Lec III

1.- More on Burger's equation
2.- Roe solver derivation
3.- Relativistic hydro on a flat background

Useful references (-2,3- includes pages from these!)

Introduction to 3+1 Numerical Relativity book by Miguel Alcubierre.

Roe solver from R. Keppens
(https://perswww.kuleuven.be/~u0016541/Talks/roe.pdf)

fail for lack of awareness of this issue. If you understand this, you will definitively be ahead! If additionally you are interested in obtaining solutions via numerical means, you must make sure things make the most sense at the analytical level. Experience (and good arguments!) says that at the numerical level, if something can go bad... it will go bad!

Note that hyperbolicity –in particular the existence of the transformation that diagonalizes the problem – imply that the problem 1 can be re-expressed as (say in 1D)

$$Q_{,t} + D^x(q,t)Q_{,x} = T(S(q,x,t)) \tag{4}$$

for $Q \equiv T(q)$ with T the transformation that diagonalizes the problem. and so it can be seen as a series of equations like

$$Q^a_{,t} + \lambda_a \partial_x Q^a = [T(S(q,x,t))]^a \tag{5}$$

with $a = 1..m$. In what follows, unless necessary we will thus discuss things with respect to a single equation of the type (5) as hyperbolicity implies we can always reduce our system to this level.

2 Towards the Euler equations, generalities

Euler equations arise from the compressible Navier-Stokes equations by neglecting viscosity and head conduction. Mathematically one of the most interesting features admitted by solutions of these equations is the presence of *shocks*.

Shocks are mathematical idealizations of the steep gradients that can be present in smooth solutions to the full Navier-Stokes equations where rapid changes occur over *very thin* regions. Any numerical effort to describe solutions to these equations must therefore be aware of this possible scenario! In fact, "naive" or "direct" discretizations of these equations typically obtain solutions which are either very smeared out or with spurious oscillations near discontinuities. We will return to numerical techniques to address these problems later; however, we first need to understand analytical properties of these equations to understand what to do.

3 Advection equation, linearly degenerate and truly non-linear equations

Consider $q(x,t)$ a generic function we want to compute, we further assume the behavior of this function obeys the simple *hyperbolic* equation,

$$q_{,t} + Aq_{,x} = 0\,, \tag{6}$$

with $q(x,t) \in \Re^m$, A a $m \times m$ matrix. The system is hyperbolic if A is diagonalizable, which allows us to view the solution in terms of propagating waves. The simplest example is the *constant coefficient one-dimensional* advection equation,

$$q_{,t} + uq_{,x} = 0\,. \tag{7}$$

The solution to this problem is given by $q(x-ut,0)$ thus any profile q has at the initial time, it is simply advected at velocity u. Of course, there are more complicated cases, for instance,

$$q_{,t} + F(q)_{,x} = 0\,. \tag{8}$$

for which a particularly simple example would be $F(q) = q^2/2$, and so $q_{,t} + qq_{,x} = 0$. While it does not look too different from equation (7), there is a lot more than meets the eye here...

3.1 Small detour, why do we even care?

Consider the function $\rho(x,t)$ describing the density of a fluid in a one-dimensional setting. The mass m in a box of extent $[x_1, x_2]$ at time t is given by

$$m = \int_{x_1}^{x_2} \rho(x,t)dx\,. \tag{9}$$

Now, if the "walls" are permeable, fluid might enter/leave the box and so m will change in time. The rate of fluid flow (flux) past any given point is $F(x,t) = \rho(x,t)u(x,t)$ so,

$$\frac{d}{dt}m = \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x_1,t)u(x_1,t) - \rho(x_2,t)u(x_2,t) = F(x_1,t) - F(x_2,t) \tag{10}$$

Integrating the above equation in both time (in $[t_1,t_2]$) one obtains,

$$\int_{x_1}^{x_2} \rho(x,t_2)dx = \int_{x_1}^{x_2} \rho(x,t_1)dx + \int_{t_1}^{t_2} \rho(x_1,t)u(x_1,t)dt - \int_{t_1}^{t_2} \rho(x_2,t)u(x_2,t)dt \tag{11}$$

Thus, $m(t_2)$ will be given by $m(t_1)$ plus/minus the amount of fluid that entered/left the domain. What do we do with this?, suppose ρ, u are differentiable (smooth), thus

$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \partial_t\left(\rho(x,t)\right)dt \tag{12}$$

$$\rho(x_2,t)u(x_2,t) - \rho(x_1,t)u(x_1,t) = \int_{x_1}^{x_2} \partial_x\left(\rho(x,t)u(x,t)\right)dx \tag{13}$$

Replacing in equation (11), we get

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \left[\partial_t\rho + \partial_x(\rho u)\right]dxdt = 0 \tag{14}$$

and so $\partial_t\rho + \partial_x(\rho u) = 0$ simply states conservation of mass. Recall that when obtaining this equation, we assumed ρ, u are differentiable, as we will discuss below, this assumption need not be justified.

