Puncture schemes

We want to solve
$$E_{\alpha\beta}[\bar h^{(1)}] = 0 \qquad x \notin \gamma \qquad (1)$$
$$E_{\alpha\beta}[\bar h^{(2)}] = 2\,\delta^2 G_{\alpha\beta}[h^{(1)}] \qquad x \notin \gamma \qquad (2)$$

subject to the (free) boundary conditions
$$\bar h^{(n)}_{\alpha\beta} = \bar h^{S(n)}_{\alpha\beta} + \bar h^{R(n)}_{\alpha\beta} \quad (3)$$ near γ, for some smooth vacuum perturbations $\bar h^{(n)}_{\alpha\beta}$

the equation of motion
$$\frac{D^2 z^\alpha}{d\tau^2} = -\tfrac{1}{2}\left(g^{\alpha\beta} - h^{R\alpha\beta}\right)\left(2\nabla_\mu h^R_{\nu\beta} - \nabla_\beta h^R_{\mu\nu}\right) u^\mu u^\nu \qquad (4)$$

and the constraint $\nabla^\beta \bar h_{\alpha\beta} = 0$. (5)

We also impose retarded BCs: no incoming waves from ∞

no outgoing waves from the BH

We enforce (3) using a puncture scheme: start with the local expansion of $\bar h^{S(n)}_{\alpha\beta}$ and truncate it at order r^p with $p \geq 1$. Call this a puncture $\bar h^{P(n)}_{\alpha\beta}$.

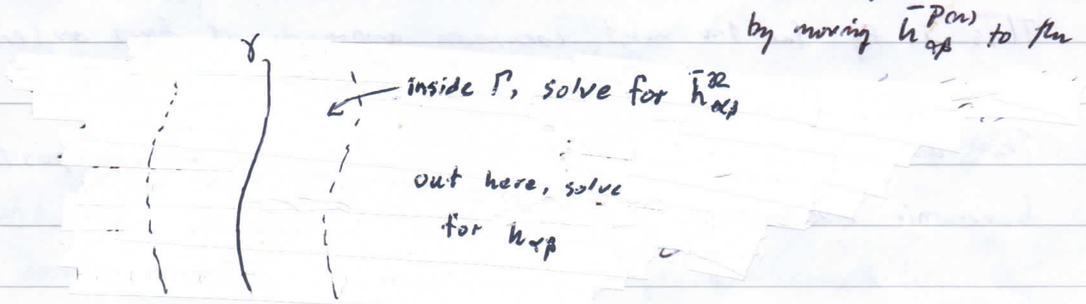
So $\bar h^{P(n)}_{\alpha\beta} = \bar h^{S(n)}_{\alpha\beta} + \mathcal{O}(r^2)$ (with $p = 1$)

Now define the residual field
$$\bar h^{\mathcal{R}(n)}_{\alpha\beta} = \bar h^{(n)}_{\alpha\beta} - \bar h^{P(n)}_{\alpha\beta}$$
$$= \bar h^{R(n)}_{\alpha\beta} + \mathcal{O}(r^2)$$

$\Rightarrow$ we can replace $h^R_{\alpha\beta}$ with $h^{\mathcal{R}}_{\alpha\beta}$ in (4)

In some region Γ around γ, convert (1) and (2) into equations for $\bar h^{\mathcal{R}(n)}_{\alpha\beta}$:

by moving $\bar h^{P(n)}_{\alpha\beta}$ to the RHS



For example, $E_{\alpha\beta}[\bar h^{(1)}] = 0 \quad \forall x \notin \gamma$
$$\Rightarrow E_{\alpha\beta}[\bar h^{P(1)} + \bar h^{\mathcal{R}(1)}] = 0 \qquad \forall x \in (\Gamma - \gamma)$$
$$\Rightarrow E_{\alpha\beta}[\bar h^{\mathcal{R}(1)}] = -E_{\alpha\beta}[\bar h^{P(1)}] \qquad \forall x \in (\Gamma - \gamma)$$
$$\Rightarrow E_{\alpha\beta}[\bar h^{\mathcal{R}(1)}] = S^{\text{eff}}_{\alpha\beta} \equiv \begin{cases} -E_{\alpha\beta}[\bar h^{P(1)}] & \forall x \in (\Gamma - \gamma) \\ -\lim_{x' \to x} E_{\alpha\beta}[\bar h^{P(1)}(x')] & \forall x \in \gamma \end{cases}$$
$$\equiv E'_{\alpha\beta}[\bar h^{P(1)}]$$

