Matched asymptotic expansions

So far we have assumed m can be approximated as a point particle. But this suggests that its gravitational field will behave as $h^{(1)}_{\alpha\beta} \sim \frac{m}{r}$

in this section, r is distance from the particle/object

$\Rightarrow$ the field blows up at the particle

If the self-force comes from some finite piece of $h^{(1)}_{\alpha\beta}$, which finite piece does it come from? We need some way to determine this.

At second and higher order, the problem is worse. Recall

$$\delta G_{\alpha\beta}[h^{(1)}] = 8\pi T^{(1)}_{\alpha\beta}$$

$$\delta G_{\alpha\beta}[h^{(2)}] = 8\pi T^{(2)}_{\alpha\beta} - \delta^2 G_{\alpha\beta}[h^{(1)}] \quad (*)$$

where $\delta^2 G_{\alpha\beta}[h^{(1)}] \sim \partial h^{(1)} \partial h^{(1)} + h^{(1)} \partial^2 h^{(1)}$

$$\sim \frac{m^2}{r^4}$$

This is too singular to be a well-defined distributional source: Its integral against a test field is $\int \delta^2 G_{\alpha\beta}\, \varphi^{\alpha\beta}\, dV \sim \iint_{r=0}^{R} \frac{m^2}{r^4} \varphi^{\alpha\beta}\, r^2 dr\, dt\, d\Omega$

some region including γ

$$\sim \int_{r=0}^{R} \frac{dr}{r^2} + \mathcal{O}(1/r)$$

$$= \infty$$

This means $(*)$ is not well defined on a region intersecting γ. This is a manifestation of a general result in GR:

- The exact (fully nonlinear) EFE with a point-particle source has no solution within a space of well-behaved functions.

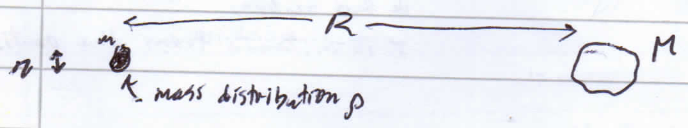
We can easily see what's going wrong: near the small object, <u>its gravity dominates over that of the large BH</u> $\Rightarrow$ in a small region around m, it does not make sense to write $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \cdots$

field of large BH

"small" perturbations due to m

To properly determine how to incorporate the small object into the EFE, we'll analyse the field in a small region around it.

Before doing that, it will be illustrative to consider the <u>Newtonian</u> case.



Say m is compact, such that its radius $r \sim m$. ← Cartesian coords x^i

ρ sources a gravitational field satisfying $\partial^i \partial_i \phi^S = 4\pi\rho$

m's "self-field"

At distances $r \gg r$, we can approximate ϕ^S with a <u>multipole expansion</u>

$$\phi^S = \frac{m}{r} + \frac{m_i n^i}{r^2} + \frac{m_{ij} n^i n^j}{r^3} + \dots$$

where n^i is a unit vector pointing radially outwards from the origin $r=0$.
$\{m_{i_1 \dots i_\ell}\}$ are ρ's multipole <u>moments</u>

m = total mass in ρ

m^i = location of c.o.m relative to $r=0$

m^{ij} = quadrupole moment

etc.

At the same time, M sources its own gravitational field.
Let's call it the "external field". In a region near m, we can write this as a Taylor series around $r=0$:

$$\phi^{ext} = \phi^{ext}(0) + r\partial_i \phi^{ext}(0) n^i + \tfrac{1}{2} r^2 \partial_i \partial_j \phi^{ext}(0) n^i n^j + \dots$$

This will be a good approximation if $r \ll R$.

called "buffer region" → So in a region $r \ll r \ll R$, we can express the total field as

$$\begin{aligned}\phi &= \phi^S + \phi^{ext} \\ &= \frac{m}{r} + \frac{m_i n^i}{r^2} + \frac{m_{ij} n^i n^j}{r^3} + \dots \\ &\quad + \phi^{ext}(0) + r\partial_i \phi^{ext}(0) n^i + \tfrac{1}{2} r^2 \partial_i \partial_j \phi^{ext}(0) n^i n^j + \dots\end{aligned}$$

Keep this example in mind!