3.2 A shocking truth, life is not so simple

Consider again the equation (8), and for simplicity take $F(q) = q^2/2$, the resulting equation is known as the Burger's equation which can also be found written as $q_{,t} + qq_{,x} = 0$. Now,

this does not seems like too different from $q_{,t} + uq_{,x} = 0$ for a general function u; it turns out however, the "small" difference introduced will lead to dramatically different behavior[2]

Why is this the case? A simple way to see this is to consider the behavior of perturbations δq over a given solution q_o for a small period of time, the equation determining this behavior is straightforwardly,

$$\delta q_{,t} + q_o \delta q_{,x} = 0 \tag{15}$$

from our previous discussion, the solution will be given by $\delta q(x,t) = \delta(x - q_o t)$. Notice that if, in particular, $0 < q_o(x_2, t = 0) < q_o(x_1, t = 0)$ $(x_1 < x_2)$ it is trivial to see that at $t = (x_2 - x_1)/(q_o(x_1, t = 0) - q_o(x_2, t = 0))$ δq is multivalued, which does not make sense. What happened here is that the *characteristics* of the solution crossed. And the solution can not be determined past this point at least. Mathematically the difference between the equation $q_{,t} + qq_{,x} = 0$ and $q_{,t} + uq_{,x} = 0$ is that the characteristic speed of the former depends on the solution itself while this is not the case for the latter. Mathematicians refer to the first one as truly non-linear while the second as linearly degenerate. For higher dimensions the analog problem arises if the characteristic speed of a given mode depends on the solutions in the subspace described by the eigenvector corresponding to such speed. If so, as in the simple Burger's example, the solution can not be easily determined past some local point (points). Physically on the other hand, this does not make sense, as we expect a unique solution to exist past these problematic points, so something went wrong somewhere. Where did we go wrong? It was in our assumption of smoothness and differentiability which we used to derive the differential form of the equation which is not valid if discontinuities develop in the solution. To address this issue we approach the problem in a different light. The new approach is to consider "weak solutions", i.e. solutions to the integral form of (11),

$$\int \int \Phi(q_{,t} + \partial_x f(q)) dx dt = 0\,, \tag{16}$$

for arbitrary (smooth and with compact support) test functions Φ, upon integration by parts (and taking the limit of the boundary –both in space and time to infinity–, one obtains,

$$\int \int (\Phi_{,t} q + f \partial_x \Phi) dx dt = \int \Phi(x,0) q(x,0) dx\,. \tag{17}$$

This approach allows for a way to deal with discontinuities in a special way (recall similar "tricks" allow to make sense of the Dirac δ "function" in a distributional sense. A detailed discussion of this theory is beyond the scope of this course. We instead describe the main aspects relevant to the numerical implementation of these type of equations.

3.3 Riemann problem and general considerations

A Riemann problem is defined by a conservation law type equation with piecewise constant data having a single discontinuity. Let's go back to Burger's equation as an example and consider initial data given by:

$$q(x, t = 0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases} \tag{18}$$

[2]The word "dramatic" is often a hyperbola used by physicists to stress an important point. It is often an exaggeration but, we can assure you... in this case it is definitively not!

For $q_l > q_r$ it is easy to see that the "left" (l) state will run into the 'right' (r) state. The solution is thus multivalued along a line defined by $x = st \equiv (q_l + q_r)/2$. The velocity of the shock s is given by the Rankine-Hugoniot jump conditions, which for one-dimensional problems is simply $s = (f(q_l) - f(q_r))/(q_l - q_r)$. The unique solution is given by,

$$q(x, t = 0) = \begin{cases} q_l & \text{if } x < st \\ q_r & \text{if } x > st \end{cases} \tag{19}$$

For $q_l < q_r$ the characteristics diverge and several weak solutions exist. The requirement that the entropy across a discontinuity increases help single out a unique solution, which is known as a rarefraction wave given by,

$$q(x, t) = \begin{cases} q_l & \text{if } x < q_l t \\ x/t & \text{if } q_l t < x < q_r t \\ q_r & \text{if } q_r t < x \end{cases} \tag{20}$$

In principle the approach we took to obtain solutions can be generalized to arbitrary dimensions, however it is expensive and cumbersome. Since we are after an approximate solution we can take a less costly approach –still based in the previous discussion– which still provides the correct solution as the discretization length is taken to 0 in a controlled manner.

4 Discretization

4.1 Finite Volumes

We are interested in truly non-linear problems which, as discussed, give rise to shocks (discontinuities on the variables describing the state of the fluid) even when the initial data is smooth. This implies that naïve discretizations based on the continuity of the functions (like some of the finite difference methods used on Project 1) are doomed to fail. There are different approaches we could take to solve this system. Here we will take a finite volume approach, meaning that we will assume that we have a mesh of grid points that define a cell structure on our spacetime (see Figure 1). In the presence of discontinuities the only way to make sense of our system of equations is to consider averages over a finite volume of the spacetime. Therefore to find the discretization we take the average of equation (44) over a spacetime cell $\mathcal{C}_i^{n+1/2}$:

$$\frac{1}{V_{\mathcal{C}_i^{n+1/2}}} \int_{\mathcal{C}_i^{n+1/2}} \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{V_{\mathcal{C}_i^{n+1/2}}} \int_{\mathcal{C}_i^{n+1/2}} \frac{\partial \mathbf{f}}{\partial x} = \frac{1}{V_{\mathcal{C}_i^{n+1/2}}} \int_{\mathcal{C}_i^{n+1/2}} \boldsymbol{\psi}, \tag{21}$$

where $\mathcal{C}_i^{n+1/2}$ is the region of spacetime defined by $(t^n, t^{n+1}) \times (x_{i-1/2}, x_{i+1/2})$, and $V_{\mathcal{C}_i^{n+1/2}} = \Delta t \Delta x$ is its volume. The resulting equation can be written as:

$$\frac{1}{\Delta t \Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t^{n+1}} \frac{\partial \mathbf{q}}{\partial t} dx dt + \frac{1}{\Delta t \Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t^{n+1}} \frac{\partial \mathbf{f}}{\partial x} dx dt = \frac{1}{\Delta t \Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t^{n+1}} \boldsymbol{\psi} dx dt. \tag{22}$$

