed on a case-by-case basis.

- Quasi-linear: linear in highest derivatives **(principal part)**, coefficients depend on the independent variables and their lower order derivatives.

- Quasi-linear PDEs allow statements on well-posedness based on properties of the principal part, EEs are quasi-linear.

- Classes of systems of hyperbolic equations which admit a well-posed intial value problem:

  - generalized wave equations (g_{ab} a smooth Lorentz metric)

$$g^{ab}(x,\phi_j,\nabla_c\phi)\nabla_a\nabla_b\phi_i = F_i(x,\phi_j,\nabla_c\phi)$$

  - strongly hyperbolic systems -> investigate in more detail ...

example: advection equation

$$\frac{\partial}{\partial t}u(\vec{x},t) + v^j \frac{\partial}{\partial x^j}u(\vec{x},t) = 0$$

- Construct general solution via Fourier transform in space:

$$\hat{u}(\vec{k},t) := \frac{1}{(2\pi)^{n/2}} \int e^{-i\vec{k}\cdot\vec{x}}\, u(\vec{x},t) d^n x \quad \Rightarrow \quad \widehat{\partial_{\vec{x}} u} = i\vec{k}\hat{u}$$

$$\partial_t \hat{u}(\vec{k},t) = -i v^j k_j \hat{u}(\vec{k},t) \quad \Rightarrow \quad \hat{u}(\vec{k},t) = e^{-i\,\vec{v}\cdot\vec{k}\,t}\hat{u}_0(\vec{k})$$

- Solution moves with speed $\vec{v}$ without changing profile:

$$u(\vec{x},t) = \frac{1}{(2\pi)^{n/2}} \int \hat{u}(\vec{k},0) e^{i\vec{k}(\vec{x}-\vec{v}t)} d^n k = u_0(\vec{x}-\vec{v}t)$$

- Fourier method works for general constant coefficient PDEs!
- Norm remains constant -> equation is well posed!
- Key idea: can solve constant coeff. case explicitly .
- Exercise: well-posedness for heat/wave/Schrödinger eq.

Constant coefficient hyperbolic systems

- First order differential systems:

$$\partial_t u^a(\vec{x},t) = A_b{}^{aj}\partial_j u^b(\vec{x},t)$$

$$\partial_t \hat{u}^a(\vec{k},t) = iA_b{}^{aj}k_j\hat{u}^b(\vec{k},t) \quad \Rightarrow \quad \hat{u}^a(\vec{k},t) = e^{i\,A_b{}^{aj}k_j\,t}\hat{u}_0^a(\vec{k})$$

- Choose direction n: $\vec{n}\cdot\vec{n} = 1,\ k = |k| \quad \Rightarrow \quad \hat{u}^a(k,\vec{n},t) = e^{i\,A_{n\,b}{}^a k\,t}\hat{u}_0^a$

- Compute matrix exponential by transforming A to Jordan form:

$$PAP^{-1} = D + N,\ N^n = 0 \quad \Rightarrow \quad e^{iAkt} = e^{iDkt}e^{iNkt} = e^{iDkt}\sum_{l=0}^{l=n-1} N^l\frac{k^l t^l}{l!}$$

- A diagonalizable & real eigenvalues: each component of u in the diagonal basis is advected with speed corresponding to (-)eigenvalue of A.
  - P.u are called “characteristic variables”.
  - Fourier domain solution is oscillatory and preserves norm.
  - Lower order terms ($u_t = A\,\partial u + Bu + C$) can result in exponential growth (frequency independent), propagation speeds and WP only depend on A (principal part = highest derivatives).

Constant coefficient hyperbolic systems

- First order differential systems:

$$\partial_t u^a(\vec{x},t) = A_b{}^{aj}\partial_j u^b(\vec{x},t)$$

$$\partial_t \hat{u}^a(\vec{k},t) = i A_b{}^{aj} k_j \hat{u}^b(\vec{k},t) \quad \Rightarrow \quad \hat{u}^a(\vec{k},t) = e^{i\,A_b{}^{aj} k_j\, t}\hat{u}^a_0(\vec{k})$$

- Choose direction n: $\vec{n}\cdot\vec{n} = 1,\ k = |k| \quad \Rightarrow \quad \hat{u}^a(k,\vec{n},t) = e^{i\,A_{n\,b}{}^{a} k\, t}\hat{u}^a_0$

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- Jordan blocks (N≠0) cause frequency (k) dependent polynomial growth - obstruction to WP!

$$\mathrm{JordanForm}(A) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- complex eigenvalues -> exponential growth (in future or past) -> ~~WP~~.

classification of hyperbolic systems

- **weakly hyperbolic:** Speeds (eigenvalues of A) all real (well posed in absence of l.o.t. in appropriate norm)

- **strongly hyperbolic:** weakly hyperbolic with complete set of eigenvectors (characteristic variables span solution space), **well posed initial value problem**

- **symmetric/symmetrizable hyperbolic:** strongly hyperbolic, and A can be diagonalized with the **same** similarity transformation P for **all** space-directions.

  - strongly hyperbolic implies symm. hyperbolic in 1D

  - admits a conserved energy: can be used to prove well-posed initial boundary value problem with appropriate BCs

- **strictly hyperbolic:** all eigenvalues are distinct

hyperbolic systems: remarks

- Quasi-linear = nonlinearities only lower order terms (e.g. Einstein equations): well-posedness carries over from equations linearized around some background solution.
- Solutions may become singular in finite time -> well-posedness only guarantees existence of solution for some small time
- local/global in time existence problem.
- first order in time system was convenient for solution procedure in Fourier domain - what happens with higher differential order systems, e.g. wave equation? -> next lecture
- Clarification of hyperbolicity of ADM, BSSN etc. has taken until 1999 -2006 [Frittelli, Reula, Sarbach, Beyer, Tiglio, Calabrese, Gundlach, Martín-García, ...]

Example: wave equation in 1D

- Start with 2nd order form: $\phi_{,tt} = c^2\phi_{,xx}$
- Can obtain a mixed first/second order form: $\phi_{,t} = c\pi, \quad \pi_{,t} = c\phi_{,xx}$
- or complete first order reduction:

$$\phi_{,t} = c\pi, \quad \phi_{,x} = \psi, \quad \pi_{,t} = c\psi_{,x}, \quad \psi_{,t} = c\pi_{,x}$$

  - where $\varphi_{,x} = \psi$ now plays the role of a constraint which is preserved by the evolution equations:
  $\partial_t(\phi_{,x} - \psi) = \partial_x\partial_t\phi - \partial_t\psi = \partial_x\pi - \partial_x\pi = 0.$
  - The evolution equation for φ decouples, and we may focus on the system of equations for ψ and π, which has the form

$$\partial_t u = A\,\partial_x u, \quad u = \{\pi, \psi\} \qquad A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

  - A has eigenvalues $\pm c$, and eigenvectors (-1,1) & (1,1), correspondingly the characteristic variables are $u_\pm = \psi \pm \pi$ and satisfy advection equations $\partial_t u_\pm = \pm c\, \partial_x u_\pm$
  - Solution preserves norm -> WP, as seen before.

