Orbital dynamics

Let's assume the particle moves on a worldline γ with coords z^α satisfying $\frac{D^2 z^\alpha}{d\tau^2} = \epsilon f_1^\alpha + \epsilon^2 f_2^\alpha + \cdots$ (this will be justified later)

At leading order we have $\frac{D^2 z^\alpha}{d\tau^2} = 0$ — i.e., geodesic motion in the background metric $g_{\alpha\beta}$

So let's start by analyzing geodesics in BH spacetimes

Geodesic orbits in Schwarzschild

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega \qquad f = 1 - \frac{2M}{r}$$

Recall that if ξ^α is a Killing vector and z^α is a geodesic, then $u^\alpha \xi_\alpha$ is constant along the geodesic. Schwarzschild has 4 Killing vectors:

$\xi^\alpha_{(t)} = \delta^\alpha_t \Rightarrow$ the orbital energy $E = -u_\alpha \xi^\alpha_{(t)}$ is constant

$\xi^\alpha_{(\phi)} = \delta^\alpha_\phi \Rightarrow$ the z-component of the angular momentum, $L_z = u_\alpha \xi^\alpha_{(\phi)}$, is constant

The other two Killing vectors correspond to rotations about the x and y axes, $\Rightarrow$ the x and y components of the AM are constant.

Since all three components of the AM are constant, we can freely set L_x and L_y to zero $\Rightarrow$ this restricts z^α to the equatorial plane (w/log). So $z^\alpha(\tau) = (t(\tau), r(\tau), \frac{\pi}{2}, \phi(\tau))$ in Schwarzschild coords

$$\left.\begin{aligned} &\text{where } \frac{dt}{d\tau} = u^t = g^{tt}u_t = -f^{-1}(-E) = E/f \\ &\text{and } \frac{d\phi}{d\tau} = u^\phi = \underset{\substack{\uparrow \\ r^{-2} \text{ because } \theta=\pi/2}}{g^{\phi\phi}} u_\phi = r^{-2}L_z \end{aligned}\right\} \begin{aligned} \dot{t} &= E/f \\ \dot{\phi} &= L_z/r^2 \end{aligned}$$

Finally, we have the conserved length,

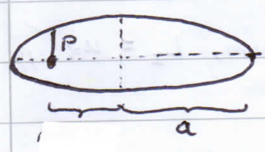
$$\begin{aligned} & g_{\alpha\beta}u^\alpha u^\beta = -1 \\ \Rightarrow\ & -f\dot{t}^2 + f^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1 \\ \Rightarrow\ & \dot{r}^2 = f\left(-1 + f\dot{t}^2 - r^2\dot{\phi}^2\right) \\ &\quad = f\left(-1 + E^2/f - L_z^2/r^2\right) \\ &\quad = E^2 - \underbrace{f\left(1 + L_z^2/r^2\right)}_{\equiv V(r)} \end{aligned}$$

We'll be interested in bound geodesics that oscillate between a minimum and a maximum. The turning points are at $V = E^2$, where $\dot{r} = 0$. We can parametrise the orbit with Keplerian-like parameters by introducing an "eccentricity" $e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$ and "semi-latus rectum" $p = \frac{2 r_{min} r_{max}}{M(r_{min} + r_{max})}$.

In terms of these variables we have $r_{min} = \frac{pM}{1+e}$ and $r_{max} = \frac{pM}{1-e}$; note $0 \le e < 1$. We can find $E(p,e)$ and $L_z(p,e)$ by solving $E^2 = V(r_{min})$ and $E^2 = V(r_{max})$

We can parametrize the radial motion as $r(\psi) = \frac{pM}{1+e\cos\psi}$ where the "radial phase" ψ goes from 0 to 2π in one complete radial cycle.

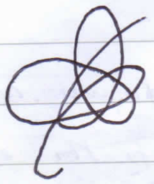
If ψ equalled ϕ, then this would describe an ellipse:



But $\frac{d\phi}{d\psi} = \frac{d\phi}{d\tau}\frac{d\tau}{dr}\frac{dr}{d\psi} = \sqrt{\frac{p}{p-6-2e\cos\psi}} > 1$

$\Rightarrow$ if $\psi \to \psi + 2\pi$, ϕ increases by more than 2π

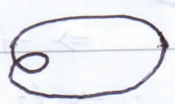
Typical:



Extreme ("zoom whirl"):

We have $r(\psi)$ and $\phi(\psi)$, need $t(\psi)$: $\frac{dt}{d\psi} = \frac{dt}{d\tau}\frac{d\tau}{dr}\frac{dr}{d\psi} = \frac{\dot t(r(\psi))}{\dot r(r(\psi))}\frac{dr}{d\psi}$

The motion has two periods, radial T_r and azimuthal T_ϕ. In general, the orbits are not closed, but if $n_r T_r = n_\phi T_\phi$ for $n_r, n_\phi \in \mathbb{Z}^+$, then they are: e.g., if $T_r = 2T_\phi$



two cycles of ϕ
one radial cycle

Geodesics in Kerr

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2+a^2+\frac{2Ma^2r}{\Sigma}\sin^2\theta\right)\sin^2\theta\, d\phi^2 - \frac{4Mar\sin^2\theta}{\Sigma}dt\,d\phi$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$.