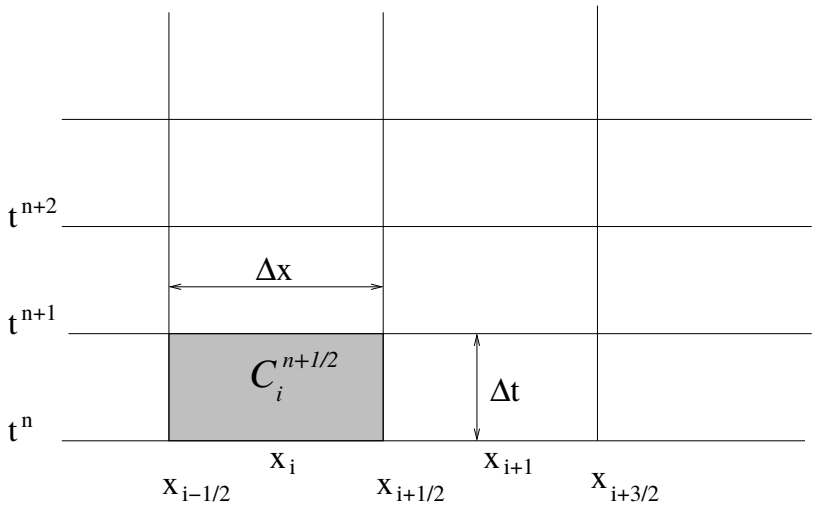

Figure 1: Cell structure of the spacetime for a finite volume discretization in one dimension. The spacetime cells $C_i^{n+1/2}$ are centred at positions $(t^{n+1/2}, x_i)$ and their volumes are $\Delta t \Delta x$.

We can partially integrate the different terms of the equation using Gauss' theorem:

$$\frac{\bar{\mathbf{q}}_i^{n+1} - \bar{\mathbf{q}}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = \widehat{\boldsymbol{\psi}}_i^{n+1/2}. \tag{23}$$

Here we have used the following definitions: the spatial averages of the conservative variables,

$$\bar{\mathbf{q}}_i^n \equiv \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(t^n, x) dx, \tag{24}$$

the temporal averages for the fluxes, also referred as the numerical fluxes,

$$\mathbf{F}_{i+1/2}^{n+1/2} \equiv \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{f}\left(\mathbf{q}(t, x_{i+1/2})\right) dt, \tag{25}$$

and the total averages over the spacetime cell for the sources,

$$\widehat{\boldsymbol{\psi}}_i^{n+1/2} \equiv \frac{1}{\Delta x \Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t^{n+1}} \boldsymbol{\psi}(t, x) dx dt. \tag{26}$$

The idea now is to use equation (23) to calculate $\{\bar{\mathbf{q}}_i^{n+1}\}$ assuming we know the values $\{\bar{\mathbf{q}}_i^n\}$. However the calculation of the numerical fluxes $\{\mathbf{F}_{i+1/2}^{n+1/2}\}$ is not as straightforward as one may think—these fluxes are averages in time, so in order to explicitly calculate them we need to already know the solution. More importantly, the values for the fluid quantities on the left side of the cell boundary $\{\bar{\mathbf{q}}_i^n\}$ and on the right side $\{\bar{\mathbf{q}}_{i+1}^n\}$ won't agree in general. The values have discontinuities and *a priori* is not clear which values to use in order to compute the numerical fluxes. One way to solve these problems is to use an idea due to Godunov that involves solving a Riemann problem at every cell boundary in order to calculate $\{\mathbf{F}_{i+1/2}^{n+1/2}\}$. For more information about Godunov methods see LeVeque [6]. In the following section we explain one such method.

4.2 Roe Solver

The solution of the full Riemann problem at every cell boundary is usually not very efficient. In most cases the overall time step to update the variables to the future time will involve some kind of iterative process, and thus exactly solving the Riemann problem at each iteration will not imply that the overall process will be solved more rapidly or accurately. The Roe solver is a solver that uses modified Riemann problems in order to compute the numerical fluxes. For a more extensive explanation of this and other approximate Riemann solvers see LeVeque [6]. The main idea is to linearize the fluxes in equation (44) as functions of $\mathbf{q}$, also assuming that the sources vanish:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial x} = 0. \tag{27}$$

Considering $\partial \mathbf{f}/\partial \mathbf{q}$ to have constant coefficients linearizes the above equation, and a solution can be obtained by diagonalizing the Jacobian matrix (see appendix for the solution of the scalar linear equation and its generalization to a system of linear equations). The numerical flux can then be written as a function of the solution to this problem. Here we write the resulting numerical fluxes directly as:

$$\mathbf{F}^{\text{Roe}}_{i+1/2} = \frac{1}{2}\left[\mathbf{f}\left(\tilde{\mathbf{p}}^{R}_{i+1/2}\right) + \mathbf{f}\left(\tilde{\mathbf{p}}^{L}_{i+1/2}\right) - \sum_{\alpha} |\lambda_\alpha| \omega_\alpha \mathbf{r}_\alpha\right]. \tag{28}$$

Some explanation of the different terms that appear in the above equation are in order. First, $(\tilde{\mathbf{p}}^{R},\ \tilde{\mathbf{p}}^{L})$, known as right and left *reconstructed variables*, are the values of the primitive variables at the boundary, $x_{i+1/2}$ calculated via some specific reconstruction (interpolation) scheme. Special care is taken in calculating the reconstructed variables in order to reduce spurious oscillations close to discontinuities. Here we use a *slope limiter* interpolation to compute the reconstructed values (see Martí and Mueller [2] and LeVeque [6] for alternate reconstruction algorithms):

$$\tilde{\mathbf{p}}^{L}_{i+1/2} = \bar{\mathbf{p}}_i + \boldsymbol{\sigma}_i \left(x_{i+1/2} - x_i\right), \tag{29}$$

$$\tilde{\mathbf{p}}^{R}_{i+1/2} = \bar{\mathbf{p}}_{i+1} + \boldsymbol{\sigma}_{i+1} \left(x_{i+1/2} - x_{i+1}\right), \tag{30}$$

where $\boldsymbol{\sigma}_i$ is given by

$$\boldsymbol{\sigma}_i = \text{minmod}\left(\mathbf{s}_{i-1/2}, \mathbf{s}_{i+1/2}\right). \tag{31}$$