Example: weakly hyperbolic system

- Consider the following simple system in 1D:

$$\partial_t u = \partial_x (u + v), \quad \partial_t v = \partial_x v$$

- In our matrix notation this becomes:

$$\partial_t u = A\, \partial_x u, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- A has only 1 proper eigenvector (1,0), with eigenvalue 1 (and "trivial" eigenvector 0), thus no complete set of eigenvectors and the system is only **weakly hyperbolic**.

- Explicit solution with frequency ω, U = (u,v):

$$u = \omega t \sin \omega (t + x)\,, \quad v = \sin \omega (t + x)$$

- Compute L^2 norm for data with u(0) = 0 and frequency ω

$$\frac{||U(t)||}{||U(0)||} = \sqrt{1 + t^2 \omega^2}$$

Linear, frequency dependent growth -> ~~WP~~

example: York-ADM in 1D

- York-ADM, $g_{ij} = g_{ij}(x,t)$ - plane wave traveling in x-direction
- gauge condition: densitized lapse: $\alpha = \sqrt{\det {}^3 g}$
- linearized around flat space:

$$\begin{aligned}
&\dot{h}_{ii} = 2K_{ii} \\
&\dot{K}_{xx} = \tfrac{1}{2}\partial_{xx} h_{xx} + \partial_{xx}\left(h_{yy} + h_{zz}\right) \\
&\dot{K}_{jj} = \tfrac{1}{2}\partial_{xx} h_{jj} \qquad (j = y, z)
\end{aligned}$$

- Jordan normal form of first order reduction: all characteristic speeds real, but 2 Jordan blocks

$$\mathrm{JordanForm}(A) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Some simple incarnations of the scalar wave equation

- Scalar WEQ defined with metric g_{ab}, may consider fixed metric, or couple scalar field to Einstein equations:

$$g^{ab}\nabla_a\nabla_b\phi = 0 \qquad G_{ab}[g] = 8\pi G T_{ab}[\phi]$$

- e.g. WEQ on Minkowski space. 1+1 dimensional problems are obtained by considering plane waves

$$g^{\mu\nu} = \eta_{\mu\nu}, \quad \phi(x,t) \to \phi_{tt} = \phi_{xx}$$

- or spherically symmetric waves

$$g^{\mu\nu} = \eta_{\mu\nu}, \quad \phi(r,t) \to \phi_{tt} = \phi_{rr} + \frac{2}{r}\phi_{rr}$$

- Scaling of variables can do miracles:

$$\tilde{\phi}(r,t) := r\phi(r,t) \to \tilde{\phi}_{tt} = \tilde{\phi}_{rr}$$

Scalar field energy in flat space

- Energy density ρ gives rise to a conserved energy E:

$$E = \int_{R^3} \rho \, d^3x \qquad \rho = \left(\frac{\partial \phi}{\partial t}\right)^2 + |\nabla \phi|^2$$

- For plane waves we get

$$\rho = \left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2$$

- Because of energy conservation, for plane waves the field strength can't decay.

- In spherical symmetry we expect decay with 1/r

$$\phi(r,t) := \frac{\tilde{\phi}(r,t)}{r}$$

Boundary conditions

- Can consider 3 distinct cases:
  - finite grid without boundaries, use periodic boundary conditions = identify end points, DONE
  - finite grid with boundaries, need to impose boundary conditions (reflecting, incoming signal, outgoing=no incoming signal)
  - infinite grid. need to "pull in" infinity with a coordinate transformation, will lead to singular equations -> investigate tomorrow

Wave equation + moving coordinates

- Restrict to plane waves in 1 space dimension:

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2$$

  - redefine x coordinate using shift (vector)

$$dt = d\tilde{t}, \quad dx = d\tilde{x} - \beta d\tilde{t}$$

  - The metric becomes

$$g_{\mu\nu} = \begin{pmatrix} (-1+\beta^2) & \beta \\ \beta & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & \beta \\ \beta & (1-\beta^2) \end{pmatrix}$$

  - Rewrite the WEQ using e.g.

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi] = 0$$

Shifted wave equation

$$
\begin{aligned}
\Box\phi &= \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi] \\
&= \partial_t[g^{t\nu}\partial_\nu\phi] + \partial_x[g^{x\nu}\partial_\nu\phi] \\
&= \partial_t[g^{tt}\partial_t\phi + g^{tx}\partial_x\phi] + \partial_x[g^{xt}\partial_t\phi + g^{xx}\partial_x\phi] \\
&= \partial_t[-\partial_t\phi + \beta\partial_x\phi] + \partial_x[\beta\partial_t\phi + (1-\beta^2)\partial_x\phi] \\
&= 0
\end{aligned}
$$

- Suggests definition of new variables:

$$\psi := \partial_x\phi \quad \pi := \partial_t\phi - \beta\partial_x\phi$$

- Evolution equations:

$$\partial_t\phi = \pi + \beta\psi$$

$$\partial_t\psi = \partial_x(\pi + \beta\psi) \qquad \partial_t\pi = \partial_x(\psi + \beta\pi)$$

characteristic variables

- matrix formulation:

$$\mathbf{u} = (\pi, \psi)^T$$

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{u}\partial_x \beta \qquad \mathbf{A} = -\begin{pmatrix} \beta & 1 \\ 1 & \beta \end{pmatrix}$$

- A is diagonalizable with eigenvalues = characteristic speeds $\lambda_1=-\beta+\alpha$, $\lambda_2=-\beta-\alpha$, and eigenvectors

$$v_1 = (1, -1)^T, \quad v_2 = (1, 1)^T$$

- characteristic variables, propagating with characteristic speeds:

$$u_1 = u_R = \frac{1}{2}(\pi - \partial_x \phi) \qquad u_2 = u_L = \frac{1}{2}(\pi + \partial_x \phi)$$

- Plot the characteristic variables in your code! Observe that these quantities propagate as expected.