To obtain a local metric of this form, we use the <u>method of matched asymptotic expansions</u>.

in this region, use

$$\tilde{g}_{\alpha\beta} = g^{obj}_{\alpha\beta}(r/M) + \epsilon H^{(1)}_{\alpha\beta}(r/M) + \epsilon^2 H^{(2)}_{\alpha\beta}(r/M) + \ldots$$

"inner expansion"

in most of spacetime,

use $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$

"outer expansion"

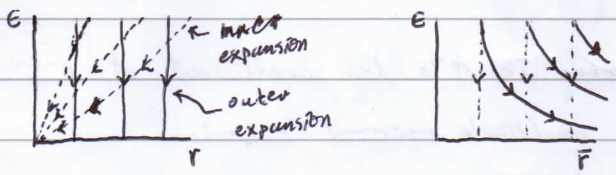
More concretely, adopt Cartesian coords (t, x^a) centred on the small object, and define the scaled coords $\bar{x}^a = x^a/\epsilon$.

The outer expansion is performed in the limit $\epsilon \to 0$ at fixed x^a (i.e., $x^a \sim \epsilon^0 \sim M$)

$$\tilde{g}_{\alpha\beta}(t, x^a, \epsilon) = g_{\alpha\beta}(t, x^a) + \epsilon h^{(1)}_{\alpha\beta}(t, x^a) + \epsilon^2 h^{(2)}_{\alpha\beta}(t, x^a) + \ldots$$

The inner expansion is performed in the limit $\epsilon \to 0$ at fixed $\bar{x}^a$ (i.e., $\bar{x}^a \sim \epsilon^0$, or $x^a \sim \epsilon \sim m$)

$$\tilde{g}_{\alpha\beta}(t, \bar{x}^a, \epsilon) = g^{obj}_{\alpha\beta}(t, \bar{x}^a) + \epsilon H^{(1)}_{\alpha\beta}(t, \bar{x}^a) + \epsilon^2 H^{(2)}_{\alpha\beta}(t, \bar{x}^a) + \ldots$$



outer: • object shrinks to zero mass and size
• external lengths fixed

inner: • object size fixed
• external lengths blow up

(<u>Note</u>: in self-consistent case, x^a is centred on the accelerated worldline γ

in Gralla-Wald case, x^a is " " " zeroth-order " γ_0)

<u>Matching condition</u>: Since these are two expansions of the same metric $g_{\alpha\beta}$, they must "match"

Let's write the outer expansion as $\tilde{g}_{\alpha\beta} = \sum_{n\geq 0} \epsilon^n \tilde{g}^{(n)}_{\alpha\beta}(r)$

and " inner " " $\tilde{g}_{\alpha\beta} = \sum_{n\geq 0} \epsilon^n \bar{g}^{(n)}_{\alpha\beta}(\bar{r})$

expand near the object → Now let's perform an inner expansion of the outer expansion: $\tilde{g}_{\alpha\beta} = \sum_{n\geq 0} \epsilon^n \sum_p \epsilon^p \bar{r}^p \tilde{g}^{(n,p)}_{\alpha\beta}$

$$= \sum_{n,p} \epsilon^n r^p \tilde{g}^{(n,p)}_{\alpha\beta}$$

expand far from the object → and an outer " " " inner " : $\tilde{g}_{\alpha\beta} = \sum_{n\geq 0} \epsilon^n \sum_p \epsilon^p/r^p \, \bar{g}^{(n,p)}_{\alpha\beta}$

$$= \sum_{n,p} \epsilon^{n+p} r^{-p} \bar{g}^{(n,p)}_{\alpha\beta}$$

These are both expansions of the same function ⇒ they should agree term by term

$$\Rightarrow \tilde{g}^{(n,p)}_{\alpha\beta} = \bar{g}^{(n+p,-p)}_{\alpha\beta}$$

(Note: these double expansions should be accurate in the buffer region $m \ll r \ll M$

One consequence: since $\bar{g}^{(n+p,-p)}_{\alpha\beta} = 0 \;\; \forall\, n+p<0$, we must have $\tilde{g}^{(n,p)}_{\alpha\beta} = 0 \;\; \forall\, p<-n$