Here:

$$\mathbf{s}_{i+1/2} = \frac{\bar{\mathbf{p}}_{i+1} - \bar{\mathbf{p}}_i}{x_{i+1} - x_i}, \tag{32}$$

and the minmod function is defined by

$$\text{minmod}(a,b) = \begin{cases} 0 & \text{if} \quad ab < 0 \\ a & \text{if} \quad |a| < |b| \ \text{ and } \ ab > 0 \\ b & \text{if} \quad |a| > |b| \ \text{ and } \ ab > 0. \end{cases} \tag{33}$$

In equation (28) we also use the characteristic structure of the Riemann problem at the $x_{i+1/2}$ interface (λ_α, ω_α, $\mathbf{r}_\alpha$). Given the Jacobian matrix:

$$\mathbf{A}|_{i+1/2} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right|_{\mathbf{q}=1/2\,(\tilde{\mathbf{q}}^L_{i+1/2}+\tilde{\mathbf{q}}^R_{i+1/2})}, \tag{34}$$

λ_α are the eigenvalues[3] of $\mathbf{A}$, $\mathbf{r}_\alpha$ are the right eigenvectors associated with the eigenvalues λ_α and ω_α are the jumps in the characteristic variables defined by

$$\tilde{\mathbf{q}}^R_{i+1/2} - \tilde{\mathbf{q}}^L_{i+1/2} = \sum_\alpha \omega_\alpha \boldsymbol{r}_\alpha. \tag{35}$$

Here $(\tilde{\mathbf{q}}^R, \tilde{\mathbf{q}}^L)$ are the values of the conservative variables calculated from the reconstructed primitive variables $(\tilde{\mathbf{p}}^R, \tilde{\mathbf{p}}^L)$. Reconstruction of the primitive variables followed by transformation to conservative variables generally yields more stable results than direct reconstruction of the conservative variables.

The final part of the Roe solver involves the update of $\{\bar{\mathbf{q}}^n_i\}$. The fact that we use an approximate Riemann solver to calculate the numerical flux makes (28), when evaluated using the spatial averages at time t^n, a first order approximation to the real numerical flux defined by (25). This is usually the case also when calculating the numerical sources (26). In order to make the time evolution second order (in the temporal discretization scale), we use a second order Runge-Kutta method to advance the solution in time:

$$\bar{\mathbf{q}}^{n+1/2} = \bar{\mathbf{q}}^n + \frac{\Delta t}{2} L(\bar{\mathbf{q}}^n) \tag{36}$$

$$\bar{\mathbf{q}}^{n+1} = \bar{\mathbf{q}}^n + \Delta t L(\bar{\mathbf{q}}^{n+1/2})\,. \tag{37}$$

Here L is defined by:

$$L(\bar{\mathbf{q}}^n) = -\frac{\mathbf{F}^{Roe}_{i+1/2}\,(\bar{\mathbf{q}}^n) - \mathbf{F}^{Roe}_{i-1/2}\,(\bar{\mathbf{q}}^n)}{\Delta x} + \widehat{\boldsymbol{\psi}}_i\,(\bar{\mathbf{q}}^n)\,. \tag{38}$$

Summarizing the approach described above involves:

- Expressing the system of equation in "conservation" form, i.e. $q^i_{,t} + \partial_j F^{ij} = 0$.
- Solve a series of (approximate) Riemann problems at cell interface.
- Calculating the characteristic structure of the system (for solving the Riemann problem).

Armed with the considerations and techniques described so far, let's turn to a problem of relevance to us.

[3]Here α (and later on also β) labels the equation number, in the fluid case since we have two equations it takes values on $\{1, 2\}$.

5 Relativistic fluid dynamics

In this project we study the evolution of a relativistic fluid with an ultrarelativistic equation of state, in slab symmetry. We will use one of the so called HRSC (High Resolution Shock Capturing) methods that are suited for the evolution of discontinuities. These methods have been proven to be useful in both the special relativistic and general relativistic cases (see Martí, et al. [2] and Font [3], respectively).

In this handout we first describe how one obtains the equations of motion for such a fluid, focusing on casting the equations in a form appropriate for discretization. In section 4 we describe the discretization *per se*, as well as the numerical method that you will use to solve the discrete equations. In 7 we describe a simpler system—Burger's equation—and provide a code that solves it using the same HRSC method described for the fluid. You will perform some simple numerical experiments with this code, before moving onto the main task of implementing the fluid solver. The last section contains notes concerning how Burger's equation code can be modified to produce a code to solve the fluid equations.

6 Equations of Motion

The equations of motion can be derived from the following conservation laws:

$$\left(\rho_o u^a\right)_{;a} = 0, \tag{39}$$

$$\left(T^{ab}\right)_{;b} = 0, \tag{40}$$

where ρ_o is the proper rest mass density in a local inertial frame, u^a is the four velocity of the fluid, and T^{ab} is the fluid's energy momentum tensor. Equation (39) expresses the conservation of baryons in the system while (40) represents the conservation of the energy and momentum.