So we get $E_{\alpha\beta}[\bar{h}^{R(1)}] = S^{\text{eff}\,(1)}_{\alpha\beta}$ $\quad x \in \Gamma$

$E_{\alpha\beta}[\bar{h}^{R(1)}] = 0$ $\quad x \notin \Gamma$

Similarly,
$$E_{\alpha\beta}[\bar{h}^{R(2)}] = S^{\text{eff}\,(2)}_{\alpha\beta} \equiv \begin{cases} 2\delta^2 G_{\alpha\beta}[h^{(1)}] - E_{\alpha\beta}[\bar{h}^{P(2)}] & x \in (\Gamma - \gamma) \\ \lim_{x' \to x} \left(2\delta^2 G_{\alpha\beta}[h^{(1)}] - E_{\alpha\beta}[\bar{h}^{P(2)}]\right)(x') & x \in \gamma \end{cases}$$
$$E_{\alpha\beta}[\bar{h}^{(2)}] = 2\delta^2 G_{\alpha\beta}[h^{(1)}] \quad x \notin \Gamma$$

when crossing into or out of Γ, we use the change of variables $\bar{h}^{(n)}_{\alpha\beta} = \bar{h}^{R(n)}_{\alpha\beta} + \bar{h}$

Physical picture: we've replaced the physical object with a puncture in spacetime

It diverges on γ, which satisfies (4)

The solution to this effective problem satisfies the original free-boundary value problem.

Mode-sum regularization

At first order, as an alternative to the puncture scheme, we can instead solve
$$E_{\alpha\beta}[\bar{h}^{(1)}] = -16\pi T^{pp}_{\alpha\beta}[\gamma]$$
for the full field, and afterward subtract $\bar{h}^{S(1)}_{\alpha\beta}$ to obtain $\bar{h}^{R(1)}_{\alpha\beta}$. This is by far the most common approach at first order.

The actual subtraction procedure relies on a decomposition into spherical-harmonic modes. Say we want to calculate $h^{R(1)}_{\alpha\beta}$ on γ. We write
$$h^{R}_{\alpha\beta}\big|_\gamma = \lim_{x \to \gamma} \left(h_{\alpha\beta}(x) - h^{S}_{\alpha\beta}(x)\right)$$
$$= \lim_{x \to \gamma} \sum_{\ell m} \left[h^{\ell m}_{\alpha\beta}(t,r) Y^{\ell m}(\theta,\phi) - h^{S,\ell m}_{\alpha\beta}(t,r) Y^{\ell m}(\theta,\phi)\right]$$

(since $h^R_{\alpha\beta}$ is C^∞ it doesn't matter which direction we take the limit from)
$$= \lim_{r \to r_p} \sum_{\ell} \left[h^{\ell}_{\alpha\beta}(r) - h^{S,\ell}_{\alpha\beta}(r)\right] \quad \text{where} \quad h^{\ell}_{\alpha\beta}(r) = \sum_m h^{\ell m}_{\alpha\beta}(t_p, r) Y^{\ell m}(\theta_p$$
$$= \sum_{\ell} \left[h^{\ell}_{\alpha\beta}(r_p) - h^{S,\ell}_{\alpha\beta}(r_p)\right]$$
$\leftarrow$ we can take the limit inside because the sum converges uniformly

In practice in the Lorenz gauge, we find $h^{S,\ell}_{\alpha\beta}(\gamma) = B \leftarrow \ell$ independent
$$\Rightarrow h^{R}_{\alpha\beta}(\gamma) = \sum_{\ell} \left[h^{\ell}_{\alpha\beta}(\gamma) - B\right]$$

Similarly, to calculate the first-order self-force, we write

$$f^{\alpha}_{(1)} = \lim_{r\to r_p^{\pm}} \sum_{\ell} \left\{ f^{\alpha,\ell}_{(1)}[h^{(1)}] - f^{\alpha,\ell}_{(1)}[h^{S(1)}] \right\}$$

doesn't matter which

$$= \sum_{\ell} \left\{ f^{\alpha,\ell\pm}_{(1)}[h^{(1)}] - f^{\alpha,\ell\pm}_{(1)}[h^{S(1)}] \right\}$$