Boundary conditions

- Putting boundary conditions on outgoing characteristic fields is not logically consistent - initial boundary value problem will not be well-posed.
- Can only put boundary conditions on incoming characteristic fields!
- Examples:
  - reflecting boundary conditions

$$\phi = \partial_t \phi = \pi = \partial_t \pi = \partial_x \psi = 0$$

  - outgoing boundary conditions: incoming signal set to zero, e.g. at left boundary:

$$u_R = 0 = \pi - \psi \quad \Rightarrow \pi = \psi$$

Scalar field coupled to gravity

- Simple form of the metric in spherical symmetry, with zero shift

$$ds^2 = -\alpha(r,t)^2 dt^2 + a(r,t)^2 dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

- Definitions:

$$\psi = \partial_r \phi, \quad \pi = \frac{a}{\alpha} \partial_t \phi$$

- Einstein equations:

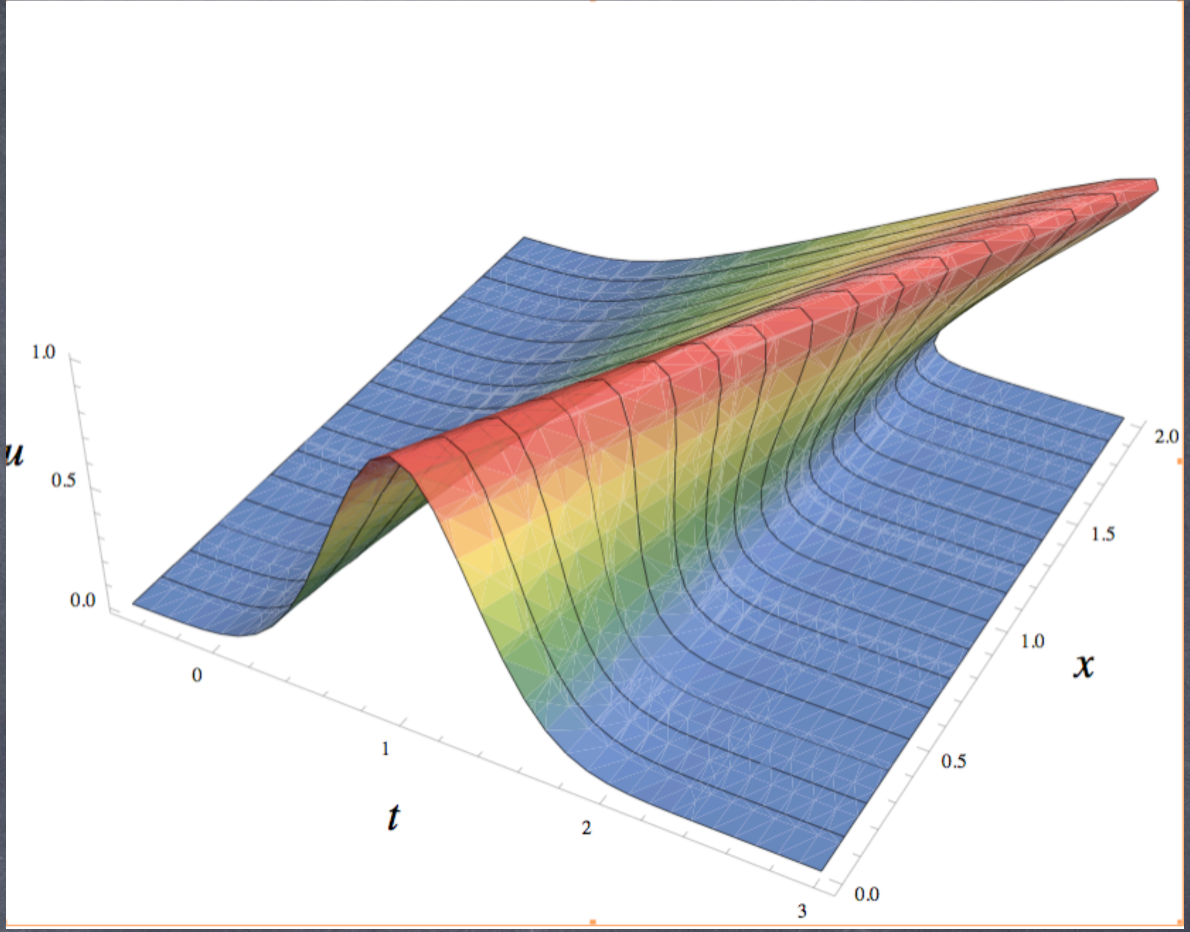
$$\frac{\partial_r \alpha}{\alpha} = \frac{\partial_r a}{a} \frac{a^2 - 1}{r} \qquad \frac{\partial_r a}{a} = \frac{1 - a^2}{2r} + \frac{r}{4} \left(\psi^2 + \pi^2 \right) \qquad \partial_t a = \frac{1}{2} r a \alpha \phi \pi$$

- Scalar field equations:

$$\partial_t \phi = \frac{\alpha}{a} \pi \qquad \partial_t \pi = \frac{1}{r^2} \left(\frac{r^2 \alpha \psi}{a} \right) \qquad \partial_t \psi = \partial_r \left(\frac{\alpha \pi}{a} \right)$$