For a perfect fluid (we will only consider an adiabatic fluid, i.e. we do not consider heat exchange or viscous terms) the stress energy tensor can be written as (see MTW [4], 22.3)

$$T^{ab} = (\rho + P)\, u^a u^b + P g^{ab}\,, \tag{41}$$

where $\rho = \rho_o\,(\epsilon + 1)$ is the energy density of the fluid, g^{ab} is the inverse metric and ϵ is the specific internal energy. In our case we consider a fluid on Minkowski spacetime (i.e. non-self-gravitating), and work in Cartesian coordinates; therefore $g^{\mu\nu} = \eta^{\mu\nu} \equiv \mathrm{diag}\{-1, 1, 1, 1\}$. In order to completely describe the fluid we need to augment the equations of motion with an equation of state (EOS) that relates the pressure to the rest of the fluid variables. In this project we will consider an *ultrarelativistic* fluid (i.e. we assume that the internal energy density of the fluid is much larger than the rest mass density—$\rho_o\epsilon \gg \rho_o$) for which the equation of state is:

$$P = (\Gamma - 1)\,\rho\,. \tag{42}$$

Here, Γ is the adiabatic index that we will be taken to be a constant in the range $(1, 2]$. With such an EOS, the rest mass density becomes dynamically irrelevant, and we can drop

equation (39) from the system. Therefore the only equations that we need to consider are (40) which we can now write as:

$$\left(T^{\mu\nu}\right)_{,\nu} = 0. \tag{43}$$

The method of solution that we will use requires that our equations of motion be cast in conservation law form, i.e. in the form

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x^i} = \boldsymbol{\psi}\,. \tag{44}$$

Here, $\mathbf{q}$ is a vector formed by the so called the *conservative* variables, $\mathbf{f}(\mathbf{q})$ is a vector of fluxes which depend on these variables (in general not explicitly) [4] and $\boldsymbol{\psi}$ are source functions.

We now impose the condition of *slab symmetry*, i.e we demand that our solutions are invariant under translations in the y and z directions. With this assumption, the dynamical variables, as well as their time derivatives, become functions of x and t alone. (We also work in a coordinate system in which the fluid velocity components, v^y and v^z, vanish).. In order to write equations (43) in conservation form it will be useful to introduce the following conservative variables, following Neilsen and Choptuik[5] (hereafter NC):

$$\tau = (\rho + P)\, W^2 - P \tag{45}$$

$$S = vW^2\,(\tau + P) \tag{46}$$

where $W = (1 - v^2)^{-1/2} = u^t$ and $v = u^x/u^t$. Using these variables, is easy to express the relevant components of the energy momentum tensor:

$$T^{tt} = \tau, \qquad T^{tx} = T^{xt} = S, \qquad T^{xx} = Sv + P. \tag{47}$$

The non-trivial equations of motion derived from (43) are:

$$\dot{\tau} + S' = 0, \tag{48}$$

$$\dot{S} + (Sv + P)' = 0, \tag{49}$$

where the dot means $\partial/\partial t$ and the prime $\partial/\partial x$. At this point it is important to stress that these *conservative* variables are not *extra* variables needed to describe the state of the fluid but a different set of state variables. A crucial part of the algorithm will be the prescription of how to transform from the conservative variables $\{\tau, S\}$ to the *primitive* variables $\{\rho, v\}$ and vice versa.

Equations (48-49) are in conservation law form where the vector $\mathbf{q}$, the physical fluxes $\mathbf{f}(\mathbf{q})$, and the source terms $\boldsymbol{\psi}$ are given by

$$\mathbf{q} = \begin{bmatrix} \tau \\ S \end{bmatrix}\,, \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} S \\ (Sv + P) \end{bmatrix}\,, \quad \boldsymbol{\psi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{50}$$

Notice that the fluxes not only depend on $\mathbf{q}$ directly but also implicitly through v and P.

[4] Actually at some points it will be more interesting to consider the fluxes to be function of the primitive variables $\mathbf{p} = \{v, \rho\}$.

Characteristic Structure for the ultrarelativistic fluid

For the case of our system of fluid equations, with $\mathbf{q}$ and $\mathbf{f}(\mathbf{q})$ are given by (50) the matrix $\mathbf{A} = (A^{\alpha}{}_{\beta})$ has the following components:

$$\begin{array}{ll} A^1{}_1 = 0, & A^1{}_2 = 1, \\ A^2{}_1 = -v^2 + (1-v^2)\,(\partial P/\partial \tau)\,, & A^2{}_2 = 2v + (1-v^2)\,(\partial P/\partial S)\,, \end{array} \tag{51}$$

where

$$\frac{\partial P}{\partial \tau} = -2\beta + \frac{(4\beta^2 + \Gamma - 1)\,\tau}{\left[4\beta^2\tau^2 + (\Gamma-1)\,(\tau^2 - S^2)\right]^{1/2}}, \tag{52}$$

$$\frac{\partial P}{\partial S} = -\frac{(\Gamma - 1)\,S}{\left[4\beta^2\tau^2 + (\Gamma-1)\,(\tau^2 - S^2)\right]^{1/2}}, \tag{53}$$

and $\beta = 1/4\,(2-\Gamma)$.