i.e., $\tilde{g}^{(n,p)}_{\alpha\beta}$ starts at order $\frac{1}{r^n}$

$$
\begin{array}{llllll}
\Rightarrow \tilde{g}^{(0)}_{\alpha\beta} = g_{\alpha\beta} & = & r^0 g^{(0)}_{\alpha\beta} & + \; r\, g^{(1)}_{\alpha\beta} & + \; r^2 g^{(2)}_{\alpha\beta} & + \cdots \\
\tilde{g}^{(1)}_{\alpha\beta} = h^{(1)}_{\alpha\beta} & = & \frac{1}{r} h^{(1,-1)}_{\alpha\beta} & + \; r^0 h^{(1,0)}_{\alpha\beta} & + \; r\, h^{(1,1)}_{\alpha\beta} & + \cdots \\
\tilde{g}^{(2)}_{\alpha\beta} = h^{(2)}_{\alpha\beta} & = & \frac{1}{r^2} h^{(2,-2)}_{\alpha\beta} & + \; \frac{1}{r} h^{(2,-1)}_{\alpha\beta} & + \; r^0 h^{(2,0)}_{\alpha\beta} & + \cdots \\
\vdots & & & & & \\
\tilde{g}^{(n)}_{\alpha\beta} = h^{(n)}_{\alpha\beta} & = & \frac{1}{r^n} h^{(n,-n)}_{\alpha\beta} & + \; \frac{1}{r^{n-1}} h^{(n,-n+1)}_{\alpha\beta} & + \; \frac{1}{r^{n-2}} h^{(n,-n+2)}_{\alpha\beta} & + \cdots \\
 & & \vdots & \vdots & \vdots & \\
 & & \shortparallel & \shortparallel & \shortparallel & \\
 & & g^{obj}_{\alpha\beta} & H^{(1)}_{\alpha\beta} & H^{(2)}_{\alpha\beta} &
\end{array}
$$

$$h^{(n,-n)}_{\alpha\beta} = \bar{g}^{(0,n)}_{\alpha\beta} = g^{obj,(n)}_{\alpha\beta}, \text{ where } g^{obj}_{\alpha\beta} = \sum_{n\geq 0} \frac{\epsilon^n}{r^n} g^{obj,(n)}_{\alpha\beta} \qquad (*)$$

i.e. the leading term in $h^{(n)}_{\alpha\beta}$ is determined by the metric of the small object if it were isolated

Recall the Newtonian case, where in ϕ^S, the coefficient of $r^{-\ell-1}$ was determined by the small object's multipole moment $m_{i_1\cdots i_\ell}$

— in an analogous way, $g^{obj,(n)}_{\alpha\beta}$ is determined by the object's moments $\underbrace{m_{i_1\cdots i_{n-1}}}_{\text{"mass moments"}}$ and $\underbrace{S_{i_1\cdots i_{n-1}}}_{\text{"spin/current moments"}}$ (and lower moments)

— but these are defined directly from (*), not from integrating over a matter distribution (so they are defined for a BH, not just for a material body)

$$\Rightarrow h^{(1)}_{\alpha\beta} \sim \frac{m}{r} + \cdots$$

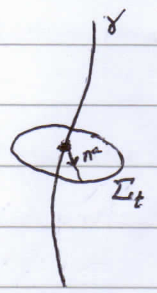
$$h^{(2)}_{\alpha\beta} \sim \frac{m^2 + m_i n^i + \epsilon_{ijk} S^j n^k}{r^2} + \cdots$$

in $h^{(3)}_{\alpha\beta}$, the quadrupole moments m_{ij} and S_{ij} appear

etc.

We now have the general form of the metric in a neighborhood of a compact object — but we haven't yet imposed the EFE. Imposing the EFE will further restrict the form of the field.

For concreteness, let's adopt <u>Fermi-Walker coordinates</u>.