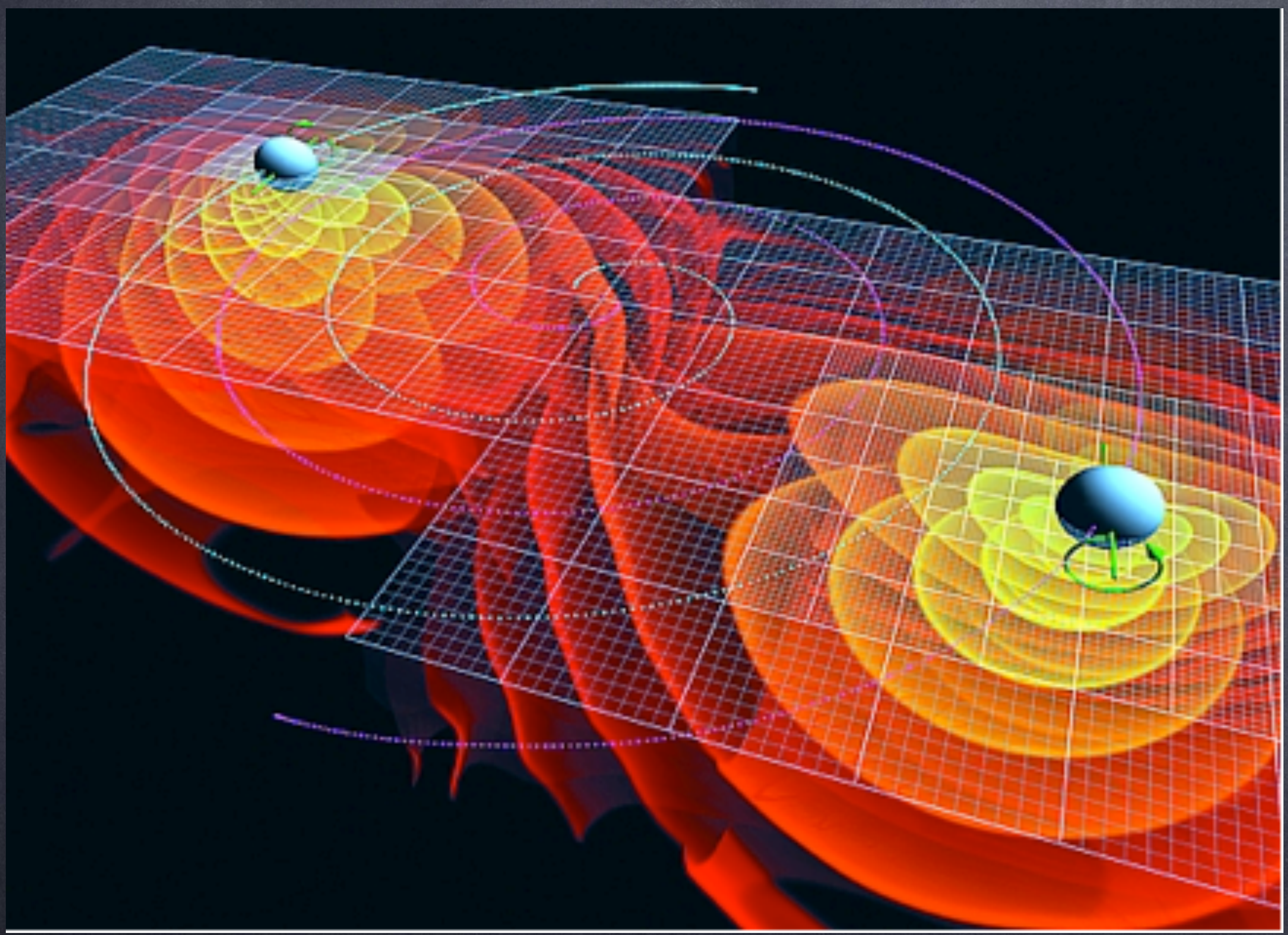
smooth and distributional solutions

- Burger's equation: $u_t = u\, u_x$.
- Characteristic speeds depend on u, peak velocity overtakes rest of the wave after some time.



- More generally: characteristics can cross, typically signifies physical breakdown of underlying PDE, like in fluid dynamics.
- Unless a PDE is linearly degenerate (speeds independent of solution), shocks can form from smooth data in a finite time.
- Vacuum EE: can be written in linearly degenerate form, do not expect physical shocks, but shocks can form due to bad gauge conditions.
- Numerical methods for fluid dynamics are dominated by methods that deal with shocks - e.g. propagate shocks at correct physical speed.
- Solutions of vacuum GR are smooth except due to bad gauges or physical singularities, high order FD or spectral ideal!

4. Numerics: Finite Differences



discretization example: wave equation

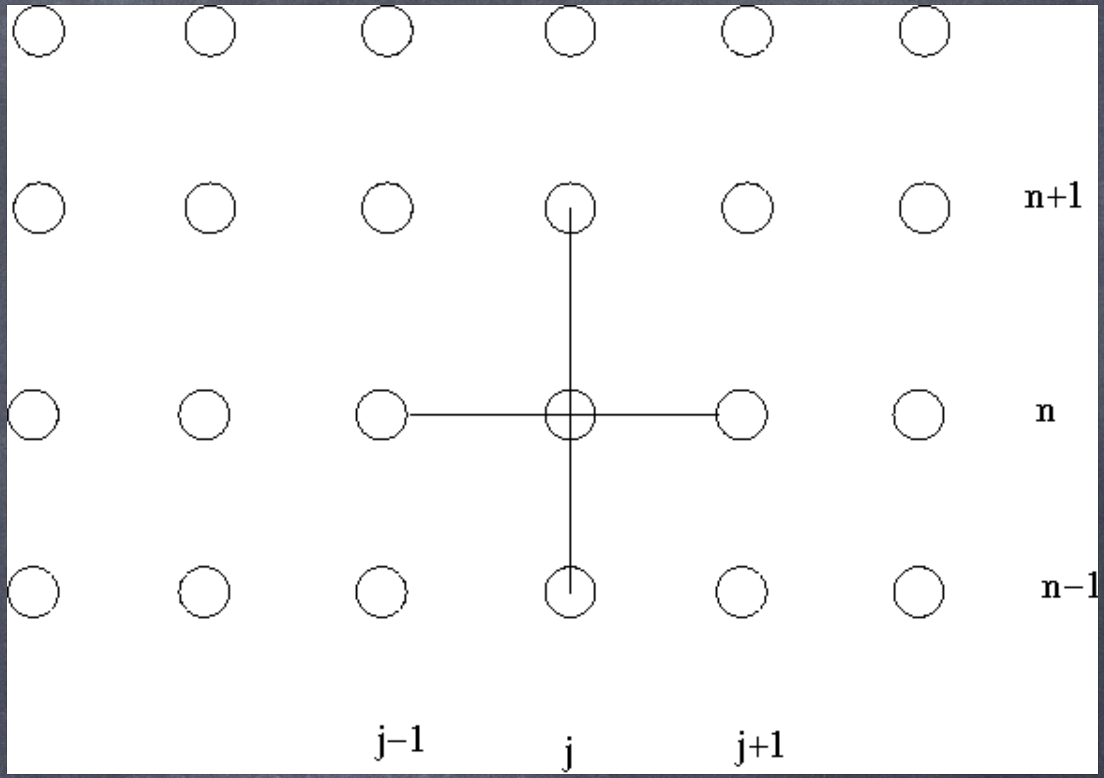
- Discretize wave equation straightforwardly to 2nd order accuracy.

$$u_{tt} = u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0.$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

- Use periodic boundaries for simplicity (all points equal!)

$$u(0,t) = u(1,t)$$



- grid: $t^n = n\Delta t, \quad x_j = (j-1)\,\Delta x, \quad \Delta t = \lambda \Delta x$