In terms of these matrix components, the eigenvalues λ_{α} are

$$\lambda_{\pm} = \frac{1}{2}\left[A^1{}_1 + A^2{}_2 \pm \sqrt{\left(A^1{}_1 - A^2{}_2\right)^2 + 4A^1{}_2 A^2{}_1}\right], \tag{54}$$

and the right eigenvectors, $\mathbf{r}_{\pm}$, are

$$\mathbf{r}_{\pm} = \begin{bmatrix} 1 \\ Y_{\pm} \end{bmatrix}, \qquad Y_{\pm} = \frac{\lambda_{\pm} - A^1{}_1}{A^1{}_2}\,. \tag{55}$$

Accordingly, the jumps, ω_{α}, are given by

$$\omega_+ = \frac{1}{d}\left[r_-[2]\left(\tilde{q}^R[1] - \tilde{q}^L[1]\right) + r_-[1]\left(\tilde{q}^L[2] - \tilde{q}^R[2]\right)\right], \tag{56}$$

$$\omega_- = \frac{1}{d}\left[r_+[1]\left(\tilde{q}^R[2] - \tilde{q}^L[2]\right) + r_+[2]\left(\tilde{q}^L[1] - \tilde{q}^R[1]\right)\right], \tag{57}$$

$$\tag{58}$$

where

$$d = r_+[1]r_-[2] - r_-[1]r_+[2]. \tag{59}$$

In the previous equations $\tilde{q}^R[1]$ stands for the first component of the vector $\tilde{\mathbf{q}}^R$ and similarly for the rest of the analogous expressions.

Calculation of the conservative/primitive variables

We have explained how the primitive variables are reconstructed at the boundaries of the cells using a slope limiter. However, in order to compute the physical fluxes, $\mathbf{f}(\tilde{\mathbf{p}}^L)$ and $\mathbf{f}(\tilde{\mathbf{p}}^R)$, that appear in equation (28), we need to compute the conservative variables as well. The conservative variables can be calculated from the primitive variables via equations (45), (46), (42) and the definition of $W = (1-v^2)^{-1/2}$.

Conversely at every half and full step in our update procedure we need to calculate the primitive variables after the conservative variables have been evolved. It is not difficult to invert the equations that define the conservative variables in order to get the primitive ones (see NC [5]):

$$P = -2\beta\tau + \sqrt{4\beta^2\tau^2 + (\Gamma - 1)(\tau^2 - S^2)} \quad (60)$$

$$\rho = P/(\Gamma - 1) \quad (61)$$

$$v = \frac{S}{\tau + P} \quad (62)$$

where $\beta = (2 - \Gamma)/4$.

Floor

Due to numerical errors (truncation error, roundoff error) the quantities that describe the fluid can sometimes take unphysical values (i.e. negative pressures, negative densities, speeds larger than one, etc,...) (see NC [5]). Effects of such errors become particularly important in "evacuated" regions, where densities are low and velocities can be very large. In order to circumvent problems associated with these errors, we force certain values to be above some threshold, that we call a *floor*. For the ultrarelativistic fluid it is convenient to floor the conservative variable τ in the following way:

$$\tau = \max\{\tau, \mathit{floor} + |S|\}, \quad (63)$$

where $|S|$ is the absolute value of S and *floor* is a small value, typically several orders of magnitude (usually 13 or 14 orders of magnitude) smaller than typical values of τ. Generally, a flooring procedure of this sort will not have an important dynamical effect (although this is something that needs to be verified empirically), and ameliorates the problems described above. We recommend application of this algorithm every time the conservative variables are updated or calculated from reconstructed primitive variables at the cell boundaries.

7 Burger's Equation

Burger's equations is an example of a non-linear scalar equation that produces shocks, even from smooth initial data (see LeVeque [6]). One form of the equation is

$$\dot{q} + q\frac{\partial q}{\partial x} = 0. \quad (64)$$

which is easy to cast into conservative form

$$\dot{q} + \left(\frac{1}{2}q^2\right)' = 0. \quad (65)$$

Using the notation introduced previously, we have

$$\mathbf{q} = q, \qquad \mathbf{f} = \frac{1}{2}q^2, \qquad \boldsymbol{\psi} = 0. \quad (66)$$

We solve this equation with a finite volume discretization and a Roe solver, as outlined in the previous section. The finite volume discretization of the equation is

$$\frac{\bar{q}_i^{n+1} - \bar{q}_i^n}{\Delta t} + \frac{F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}}{\Delta x} = 0\,, \tag{67}$$

where $\bar{q}_i^n$ is the spatial average defined by equation (24) and $F_{i+1/2}^{n+1/2}$ is the numerical flux. We now focus on a description of the characteristic structure of the equation that will allow us to compute the Roe flux. Since (64) is a scalar equation the Jacobian matrix $\mathbf{A}$ is also a scalar

$$\mathbf{A} = q, \tag{68}$$

the eigenvalue is the same scalar, $\lambda = q$, and we can take the right eigenvector to be 1. Finally, the jump ω is just the difference of q across the cell boundary,

$$\omega = q^R - q^L\,. \tag{69}$$

Thus, we can write the Roe numerical flux as:

$$F_{i+1/2}^{\text{Roe}} = \frac{1}{2}\left[f\left(q^L\right) + f\left(q^R\right) - |\lambda|\, r\, \omega\right]_{i+1/2}, \tag{70}$$

where $[\cdots]_{i+1/2}$ means that the quantities within the bracket are evaluated at $x_{i+1/2}$. We then solve (67) using the following time-stepping procedure:

```
1) Calculate the Roe numerical fluxes at the cell boundaries.
2) Update the variables to the half time step using equation (47) with
   Delta t=Delta t/2.
3) Use the quantities at the half time step to compute the Roe numerical fluxes.
4) Do a full step to update to the future time step using the numerical fluxes
   calculated in 3).
```

The previous pseudo code describes the use of the Roe solver within a second order Runge-Kutta time stepping scheme. The overall method should be second order if no shocks are developed, except in the vicinity of extrema of the dynamical variable, where the slope limiting interpolation will generally degrade the solution to first order.