In practice we find $f^{\alpha,\ell\pm}_{(1)}[h^{S(1)}] = (\ell+\tfrac{1}{2})A^{\alpha}_{\pm} + B$ different B

$$\Rightarrow f^{\alpha}_{(1)} = \sum_{\ell} \left\{ f^{\alpha,\ell\pm}_{(1)}[h^{(1)}] - (\ell+\tfrac{1}{2})A^{\alpha}_{\pm} - B \right\}$$

In general (in the Lorenz gauge), a quantity $I[h]$ constructed from k derivatives of $h_{\alpha\beta}$ behaves as

$$I[h^{S(1)}] \sim \frac{1}{r^{k+1}} \quad \text{in 4D}$$

and $I^{\ell}[h^{S(1)}] \sim (\ell+\tfrac{1}{2})$ in a mode decomposition

Example: first-order calculation in flat spacetime

Since $\gamma \approx$ a geodesic, let's approximate γ as a straight line.

recall general result first

In Fermi-Walker coords, we have $\bar{h}^{(1)}_{\alpha\beta} = \bar{h}^{S(1)}_{\alpha\beta} + \bar{h}^{R(1)}_{\alpha\beta}$

(from matched asymptotic expansions) where $\bar{h}^{S(1)}_{tt} = \frac{4m}{s}$, neglecting acceleration terms

$\bar{h}^{S(1)}_{ti} = 0$

$\bar{h}^{S(1)}_{ij} = 0$

$\bar{h}^{R(1)}_{\alpha\beta}$ is a smooth solution to $(-\partial_t^2 + \partial^i\partial_i)\bar{h}^{R(1)}_{\alpha\beta} = 0$

constant position

Let's switch to another coordinate system (t, x^i) in which $z^{\mu}_0 = (t, x^i_0)$

In these coords, let's calculate $\bar{h}^{R(1)}_{\alpha\beta}$ using a puncture scheme: First decompose into harmonics: $\bar{h}^{S(1)}_{tt} = \sum_{\ell m} \frac{16\pi m}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^{*}_{\ell m}(\theta_0,\phi_0)\, Y_{\ell m}(\theta,\phi)$ where $r_< = \min(r, r_0)$, $r_> = \max(r, r_0)$

impose staticity and BCs and mode decomposition first

$$(-\partial_t^2 + \partial^i\partial_i)\bar{h}^{R(1)}_{\alpha\beta} = -(-\partial_t^2 + \partial^i\partial_i)\bar{h}^{P(1)}_{\alpha\beta}$$

Becomes

$$\left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2}\right]\bar{h}^{R(1)\ell m}_{\alpha\beta} = -\left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2}\right]\bar{h}^{P(1)\ell m}_{\alpha\beta} = 0$$

$$\Rightarrow \bar{h}^{R(1)\ell m}_{\alpha\beta} = C^{\ell m}_{\alpha\beta} r^{\ell} + D^{\ell m}_{\alpha\beta}\frac{1}{r^{\ell+1}}$$

And that $\Gamma = \mathbb{R}^4$

regularity at $r=0 \Rightarrow D^{\ell m}_{\alpha\beta} = 0$

and decay at $r\to\infty \Rightarrow C^{\ell m}_{\alpha\beta} = 0$

$\Rightarrow \bar{h}^{R(1)\ell m}_{\alpha\beta} = 0$

$\therefore \bar{h}^{R(1)}_{\alpha\beta} = 0$

~~Current/s~~

Current status

First order: von de Meent has calculated the self-force on generic bound geodesics in Kerr. His calculation uses mode-sum regularization in a "radiation gauge"

Warburton and others have simulated inspirals in Schwarzschild, using mode-sum regularization in the Lorenz gauge

Second order: Pound, Wardell, Warburton, Miller have calculated some quantities at second order for quasicircular orbits in Schwarzschild. Our calculation uses a puncture scheme in the Lorenz gauge, combined with a two-timescale expansion of the field equations