This only has two Killing vectors, $\xi^\alpha_{(t)} = \delta^\alpha_t$ and $\xi^\alpha_{(\phi)} = \delta^\alpha_\phi$

$\Rightarrow$ two constants of motion, $E = -u_t$ and $L_z = u_\phi$

But there is a hidden symmetry associated with the Killing tensor $K_{\alpha\beta} = 2\Sigma\, l_{(\alpha}n_{\beta)} + r^2 g_{\alpha\beta}$ where l^α and n^α are the outgoing and ingoing principle null vectors

$$l^\alpha = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right) \xrightarrow[a\to 0]{M\to 0} l^\alpha = (1,1,0,0)$$

$$n^\alpha = \left(\frac{r^2+a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right) \qquad n^\alpha = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0\right)$$

exercise

In analogy with $\nabla_{(\alpha}\xi_{\beta)} = 0$, $K_{\alpha\beta}$ satisfies $\nabla_{(\alpha}K_{\beta\gamma)} = 0$.
We can check that $K_{\alpha\beta}u^\alpha u^\beta$ is a constant of motion:

$$u^\gamma\nabla_\gamma(K_{\alpha\beta}u^\alpha u^\beta) = \underbrace{u^\gamma\nabla_\gamma K_{\alpha\beta}u^\alpha u^\beta}_{\nearrow 0} + K_{\alpha\beta}(\underbrace{u^\gamma\nabla_\gamma u^\alpha}_{\nearrow 0} u^\beta + u^\alpha \underbrace{u^\gamma\nabla_\gamma u^\beta}_{\nearrow 0})$$
$$= 0.$$

$C = K_{\alpha\beta}u^\alpha u^\beta$ is called the Carter constant. We often work instead with $Q = C - (L_z - aE)^2$, also called the Carter constant. In the $a \to 0$ limit, $C \to L_x^2 + L_y^2 + L_z^2$ and $Q \to L_x^2 + L_y^2$

In terms of these constants of motion, $E = -g_{tt}\dot{t} - g_{t\phi}\dot{\phi}$, $L_z = g_{\phi t}\dot{t} + g_{\phi\phi}\dot{\phi}$, and $C = K_{tt}\dot{t}^2 + 2K_{tr}\dot{t}\dot{r} + 2K_{t\phi}\dot{t}\dot{\phi} + K_{rr}\dot{r}^2 + K_{r\phi}\dot{r}\dot{\phi} + K_{\theta\theta}\dot{\theta}^2 + K_{\phi\phi}\dot{\phi}^2$,

along with $g_{\alpha\beta}u^\alpha u^\beta = g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 = -1$

Rearranging for $\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}$, we get $\Sigma\dot{t} = E\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right] + aL_z\left(1 - \frac{r^2+a^2}{\Delta}\right) \equiv V_t(r,\theta)$

$$(\Sigma\dot{r})^2 = \left[E(r^2+a^2) - aL_z\right]^2 - \Delta\left[r^2 + (L_z - aE)^2 + Q\right] \equiv V_r(r)$$

$$(\Sigma\dot{\theta})^2 = Q - \cot^2\theta\, L_z^2 - a^2\cos^2\theta\,(1-E^2) \equiv V_\theta(\theta)$$

Rewrite

$$\text{and } \Sigma\dot{\phi} = \csc^2\theta\, L_z + aE\left(\frac{r^2+a^2}{\Delta} - 1\right) - \frac{a^2 L_z}{\Delta} \equiv V_\phi(r,\theta)$$

Notice that the derivatives are all of the form $\Sigma \frac{d}{d\tau}$ $\Rightarrow$ we can introduce a new parameter λ satisfying $\frac{d\tau}{d\lambda} = \Sigma$, such that

$$\left(\frac{dr}{d\lambda}\right)^2 = V_r(r),\quad \left(\frac{d\theta}{d\lambda}\right)^2 = V_\theta(\theta),\quad \frac{dt}{d\lambda} = V_t(r,\theta),\quad \frac{d\phi}{d\lambda} = V_\phi(r,\theta)$$

the radial and polar motion are decoupled! They oscillate btwn min and max,

Like in Schwarzschild, we can write

$$r = \frac{PM}{1+e\cos\psi_r}$$

for some "radial phase" ψ_r. $r_{max} = \frac{PM}{1-e}$ and $r_{min} = \frac{PM}{1+e}$ satisfy $V_r(r_{min}) = 0$ and $V_r(r_{max}) = 0$

Similarly, $\cos\theta = (\cos\theta)_{max}\cos\psi_\theta$ for a "polar phase" ψ_θ.