The calculation of the numerical fluxes involve the following steps:

```
1) Calculate the left and right reconstructed variables at the cell boundary.
2) Calculate the characteristic structure: eigenvalues, right eigenvectors
   and jumps in the characteristic variables.
4) Calculate the physical fluxes F for the left and right reconstructed
   variables.
3) Calculate the Roe numerical flux using equation (50).
```

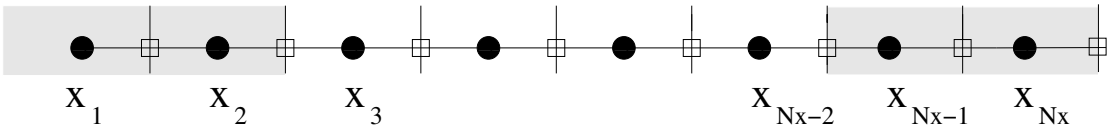

Figure 2: Spatial cell structure for a grid with N_x cells and two ghost cells ($N_g = 2$) per boundary. The solid circles lie at the spatial locations of the grid cell centres, and coincide with the grid points generated by the RNPL code. Shaded areas represent ghost cells where dynamical variables are updated according to the boundary conditions we impose. The squares and vertical lines denote the cell boundaries, and are the locations at which the reconstructed variables and numerical fluxes are calculated.

Boundary Conditions and Cell Structure

In order to impose boundary conditions we make use of *ghost cells*. These cells are *not* updated using the equations of motion, but rather are set according to the specific boundary conditions that we wish to impose. The boundary conditions that we impose are a first order approximation to outgoing boundary conditions (often called *outflow conditions* in the fluid literature). We implement these conditions simply by setting the ghost cell values to the value in the last regular cell:

$$q_1 = q_3 \tag{71}$$

$$q_2 = q_3 \tag{72}$$

$$q_{N_x-1} = q_{N_x-2} \tag{73}$$

$$q_{N_x} = q_{N_x-2} \tag{74}$$

Here N_x is the number of cells in the entire grid (including ghost cells).

Note that in order to update the interior points, i.e. the x_i with $i = N_g + 1, ..., N_x - N_g$ we need to calculate the numerical fluxes at positions $x_{i+1/2}$ with $i = N_g, ..., N_x - N_g$.

Finite Difference

Due to the simplicity of equation (64), it is also straightforward to solve using a finite difference approximation that uses an *upwind stencil* (an upwind stencil is one which uses information only a specific characteristic direction, relative to the point at which the approximation is applied). Interestingly, however, if we discretize Burger's equation in the form (64) using such a technique

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} q_i^{n+1/2} \left(q_i^{n+1/2} - q_{i-1}^{n+1/2} \right), \tag{75}$$

we find that the shock speeds obtained are erroneous, even in the continuum limit. In this instance, the problem can be solved by discretizing the conservative form of the equation, (65):

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} \left[\frac{1}{2} \left(q_i^{n+1/2} \right)^2 - \frac{1}{2} \left(q_{i-1}^{n+1/2} \right)^2 \right]. \tag{76}$$

Roe solver

keppens@rijnh.nl

- modern high resolution, shock-capturing schemes for Euler
  - $\Rightarrow$ capitalize on known solution of the Riemann problem
  - $\Rightarrow$ originally developed by Godunov
- always use conservative scheme of form

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) = 0$$

  - $\Rightarrow$ cell values U_i change through fluxes across cell edges
  - $\Rightarrow$ edge-centered numerical flux $F_{i+1/2}(U_{i-p}, U_{i-p+1}, \ldots, U_{i+q})$

The Godunov scheme

- values U_i^n for time $t = t_n$
  - $\Rightarrow$ consider piecewise constant values in cells
  - $\Rightarrow$ serve as initial condition to solve $U_t + \left(F(U)\right)_x = 0$ for $t > t_n$
  - $\Rightarrow$ restrict timestep to $\Delta t_{n+1} < \frac{\Delta x}{2 \max|\lambda|}$
  - $\Rightarrow$ with λ eigenvalue of flux Jacobian F_U
  - $\Rightarrow$ then exact solution given by solving RP at cell interfaces
  - $\Rightarrow$ restriction on timestep ensures no wave interaction within Δt_{n+1}
- Godunov scheme
  - $\Rightarrow$ denote exact RP solution for state U_i^n and U_{i+1}^n as $\hat{U}\left(\frac{x - x_{i+1/2}}{t}, U_i^n, U_{i+1}^n\right)$
  - $\Rightarrow$ numerical flux

$$F_{i+1/2}\left(U_i, U_{i+1}\right) = F(\hat{U}\left(0, U_i^n, U_{i+1}^n\right))$$

  - $\Rightarrow$ need an exact Riemann solver

The Roe solver

- due to piecewise constant representation

  $\Rightarrow$ Godunov scheme 1st order accurate

- exact solution to RP is complicated

  $\Rightarrow$ scheme is not exact due to piecewise constant representation

  $\Rightarrow$ might as well solve RP in approximate fashion

- schemes exploiting *approximate Riemann solver*

  $\Rightarrow$ use linearization of the nonlinear problem

  $\Rightarrow$ recall: exact solution for linear hyperbolic system known

- Roe suggested following procedure

  $\Rightarrow$ for system $U_t + (F(U))_x = 0$

  $\Rightarrow$ consider again RP with U_l and U_r states

  $\Rightarrow$ solve linear Riemann problem where

  $$U_t + AU_x = 0, \;\; A \in \mathcal{R}^{3\times 3}$$

  $\Rightarrow$ constant matrix $A = A\,(U_l, U_r)$ must satisfy conditions

$$F(U) - F(V) = A(U,V)\,(U - V)$$
$$A(U,V) \rightarrow F_U(U) \text{ as } V \rightarrow U$$
$$A(U,V) \text{ has only real eigenvalues}$$
$$A(U,V) \text{ has complete system of eigenvectors}$$