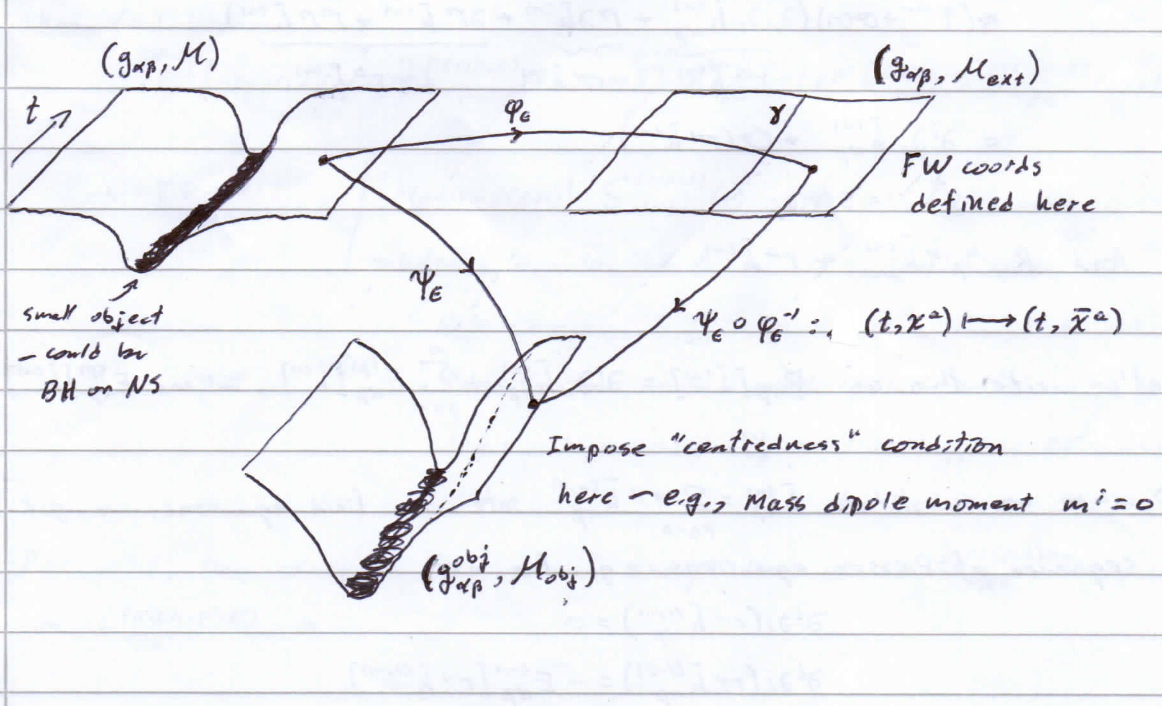
Let t be proper time τ on γ (as measured in $g_{\alpha\beta}$). At each t, send out spatial geodesics orthogonal (in $g_{\alpha\beta}$) to u^α. These span a surface Σ_t. Label each geodesic with a unit vector n^a (defined at $\Sigma_t \cap \gamma$, in the tangent space of Σ_t). Let r be the proper distance along the geodesic. Then $x^a = r n^a$ define coords on Σ_t, and (t, x^a) define coords in a neighborhood of γ. In these coords,

$$\left.\begin{aligned} g_{tt} &= -\left[1 + 2a_i x^i + (a_i x^i)^2 + R_{titj}(t)\, x^i x^j + \mathcal{O}(r^3)\right] \\ g_{ta} &= -\tfrac{2}{3} R_{tiaj}(t)\, x^i x^j + \mathcal{O}(r^3) \\ g_{ab} &= \delta_{ab} - \tfrac{1}{3} R_{aibj}(t)\, x^i x^j + \mathcal{O}(r^3) \end{aligned}\right\} \quad g_{\alpha\beta}|_\gamma = \eta_{\alpha\beta}$$

↖ Riemann tensor of $g_{\alpha\beta}$ on γ.

Here $a^\alpha = \dfrac{D^2 z^\alpha}{d\tau^2}$ is γ's proper acceleration in $g_{\alpha\beta}$.

How is γ related to the "center" of the object?



Strategy: • work in self-consistent framework, so γ is the self-accelerated wo[rldline]

• solve
$$\left.\begin{aligned} E_{\alpha\beta}[\bar{h}^{(1)}] &= 0 \\ E_{\alpha\beta}[\bar{h}^{(2)}] &= 2\delta^2 G_{\alpha\beta}[h^{(1)}] \\ &\vdots \end{aligned}\right\}\ \text{EFE \underline{outside} object, in vacuum}$$

order by order in r, without constraining γ

• Substitute the solution into the gauge condition
$$\nabla^\beta(\epsilon \bar{h}^{(1)}_{\alpha\beta} + \epsilon^2 \bar{h}^{(2)}_{\alpha\beta} + \cdots) = 0$$
along with the expansion $a^\alpha = a_0^\alpha + \epsilon a_1^\alpha + \cdots$

$\Rightarrow$ obtain equations for each a_n^α

From this, we will find out (i) the physically correct form of the metric near γ
(ii) how γ moves in response to the metric

To start, note that a spatial derivative, $\frac{\partial}{\partial x^i}$, lowers the power of r by one; $\partial_i r^p = p r^{p-1} \underbrace{\partial_i r}_{n_i}$ where $n_i = \delta_{ij} n^j$