- leapfrog algorithm:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = (u_{tt})_j^n + O(\Delta t^2)$$

$$\lambda = \frac{\Delta t}{\Delta x}$$

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

discretization example: wave equation II

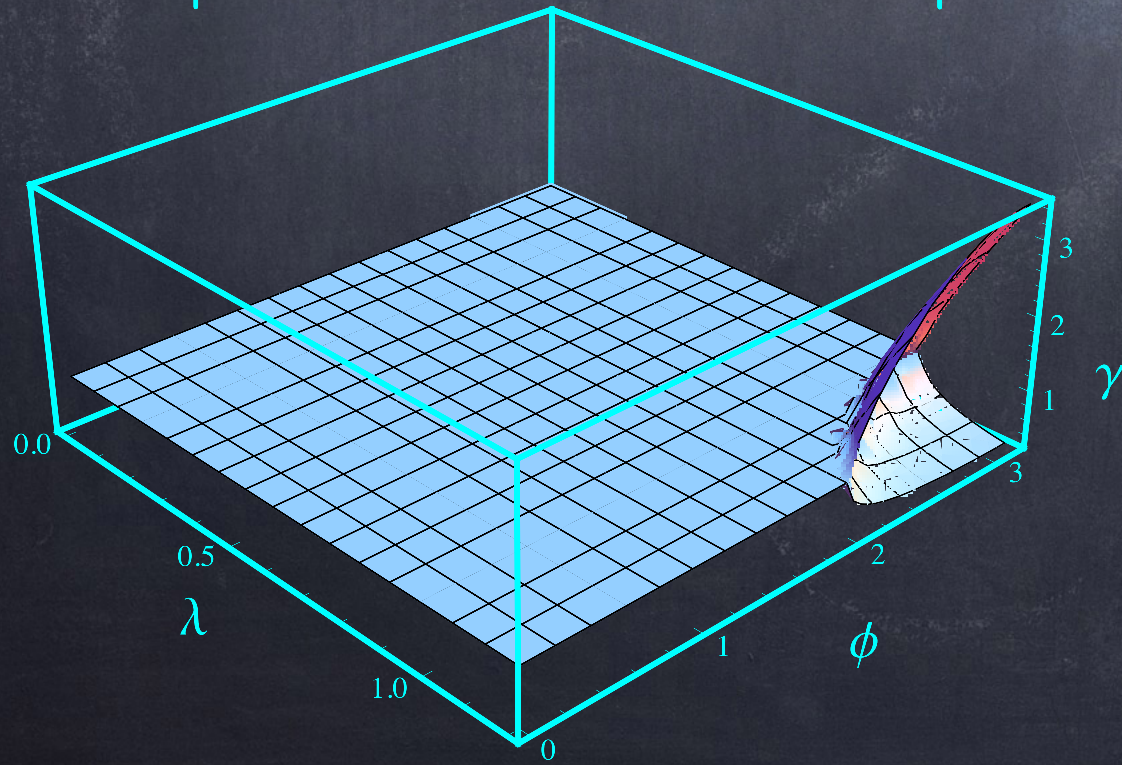
- Superpose solution of Fourier modes $e^{i \omega j \Delta x}$ -> $u = q^n e^{i \omega j \Delta x}$
  - wave number/frequency , $|\omega \Delta x| \leq$ pi
  - call q amplification factor, q > 1 => unstable algorithm
- for a smooth solution, the “signal” is concentrated at small $\xi = \omega \Delta x$
- insert ansatz into discretization

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

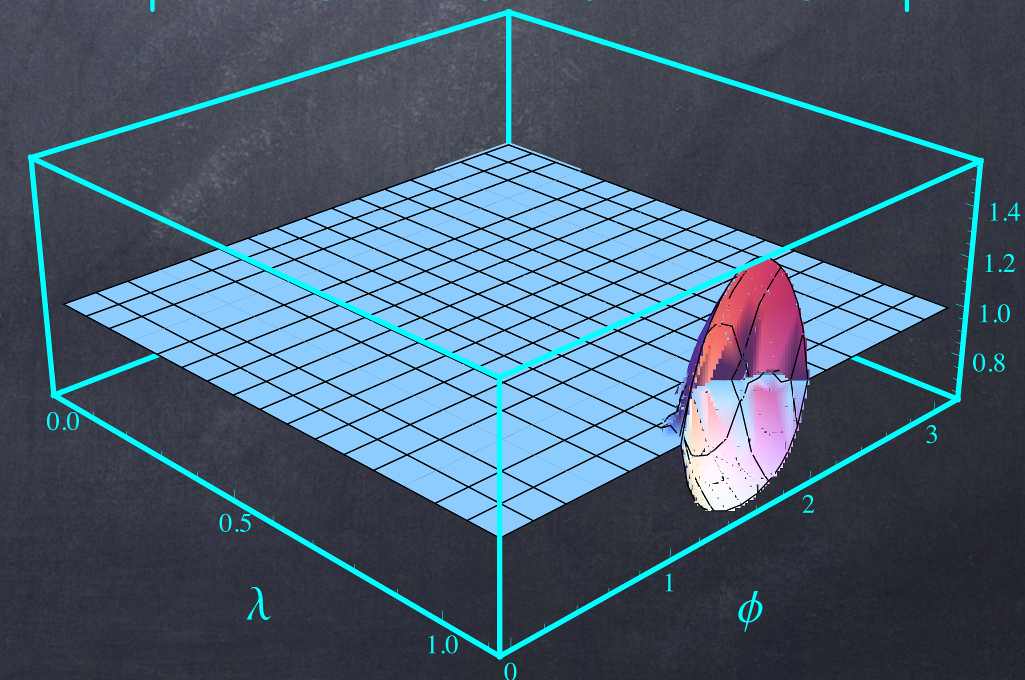
- -> obtain quadratic equation:

$$\xi \left(2\lambda^2 - 2\lambda^2 \cos(k\Delta x) - 2\right) + \xi^2 + 1 = 0$$

amplification factors for wave eq.