The radial and polar motions have periods $\Lambda_r = 2\int_{r_{min}}^{r_{max}} \frac{d\lambda}{dr}\,dr = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{V_r}}$

and $\Lambda_\theta = 2\int_{\theta_{min}}^{\pi-\theta_{min}} \frac{d\lambda}{d\theta}\,d\theta = 2\int_{\theta_{min}}^{\pi-\theta_{min}} \frac{d\theta}{\sqrt{V_\theta}}$

The azimuthal motion has a period $\Lambda_\phi = \int_0^{2\pi} \frac{d\lambda}{d\phi}\,d\phi = \int_0^{2\pi} \frac{d\phi}{V_\phi}$

If $n_r\Lambda_r = n_\theta\Lambda_\theta$, then there is an "intrinsic" resonance, which has a significant impact on the inspiral

because the dynamics depend directly on r, θ, while ϕ is an "extrinsic" parameter

We have $r(\psi_r)$ and $\theta(\psi_\theta)$, find $\lambda(\psi_r)$ and $\lambda(\psi_\theta)$ from

$$\frac{d\lambda}{d\psi_r} = \frac{d\lambda}{dr}\frac{dr}{d\psi_r} = \frac{\pm 1}{\sqrt{V_r}}\frac{dr}{d\psi_r}$$

$$\text{and } \frac{d\lambda}{d\psi_\theta} = \frac{\pm 1}{\sqrt{V_\theta}}\frac{d\theta}{d\psi_\theta}$$

$\Rightarrow$ Find $t(\lambda)$ and $\phi(\lambda)$ from $r(\lambda)$ and $\theta(\lambda)$

Orbital evolution

How do we account for the forcing terms f_n^α? First, note that the worldline γ and the metric perturbations $h^{(n)}_{\alpha\beta}$ are coupled

$\Rightarrow$ How we describe the accelerated γ will affect how we describe $h^{(n)}_{\alpha\beta}$

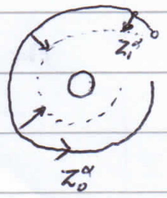
There are several approaches in the literature:

I. Gralla-Wald (GW 2008, Gralla 2012)

$$\underbrace{z^\alpha(\tau,\epsilon)}_{\text{curve}} = \underbrace{z_0^\alpha(\tau) + \epsilon z_1^\alpha(\tau)}_{\text{vectors}} + \epsilon^2 z_2^\alpha(\tau) + \cdots$$

$$h_{\alpha\beta}(x,\epsilon) = \epsilon h^{(1)}_{\alpha\beta}(x; z_0) + \epsilon^2 h^{(2)}_{\alpha\beta}(x; z_0, z_1) + \cdots$$

independent of ϵ



$$\delta G_{\alpha\beta}[h^{(1)}] = 8\pi T^{(1)}_{\alpha\beta}[z_0]$$

$$\delta G_{\alpha\beta}[h^{(2)}] = 8\pi T^{(2)}_{\alpha\beta}[z_0, z_1] - \delta^2 G_{\alpha\beta}[h^{(1)}]$$

$$\vdots$$

$$\frac{D^2 z_0^\alpha}{d\tau^2} = 0, \quad \frac{D^2 z_1^\alpha}{d\tau^2} = f_1^\alpha[h^{(1)}] + R^\alpha{}_{\mu\beta\nu}\, u_0^\mu z_1^\beta u_0^\nu, \ \ldots \qquad u_0^\mu = \frac{dz_0^\mu}{d\tau}$$

solve for z_0^μ, then $h^{(1)}_{\alpha\beta}$, then z_1^μ, then $h^{(2)}_{\alpha\beta}$, etc.

Note: $z_1^\alpha, z_2^\alpha, \ldots$ grow large with time

$\Rightarrow$ this approximation breaks down well before the radiation-reaction time $t_{rr} \sim 1/\epsilon$

II. Self-consistent (Pound 2009, 2012)

don't expand z^α

$$h_{\alpha\beta}(x,\epsilon) = \epsilon h^{(1)}_{\alpha\beta}(x;\gamma) + \epsilon^2 h^{(2)}_{\alpha\beta}(x;\gamma) + \cdots$$

depend on ϵ

$$E_{\alpha\beta}[\bar{h}^{(1)}] = -16\pi T^{(1)}_{\alpha\beta}[\gamma]$$

$$E_{\alpha\beta}[\bar{h}^{(2)}] = -16\pi T^{(2)}_{\alpha\beta}[\gamma] + 2\delta^2 G_{\alpha\beta}[h^{(1)}]$$

$$\vdots$$

$$\frac{D^2 z^\alpha}{d\tau^2} = \epsilon f_1^\alpha[h^{(1)}] + \epsilon^2 f_2^\alpha[h^{(1)}, h^{(2)}] + \cdots$$

solve as coupled system

$$\left[+ \text{ constraint } \nabla^\beta(\epsilon \bar{h}^{(1)}_{\alpha\beta} + \epsilon^2 \bar{h}^{(2)}_{\alpha\beta} + \cdots) = 0\right]$$