(check: $r = \sqrt{\delta_{ab} x^a x^b} \Rightarrow \partial_i r = \frac{1}{2} r^{-1}(\delta_{ab}\delta^a_i x^b \times 2) = \frac{x_i}{r} = n_i$) $\quad x_i = \delta_{ij} x^j$

We can use this to considerably simplify the structure of the equations:
$$\begin{aligned}
\Box \bar{h}^{(n)}_{\alpha\beta} &= g^{\mu\nu} \nabla_\mu \nabla_\nu \bar{h}^{(n)}_{\alpha\beta} \\
&\approx (\eta^{\mu\nu} + \mathcal{O}(r))\Big(\underbrace{\partial_\mu\partial_\nu \bar{h}^{(n)}_{\alpha\beta}}_{\sim r^{-2}\bar{h}^{(n)}} + \underbrace{\Gamma\partial\bar{h}^{(n)}}_{\sim r^{-1}\bar{h}^{(n)}} + \underbrace{\partial\Gamma\,\bar{h}^{(n)} + \Gamma\Gamma\,\bar{h}^{(n)}}_{\sim r^0 \bar{h}^{(n)}}\Big) \\
&= \partial^i\partial_i \bar{h}^{(n)}_{\alpha\beta} + \mathcal{O}(r^{-1}\bar{h}^{(n)})
\end{aligned}$$
$\partial^i = \delta^{ij}\partial_j$

And $R_\alpha{}^\mu{}_\beta{}^\nu \bar{h}^{(n)}_{\mu\nu} \sim r^0 \bar{h}^{(n)}$

Let's write this as $E_{\alpha\beta}[\bar{h}^{(n)}] = \partial^i\partial_i \bar{h}^{(n)}_{\alpha\beta} + \sum_{p=-1}^{\infty} E^{(p)}_{\alpha\beta}[\bar{h}^{(n)}]$, where $E^{(p)}_{\alpha\beta}[\bar{h}^{(n)}] \propto r^p \bar{h}^{(n)}$

$\Rightarrow$ when we substitute $\bar{h}^{(n)}_{\alpha\beta} = \sum_{p\geq -n} r^p \bar{h}^{(n,p)}_{\alpha\beta}$ into the field equations, we get a sequence of Poisson equations. e.g., for $n=1$,
$$\begin{aligned}
\partial^i\partial_i(r^{-1}\bar{h}^{(1,-1)}_{\alpha\beta}) &= 0 \\
\partial^i\partial_i(r^0 \bar{h}^{(1,0)}_{\alpha\beta}) &= -E^{(-1)}_{\alpha\beta}[r^{-1}\bar{h}^{(1,-1)}] \\
\partial^i\partial_i(r \bar{h}^{(1,1)}_{\alpha\beta}) &= -E^{(-1)}_{\alpha\beta}[r^0 \bar{h}^{(1,0)}] - E^{(0)}_{\alpha\beta}[r^{-1}\bar{h}^{(1,-1)}]
\end{aligned}$$

Or in general, $\partial^i\partial_i\left(r^p\bar{h}^{(1,p)}_{\alpha\beta}\right) = -\sum\limits_{p'=-1}^{p-1} E^{(p-2-p')}_{\alpha\beta}\left[r^{p'}\bar{h}^{(1,p')}\right]$ (*)

To solve these equations, it's useful to expand each $\bar{h}^{(n,p)}_{\alpha\beta}$ in spherical harmonics, which are eigenfunctions of the Laplacian: $\partial^i\partial_i Y_{\ell m} = -\frac{\ell(\ell+1)}{r^2}Y_{\ell m}$
(These are defined on spheres around m, not around M).

Rather than using $Y_{\ell m}$, it's easier to use $\hat{n}^L = n^{\langle L\rangle} = n^{\langle i_1}n^{i_2}\cdots n^{i_\ell\rangle}$ ← symmetric trace-free (STF)

These also satisfy $\partial^i\partial_i\hat{n}^L = -\frac{\ell(\ell+1)}{r^2}\hat{n}^L$.