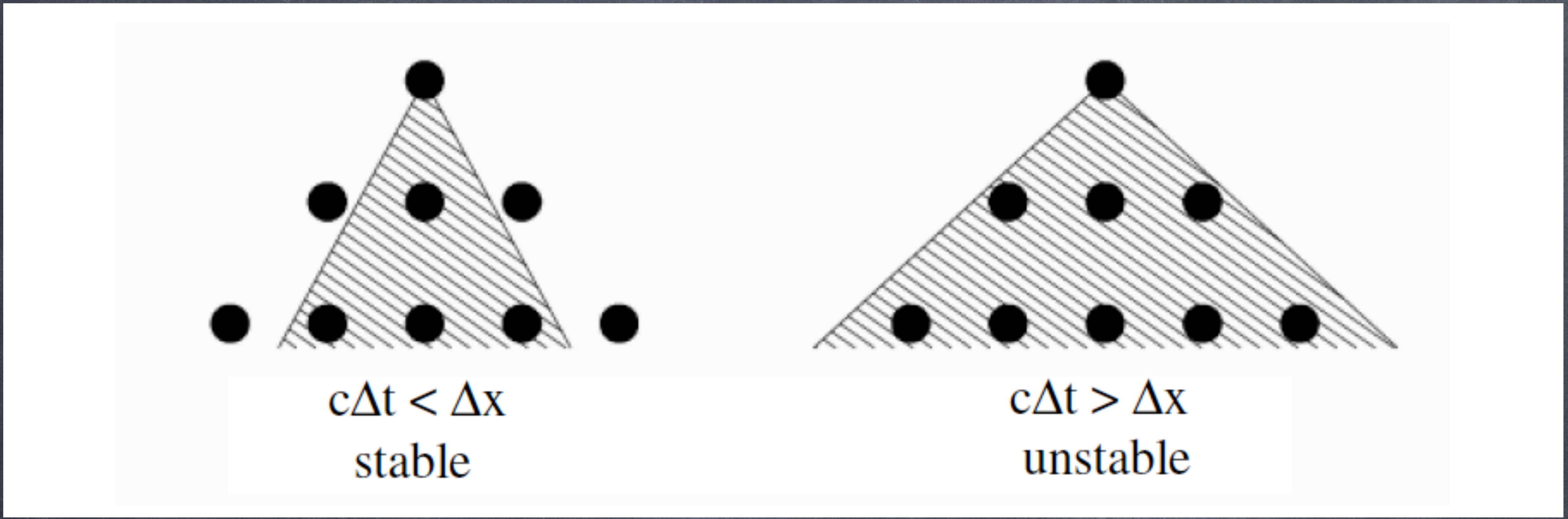


amplification factors for advection eq.



Courant-Friedrichs-Lewy condition

- Explicit time stepping schemes impose limits on Δt.



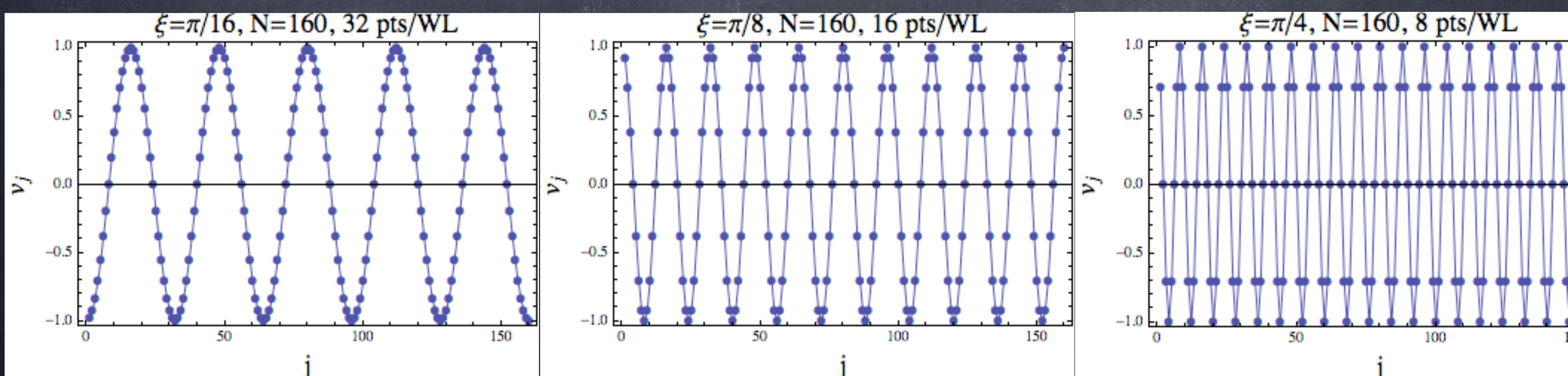
- Geometric interpretation: the numerical domain of dependence should include the physical domain of dependence. If the physical DoD is larger, we can't converge to the correct solution, since relevant physical information is neglected. Lax => unstable

- necessary but not sufficient

- parabolic: const. $\Delta t < \Delta x^2$ -> use implicit methods

Of grids and frequencies

- consider equispaced grid in d dimensions, tensor product of 1-D grids $x_j = j\,h$, $j = 0,1,\ldots,N-1$
  - inner product $(u,v)_h = \Sigma\, u_j\, v_j\, h^d$, $\|v\|_h = (v,v)^{1/2}$
  - Stability: $\exists\ K, \alpha$: $\|v^n\|_h \leq K\, e^{\alpha\, t_n}\, \|v^0\|_h\ \forall\ n$: $t_n = n\, k = n\, \Delta t$, $\forall\ v^0$
- can represent frequencies $\omega_j = -N/2 +1, \ldots, N/2$, $\xi_j = \omega_j\, h = -\pi + 2\pi/N, -\pi + 4\pi/N, \ldots, \pi$ (N even, highest frequency represented)
- grid function v at time step n: $$v_j^n = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_\omega e^{i\omega x_j}\, \hat{v}^n(\omega)$$
- when smooth functions are represented and well resolved (many gridpoints per wavelength) on the grid, "signal" is concentrated at low frequencies.



Deriving finite difference stencils

- Taylor expansions, or approximating polynomials.
- Example: derive second order centered finite difference stencils.
- -> need to approximate solution by second order polynomial.

$$f(x) = a_0 + a_1 x + a_2 x^2$$

- 3 coefficients a_i -> need 3 gridpoints to define their values.
- Consider grid X = {-h, 0, h} -> equations:

$$\{f(-h) = f_1,\, f(0) = f_2,\, f(h) = f_3\}$$

- solution:

$$a_0 = f_2, a_1 = \frac{f_3 - f_1}{2h}, a_2 = -\frac{f_3 + 2f_2 + f_1}{2h^2}$$

- Take derivatives of f to obtain stencil coefficients, and set x=0:

$$\partial_x f \approx \frac{f_3 - f_1}{2h} \qquad \partial_{xx} f \approx \frac{f_3 - 2f_2 + f_1}{h^2}$$

General algorithms to compute finite difference stencils

B. Fornberg (1988)

- **Equispaced grids: one-liner in Mathematica:**

  ```
  CoefficientList[Normal[Series[x^s Log[x]^m, {x, 1, n}]/h^m], x]
  ```

  https://amath.colorado.edu/faculty/fornberg/Docs/sirev_cl.pdf

- **General (non-uniform) grids: Recursive formula**

- https://reference.wolfram.com/language/tutorial/NDSolveMethodOfLines.html

- http://web.media.mit.edu/~crtaylor/calculator.html

Method of Lines

- Direct space-time discretizations are hard to generalize to higher order, and stability has to be analyzed case by case.
- Convert PDEs to coupled ODEs, discretize space and time separately. Example:

$$\partial_t u(x,t) + \partial_x u(x,t) = 0 \quad \rightarrow \quad \partial_t u(i,t) = -\frac{u(i+1,t) - u(i-1,t)}{2\Delta x}$$