Accurately tracks z^μ $\Rightarrow$ accurate on rad-reaction time

III, Two-timescale approximation

Let $J_\alpha = \{E, L, C\}$

and $\psi^\alpha = \{\psi^r, \psi^\theta, \psi^\phi\}$

When we account for the self-force, J_α evolves (slowly) with time:

$$J_\alpha = J^0_\alpha(\tilde{t}) + \epsilon J^1_\alpha(\tilde{t}) + \cdots$$

where $\tilde{t} \equiv \epsilon t$ is "slow time", $\tilde{t} \sim \epsilon^0$ when $t \sim t_{rr} \sim 1/\epsilon$

The phases evolve according to $\dfrac{d\psi^\alpha}{dt} = \Omega^\alpha = \Omega^\alpha_0(J^0(\tilde{t})) + \epsilon \Omega^\alpha_1(J^0(\tilde{t}), J^1(\tilde{t})) + \cdots$

$+\sqrt{\epsilon}$ for resonance

$$\Rightarrow \psi^\alpha = \int \Omega^\alpha dt = \frac{1}{\epsilon}\Big(\underbrace{\int \Omega^\alpha_0 d\tilde{t}}_{\text{called "adiabatic" order}} + \epsilon \underbrace{\int \Omega^\alpha_1 d\tilde{t}}_{\text{"post-1-adiabatic" order}} + \mathcal{O}(\epsilon^2)\Big)$$

this expansion is also valid for the GW phase

We write $z^\alpha = z^\alpha_0(J^\beta_0, \psi^\beta) + \epsilon z^\alpha_1(J^\beta_0, J^\beta_1, \psi^\beta) + \cdots$

This has the same dependence on J^β_0 and ψ^β as a geodesic would have. But it isn't actually a geodesic, because J^β_0 and ψ^β have non-geodesic dependence on time

and $h_{\alpha\beta} = \epsilon h^{(1)}_{\alpha\beta}(x^A, \tilde{t}; J^\gamma_0, \psi^\gamma) + \epsilon^2 h^{(2)}_{\alpha\beta}(x, \tilde{t}; J^\gamma_0, J^\gamma_1, \psi^\gamma) + \cdots$, where $x^A = (r, \theta, \ldots$

st z^α_n and $h^{(n)}_{\alpha\beta}$ are periodic in ψ^γ

i.e. $h^{(n)}_{\alpha\beta} = \sum_{k_\alpha \in \mathbb{Z}} h^{(n,k)}_{\alpha\beta}(x^A, \tilde{t})\, e^{-ik_\alpha \psi^\alpha}$

$\Rightarrow$ the exponentials factor out:

$$\left.\begin{aligned} \delta G^{(0,k)}_{\alpha\beta}[h^{(1,k)}] &= 8\pi T^{(1,k)}_{\alpha\beta}[J^\gamma_0] \\ \delta G^{(0,k)}_{\alpha\beta}[h^{(2,k_\alpha)}] &= 8\pi T^{(2,k_\alpha)}_{\alpha\beta} - \delta^2 G^{(0,k_\alpha)}_{\alpha\beta}[h^{(1,k_\alpha)}] \\ &\quad - \underbrace{\delta G^{(1,k_\alpha)}_{\alpha\beta}}_{\text{contains one } \frac{\partial}{\partial\tilde{t}} \text{ or one } \Omega^\alpha_1}[h^{(1,k_\alpha)}] \end{aligned}\right\} \text{independent of } \psi^\alpha$$

$$\vdots$$

Note: $\dfrac{d}{dt} f(\tilde{t}) = \epsilon \dfrac{d}{d\tilde{t}} f(\tilde{t})$

$\dfrac{D^2 z^\alpha}{d\tau^2} = f^\alpha$ becomes $\dfrac{dJ^\alpha_0}{d\tilde{t}} = F^\alpha_1[h^{(1,k_\alpha)}]$, $\dfrac{dJ^\alpha_1}{d\tilde{t}} = F^\alpha_2[h^{(2,k_\alpha)}, h^{(1,k_\alpha)}]$, ...

and $\psi^\alpha = \dfrac{1}{\epsilon}\left(\int \Omega^\alpha_0 d\tilde{t} + \epsilon \int \Omega^\alpha_1 d\tilde{t} + \cdots\right)$