We write

$$\bar{h}^{(n,p)}_{\alpha\beta} = \sum_{\ell\geq 0}\bar{h}^{(n,p,\ell)}_{\alpha\beta L}(t)\,\hat{n}^L \qquad \left(\bar{h}^{(n,p,\ell)}_{\alpha\beta i_1\cdots i_\ell}(t)\,\hat{n}^{i_1\cdots i_\ell} = \sum_{m=-\ell}^{+\ell}\bar{h}^{(n,p,\ell,m)}_{\alpha\beta}(t)\,Y^{\ell m}\right)$$

$$\Rightarrow \partial^i\partial_i\left(r^p\bar{h}^{(n,p)}_{\alpha\beta}\right) = \sum_{\ell\geq 0}\underbrace{\partial^i\partial_i\left(r^p\hat{n}^L\right)}_{\partial_i\left(p r^{p-1}n^i\hat{n}^L + r^p\partial^i\hat{n}^L\right)}\bar{h}^{(n,p,\ell)}_{\alpha\beta L}(t)$$

$$= p(p-1)r^{p-2}\underbrace{n_i n^i}_{1}\hat{n}^L + p r^{p-1}\Big(\underbrace{\partial_i n^i}_{2/r}\hat{n}^L + \underbrace{n^i\partial_i\hat{n}^L}_{\partial_r\hat{n}^L = 0}\Big)$$

$$+ p r^{p-1}\underbrace{n_i\partial^i\hat{n}^L}_{0} + r^p\underbrace{\partial^i\partial_i\hat{n}^L}_{-\frac{\ell(\ell+1)}{r^2}\hat{n}^L}$$

$$\Rightarrow \partial^i\partial_i\left(r^p\bar{h}^{(n,p)}_{\alpha\beta}\right) = \sum_{\ell\geq 0} r^{p-2}\left[p(p+1)-\ell(\ell+1)\right]\hat{n}^L\,\bar{h}^{(n,p,\ell)}_{\alpha\beta L}(t)$$

If we also expand $E^{(p)}_{\alpha\beta}[\bar{h}]$ in harmonics, $E^{(p)}_{\alpha\beta}[\bar{h}] = \sum_\ell E^{(p,\ell)}_{\alpha\beta L}[\bar{h}]\,\hat{n}^L$,
then (*) becomes

$$r^{p-2}\left[p(p+1)-\ell(\ell+1)\right]\bar{h}^{(1,p,\ell)}_{\alpha\beta L}(t) = -\sum_{p'=-1}^{p-1}E^{(p-2-p')}_{\alpha\beta L}\left[r^{p'}\bar{h}^{(1,p')}\right] \equiv S^{(1,p,\ell)}_{\alpha\beta L}(t)\,r^{p-2}$$

$$\Rightarrow \bar{h}^{(1,p,\ell)}_{\alpha\beta L}(t) = \begin{cases}\left[p(p+1)-\ell(\ell+1)\right]^{-1}S^{(1,p,\ell)}_{\alpha\beta L}(t) & \text{if } p(p+1)\neq\ell(\ell+1)\\ \text{arbitrary function of } t & \text{if } p(p+1)=\ell(\ell+1)\end{cases}$$

← Note: assumes $S^{(1,p,\ell)}_{\alpha\beta L} = 0$ for these cases; if not, these functions remain arbitrary, but we have to introduce $\log r$ terms into $h^{(n)}_{\alpha\beta}$ — this happens for $n>1$.

For $n>1$, the story is the same, except the source $S^{(n,p,\ell)}_{\alpha\beta L}$ depends on $\bar{h}^{(n'<n,p',\ell')}_{\alpha\beta L'}$.

<u>Conclusion</u>: every $\hat{h}^{(n,p,\ell)}_{\alpha\beta L}(t)$, $\forall\, n,p,\ell$, ends up algebraically determined by the modes $\hat{h}^{(n,p,\ell_p)}_{\alpha\beta L_p}(t)$ satisfying $p(p+1)=\ell_p(\ell_p+1)$

What are these special modes?

For $p<0$, they appear in the metric as $\dfrac{\hat{h}^{(n,-\ell-1,\ell)}_{\alpha\beta L}\,\hat{n}^L}{r^{\ell+1}}$ ← $p=-\ell-1$ is the unique soln. to $p(p+1)=\ell(\ell+1)$ for p

— this is just like the terms in ϕ^S that we saw in the Newtonian case.