- Integrate ODEs with any stable ODE integrator.
  - explicit: subject to time step conditions, e.g. RK3, RK4, ...
  - implicit: no or negligible time step restriction for stability
- Easy to plug in different time integrators, space discretizations, boundary conditions, ... Flexibility and robustness are key virtues in scientific computing!
- **First order constant coefficient hyperbolic systems are stable with centered finite differencing and simple time step restriction.**

Finite difference stencils in Fourier space

- Example: second order centered finite difference stencils.

$$\partial_x f \approx \frac{f_3 - f_1}{2h} \qquad \partial_{xx} f \approx \frac{f_3 - 2f_2 + f_1}{h^2}$$

- apply them to a wave of frequency ω:

$$f(x) = e^{i\omega x}$$

- Apply finite difference operator to function:

$$\frac{e^{ihw} - e^{-ihw}}{2h}$$

- Simplify expression

$$\hat{D}_2 = \frac{i\sin(hw)}{h}$$

Numerical stability for first order hyperbolic systems

- P : linear constant coefficient differential operator

$$\partial_t u = P\left(\partial_x\right) u \qquad \hat{P}(i\omega): \quad \partial/\partial x_j \to i\omega_j = i\frac{\xi_j}{h} \quad (\text{i.e.} \;\; \hat{P} = i\omega_i A^i)$$

- WP is equivalent to $|e^{\hat{P}(i\omega)t}| \leq K e^{\alpha t}$ -> need $\hat{P}$ diagonalizable
- discretize, e.g. 2nd order centered: $\partial_x \Rightarrow \frac{i}{h}\sin\xi$ (exercise!)
- n-th order Runge Kutta: $v^{n+1} = Q\, v^n = p(\Delta t\, P) v^n \quad p(x) = \sum_{l=0}^{l=n} \frac{x^l}{l!}$
- Fourier: $\hat{v}^{n+1}(\xi) = \hat{Q}(\xi)\, v^n(\xi) = p(\Delta t\, \hat{P}(\xi))\hat{v}^n(\xi)$
- now we can solve: $\hat{v}^n(\xi) = \hat{Q}(\xi)^n\, v^0(\xi)$
- amplification matrix $\hat{Q}$ diagonalizable if $\hat{P}$ is!
- stability if eigenvalues satisfy: $|q_\mu| \leq 1$, $q_\mu = p(\Delta t\, p_\mu)$
- PDE does not explicitly depend on direction or dimension d

$$\lambda = \frac{\Delta t}{\Delta x} \leq \frac{\alpha_0}{\sigma(A)\sqrt{d}}, \quad \alpha_0 = 2(ICN), \sqrt{3}(RK3), \sqrt{8}(RK4)$$

nonlinear systems and dissipation

- Numerical schemes for quasi-linear hyperbolic PDEs: can use the same numerical methods, but need to dissipate high frequency modes to achieve numerical stability.
- Standard procedure: add Kreiss-Oliger dissipation for 2r-2 accurate scheme, dissipation strength σ > 0:

$$\partial_t u \to \partial_t u + Qu, \quad Q_{2r} = \sigma \frac{(-\Delta x)^{2r-1}}{2^{2r}} (D_+)^r (D_-)^r$$

  - does not degrade convergence order!
- Adding too much dissipation decreases time-step limit (makes equations behave more and more like heat equation).
- Artificial dissipation in fluid dynamics has traditionally been used to smear out shocks, superseded by "High resolution shock capturing" methods.

Second order in space systems: motivation

- Can we discuss well-posedness for second order in space systems like YADM and g-harmonic without first order reduction?
- Reduction to first order in time -> new evolution equations
- Reduction to first order in space -> new evolution & constraint equations.
  - enlarges solution space, new unphysical d.o.f. may give rise to instabilities (remember EM on curved background).
- General theory for WP of 2nd order in space only > 2004
  - How about accuracy of 1st vs. 2nd order in space?
- generalized wave equations: WP

example: mixed order wave equation

- Time domain: $h_{,t} = k, \quad k_{,t} = h_{,xx}$
- Frequency domain, t -> ω: $\hat{h}_{,t} = \hat{k}, \quad \hat{k}_{,t} = -\omega^2 \hat{h}$
- Introduce new variable λ as the square root of $h_{,xx}$:

$$\hat{\lambda} := i\omega\hat{h} \quad \Rightarrow \quad \hat{\lambda}_{,t} = i\omega\hat{k}, \quad \hat{k}_{,t} = i\omega\hat{\lambda} \quad \hat{h}_{,t} = \hat{k}$$

$$\partial_t \begin{pmatrix} h \\ k \\ \lambda \end{pmatrix} = A \begin{pmatrix} h \\ k \\ \lambda \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & i\omega \end{pmatrix}$$

- Characteristic speeds are -1,1,0; problem is symmetric hyperbolic and well posed in the norm (L^2 does not always work!):

$$||u||^2 = \int \left(|h|^2 + |k|^2 + |\partial_x h|^2\right) dx$$

- In the Fourier domain this system could be treated in analogy with first order in space systems, using a "pseudo-differential reduction" - but variables play different roles depending on how often they are differentiated.
- In the discrete case, we will have to choose an appropriate discretization for the derivative in the norm!

second order in space hyperbolic systems

- normal form: P takes second derivatives of u, but not v.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}, \quad P = \begin{pmatrix} A^i\partial_i + B & C \\ D^{ij}\partial_i\partial_j + E^i\partial_i + F & G^i\partial_i + J \end{pmatrix}$$

- Second order principal symbol $\hat{P} = \begin{pmatrix} i\omega A^n & C \\ -\omega^2 D^{nn} & i\omega G^n \end{pmatrix}$
- Analyze WP & numerical stability by pseudo-differential reduction (first order reduction in Fourier space).
- WP reduces to diagonalizability of $\hat{P}_{\text{reduced}} = i\omega \begin{pmatrix} A^n & C \\ D^{nn} & G^n \end{pmatrix}$
- Discrete stability is **not** implied by WP + centered FD + small Δt
  - $\partial_{xx} = \partial_x \partial_x$ does not carry over from continuum, e.g.