In fact, $\hat{h}^{(n,-\ell-1,\ell)}_{\alpha\beta L}$ can be written purely in terms of either a moment of $g^{obj}_{\alpha\beta}$ or a correction to such a moment

For $p\geq 0$, these modes appear in the metric as $r^\ell\,\hat{h}^{(n,\ell,\ell)}_{\alpha\beta L}\,\hat{n}^L$ ← $p=\ell$ is the unique soln to $p(p+1)=\ell(\ell+1)$ for $p\geq 0$

— this is just like the terms in ϕ^{ext}

Motivated by these analogies, let's define

$$h_{\alpha\beta}=h^S_{\alpha\beta}+h^R_{\alpha\beta}$$

where $h^R_{\alpha\beta}$ is the piece of the locally constructed soln. involving <u>only</u> the modes $\hat{h}^{(n,\ell,\ell)}_{\alpha\beta L}$ (and linear and nonlinear combinations of them), and $h^S_{\alpha\beta}$ contains all the dependence on the modes $\hat{h}^{(n,-\ell-1,\ell)}_{\alpha\beta L}$ (though it also contains nonlinear combinations of them with $\hat{h}^{(n,\ell,\ell)}_{\alpha\beta L}$ modes).

<u>Properties</u>:

- $h^R_{\alpha\beta}$ is smooth at $r=0$, and it satisfies the vacuum equations $E_{\alpha\beta}[\bar{h}^{R(1)}]=0$, $E_{\alpha\beta}[\bar{h}^{R(2)}]=2\delta^2G_{\alpha\beta}[\bar{h}^{R(1)}]$, ..., including at $r=0$, to all orders in ϵ. Locally, the metric $g^{eff}_{\alpha\beta}=g_{\alpha\beta}+h^R_{\alpha\beta}$ is indistinguishable from an "external" metric. We call it the "effective external metric"
- $h^S_{\alpha\beta}$ involves all the local dependence on m's multipole structure. We can think of it as a self-field. But note that it doesn't satisfy "nice" equations: for $r\neq 0$, $E_{\alpha\beta}[\bar{h}^{S(1)}]=0$, $E_{\alpha\beta}[\bar{h}^{S(2)}]=2\delta^2G_{\alpha\beta}[h^{(1)}]-2\delta^2G_{\alpha\beta}[h^{R(1)}]$

We could other pairs $h^{S'}_{\alpha\beta}$ and $h^{R'}_{\alpha\beta}$ with the same properties, but this pair arises most naturally from the algorithm used to find the local solution.

Summary of results

Working through the algorithm and imposing the gauge condition, one finds the following structure:

$$\left.\begin{aligned}
\bar{h}^{S(1)}_{tt} &\approx \frac{4m}{r} + \text{"}m a_i n^i\text{"} + \text{"}mr(a_i a_j + R_{titj})n^i n^j\text{"} + \mathcal{O}(r^2) \\
\bar{h}^{S(1)}_{ta} &\approx \text{"}mr(\dot{a}_a + \text{Riemann term})\text{"} + \mathcal{O}(r^2) \\
\bar{h}^{S(1)}_{ab} &\sim mr(a_a a_b + \text{Riemann}) + \mathcal{O}(r^2)
\end{aligned}\right\}$$

known to order r^2 (inclusive) – or to r^4, neglecting acceleration

$$\left.\begin{aligned}
\bar{h}^{S(2)}_{tt} &\approx \frac{3m^2}{r^2} + \frac{mh^{R(1)}}{r} + \mathcal{O}(r^0) \\
\bar{h}^{S(2)}_{ta} &\approx \frac{2\epsilon_{aij} n^i S^j}{r^2} + \frac{mh^{R(1)}}{r} + \mathcal{O}(r^0) \\
\bar{h}^{S(2)}_{ab} &\sim -\frac{7m^2(\hat{n}_{ab} + \frac{1}{3}\delta_{ab})}{r^2} + \frac{mh^{R(1)}}{r} + \mathcal{O}(r^0)
\end{aligned}\right\}$$

known to order r^2 (inclusive)

$\bar{h}^{R(n)}_{\alpha\beta}$ is known in terms of the "special modes" $\bar{h}^{(n,p,p)}_{\alpha\beta P}(t)$, but those modes are not themselves determined by the local analysis near the small object; they are only fixed when boundary conditions are imposed at large distances.