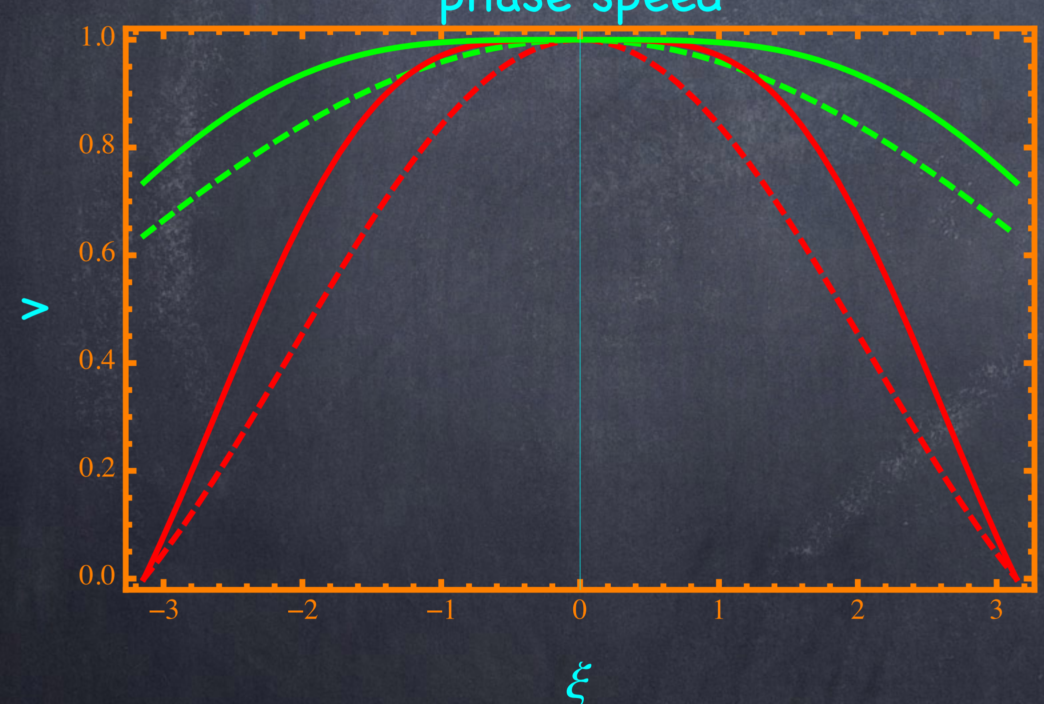
$$\hat{D}^{(2)} = -\frac{4}{\Delta x^2}\sin^2\frac{\xi}{2} \neq \left(\frac{i}{\Delta x}\sin\xi\right)^2$$

- discrete norm: $\|u\|_h^2 + \|v\|_h^2 + \sum_{i=1}^{d} \|D_{+i}u\|_h^2, \; D_+ v_j = \dfrac{v_{j+1} - v_j}{\Delta x}$

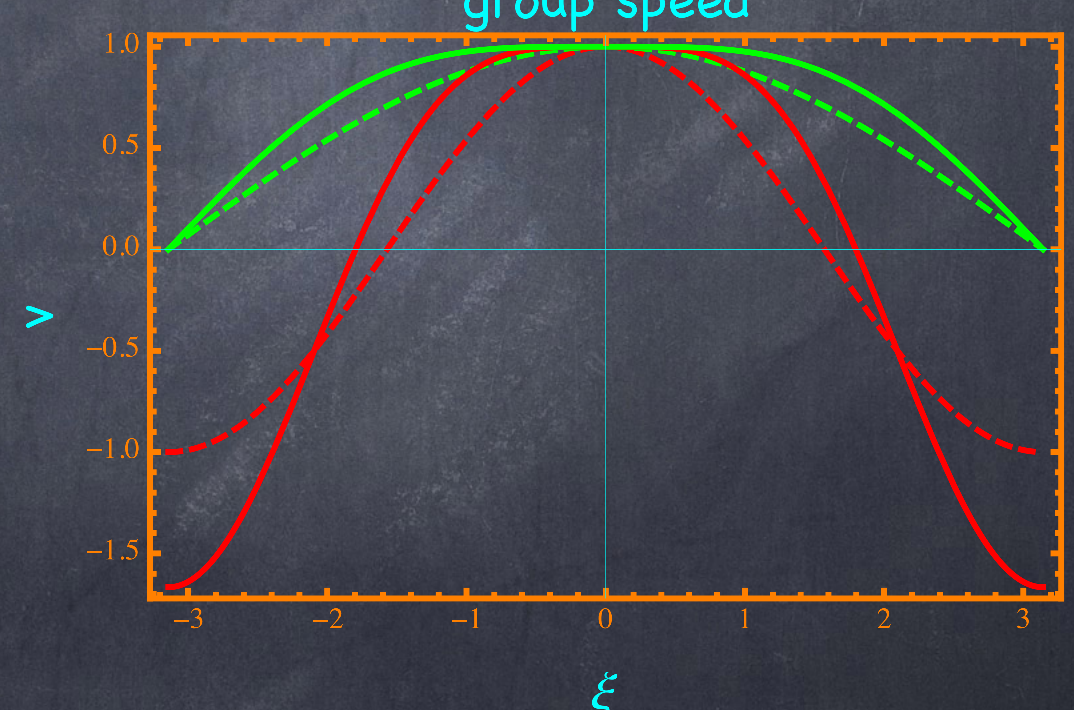
comparison 1st vs 2nd order in space

- $\lambda(\xi)$ eigenval. of $\hat{P}(\xi)$
- phase velocity $v_p = i\dfrac{\lambda}{\omega}$
- group vel. $v_g = i\dfrac{d\lambda}{d\omega}$

	2nd order accurate	
	advective	wave
v_p	$\frac{\sin\xi}{\xi} \approx 1 - \frac{\xi^2}{6} + O(\xi^4)$	$\frac{2}{\xi}\sin\frac{\xi}{2} \approx 1 - \frac{\xi^2}{24} + O(\xi^4)$
v_g	$\cos\xi \approx 1 - \frac{\xi^2}{2} + O(\xi^4)$	$\cos\frac{\xi}{2} \approx 1 - \frac{\xi^2}{8} + O(\xi^4)$
C.I.	α_0	$\alpha_0/2$
u.m.	$0, \pi$	0
f.u.m.	$\pm\frac{\pi}{2} \approx \pm 1.571$	π



advection eq.
wave eq.
- - 2nd order
--- 4th order



- modes with speeds of the wrong sign will come out of BHs!
- second order in space systems have high frequency damping built in!