The EOM is found to be

$$\frac{D^2 Z^\alpha}{d\tau^2} = -\frac{1}{2}(g^{\alpha\beta} + u^\alpha u^\beta)(2h^{R(1)}_{\beta\mu;\nu} - h^{R(1)}_{\mu\nu;\beta})u^\mu u^\nu + \overbrace{\frac{1}{2m} R^\alpha{}_{\beta\mu\nu} u^\beta S^{\mu\nu}}^{\text{Mathisson-Papapetrou term}} + \mathcal{O}(\epsilon^2)$$

$S^{\mu\nu} \leftarrow \delta^\mu_a \delta^\nu_b \epsilon^{ab}{}_c S^c$

This is for a generic compact object. For an object with $S_i = m_{ij} = S_{ij} = 0$, the EOM is known to second order:

$$\frac{D^2 Z^\alpha}{d\tau^2} = -\frac{1}{2}(g^{\alpha\beta} + u^\alpha u^\beta)(g_\beta{}^\gamma - h^R_\beta{}^\gamma)(2h^R_{\gamma\mu;\nu} - h^R_{\mu\nu;\gamma})u^\mu u^\nu + \mathcal{O}(\epsilon^3)$$

where $h^R_{\alpha\beta} = \epsilon h^{R(1)}_{\alpha\beta} + \epsilon^2 h^{R(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)$.

These results all come directly from the EFE in the region near the small object (along with a "centredness" condition on γ). We can write them in a more suggestive form:

$$\frac{D^2_{\text{eff}} Z^\alpha}{d\tau^2_{\text{eff}}} = \frac{1}{2m} R^\alpha{}_{\beta\mu\nu} S^{\mu\nu} + \mathcal{O}(\epsilon^2) \quad (*) \qquad \text{and} \qquad \frac{D^2_{\text{eff}} Z^\alpha}{d\tau^2_{\text{eff}}} = \mathcal{O}(\epsilon^3) \quad (**)$$

where $\frac{D_{eff}}{d\tau_{eff}}$ is the covariant derivative along γ, $u^{\alpha}_{eff}\nabla^{eff}_{\alpha}$, where τ_{eff} and ∇^{eff}_{α} are defined with respect to the effective metric $g^{eff}_{\alpha\beta} = g_{\alpha\beta} + h^{R}_{\alpha\beta}$.

(*) is the EOM for a spinning test body in $g^{eff}_{\alpha\beta}$

(**) is the EOM for a test mass in $g^{eff}_{\alpha\beta}$

— this furthers the interpretation of $g^{eff}_{\alpha\beta}$ as the "external" metric from the perspective of the small object

One can also show that if $\tilde{g}_{\alpha\beta}$ is causal (i.e., satisfies retarded boundary conditions), then $g^{eff}_{\alpha\beta}$ is also causal when evaluated at a point on γ (i.e., it only depends on the causal past of that point). This again makes $g^{eff}_{\alpha\beta}$ seem like a "physical" ~~ex~~ external metric. But note that at points off γ, $g^{eff}_{\alpha\beta}$ is <u>not</u> causal; it is only an effective external metric, not the physical one.

Can we recover the point-particle approximation? Yes! (for $n=1$)

We have $\bar{h}^{(1)}_{\alpha\beta} = \frac{4m}{r} \delta^t_\alpha \delta^t_\beta + O(r^0)$

Let's take this to hold for all $r>0$

— this doesn't affect the field for $r \gg m$; it just replaces the true field in the small region $r \sim m$ (think back to the Newtonian case again)

Now define $T^{(1)}_{\alpha\beta} \equiv -\frac{1}{16\pi} E_{\alpha\beta}[\bar{h}^{(1)}]$

Since $\bar{h}^{(1)}_{\alpha\beta}$ is integrable (i.e. $\int |\bar{h}^{(1)}_{\alpha\beta}| dV < \infty$), $E_{\alpha\beta}[\bar{h}^{(1)}]$ is well defined as a distribution. To find out what it is, integrate against a test field:

$$-\frac{1}{16\pi}\int E_{\alpha\beta}[\bar{h}^{(1)}]\varphi^{\alpha\beta}\,dV = -\frac{1}{16\pi}\int \bar{h}^{(1)}_{\alpha\beta} E^{\alpha\beta}[\varphi]\,dV$$
$$\vdots$$
$$= m\varphi^{tt}(t,0)$$
$$= m\,\varphi_{\alpha\beta}(z)\,u^\alpha u^\beta$$

$$\Rightarrow T^{\alpha\beta} = m\int_\gamma u^\alpha u^\beta \frac{\delta^4(x-z)}{\sqrt{-g}}\,d\tau$$

$\therefore$ the physical field $\bar{h}^{(1)}_{\alpha\beta}$ is identical to the field sourced by a point mass m moving on γ.