- conditions ensure

  $\Rightarrow$ exact RP solution obtained for initial condition: single shock or CD

  $\Rightarrow$ consistency with original equation

  $\Rightarrow$ solvability of linear Riemann problem

- Roe scheme given by numerical 'Roe' flux

$$F_{i+1/2}\left(U_i, U_{i+1}\right) = A_{i+1/2}(U_i, U_{i+1})\hat{U}\left(0, U_i^n, U_{i+1}^n\right)$$

  $\Rightarrow \hat{U}$ is exact solution of *linear* Riemann problem

  $\Rightarrow$ Roe matrix $A_{i+1/2}$

- derive explicit expressions for Roe flux as follows

  $\Rightarrow$ denote right eigenvectors $\mathbf{r}_p$

  $\Rightarrow$ decompose $U_i = \Sigma\beta_p\mathbf{r}_p$ and $U_{i+1} = \Sigma\gamma_p\mathbf{r}_p$

  $\Rightarrow$ then define coefficients α_p from

$$U_{i+1} - U_i = \Sigma\left(\gamma_p - \beta_p\right)\mathbf{r}_p \equiv \Sigma\,\alpha_p\mathbf{r}_p$$

  $\Rightarrow$ solution of RP is

$$\hat{U} = \sum_{\lambda_p>0}\beta_p\mathbf{r}_p + \sum_{\lambda_p<0}\gamma_p\mathbf{r}_p$$

  $\Rightarrow$ can be written alternatively as

$$\begin{aligned}\hat{U} &= \Sigma\beta_p\mathbf{r}_p + \Sigma_{\lambda_p<0}(\gamma_p - \beta_p)\mathbf{r}_p \\ &= U_i + \Sigma_{\lambda_p<0}\alpha_p\mathbf{r}_p\end{aligned}$$

  $\Rightarrow$ or similarly

$$\begin{aligned}\hat{U} &= \Sigma\gamma_p\mathbf{r}_p + \Sigma_{\lambda_p>0}(\beta_p - \gamma_p)\mathbf{r}_p \\ &= U_{i+1} - \Sigma_{\lambda_p>0}\alpha_p\mathbf{r}_p\end{aligned}$$

- combine into one expression

$$\hat{U} = \frac{U_i + U_{i+1}}{2} + \frac{1}{2}\left[\sum_{\lambda_p<0} - \sum_{\lambda_p>0}\right]\alpha_p \mathbf{r}_p$$

- The Roe flux then becomes

$$F_{i+1/2} = A_{i+1/2}\hat{U} = \frac{1}{2}A_{i+1/2}\left(U_i + U_{i+1}\right) + \frac{1}{2}\left[\sum_{\lambda_p<0} - \sum_{\lambda_p>0}\right]\alpha_p A_{i+1/2}\mathbf{r}_p$$

$\Rightarrow$ since $A_{i+1/2}\mathbf{r}_p = \lambda_p \mathbf{r}_p$ this yields

$$F_{i+1/2} = \frac{1}{2}A_{i+1/2}\left(U_i + U_{i+1}\right) - \frac{1}{2}\Sigma \mid \lambda_p \mid \alpha_p \mathbf{r}_p$$

$\Rightarrow$ due to first Roe condition and $F(-U) = -F(U)$ written as

$$F_{i+1/2} = \frac{1}{2}\left(F(U_i) + F(U_{i+1})\right) - \frac{1}{2}\Sigma \mid \lambda_p \mid \alpha_p \mathbf{r}_p$$

sented here will be very brief. There exists, however, a large literature dedicated exclusively to the study of fluid mechanics, most notably the beautiful book of Landau and Lifshitz [184]. There are also many books that discuss the numerical treatment of the hydrodynamic equations, particularly in the non-relativistic case. In the case of relativity there is the recent book by Wilson and Mathews on relativistic numerical hydrodynamics [300], and the review papers by Marti and Müller [200], and Font [129].

7.2 Special relativistic hydrodynamics

The starting point to the study of relativistic hydrodynamics is the stress-energy tensor for a perfect fluid, *i.e.* a fluid with zero viscocity and no heat conduction. Such a stress-energy tensor has already been introduced in equation (1.12.4) of Chapter 1, and has the form

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, \eta_{\mu\nu} \ , \tag{7.2.1}$$

where u^μ is the 4-velocity of the fluid elements (the average 4-velocity of the particles), ρ and p are the energy density and pressure as measured in the fluid's rest frame, and where, for the moment, we have assumed that we are in special relativity so the metric is given by the Minkowski tensor $g_{\mu\nu} = \eta_{\mu\nu}$.

The stress-energy tensor above is usually written in a simplified form by first separating the total energy density ρ into contributions coming from the *rest mass energy density* ρ_0 and the internal energy:

$$\rho = \rho_0\,(1 + \epsilon) \ , \tag{7.2.2}$$

where ϵ is the *specific internal energy* (internal energy per unit mass) of the fluid. Let us now introduce the so-called *specific enthalpy* of the fluid defined as[71]

$$h := 1 + \epsilon + \frac{p}{\rho_0} \ . \tag{7.2.3}$$

In terms of h, the stress-energy tensor then takes the simple form

$$T_{\mu\nu} = \rho_0 h\, u_\mu u_\nu + p\, \eta_{\mu\nu} \ . \tag{7.2.4}$$

The rest mass energy density is also often written in terms of the particle number density n as

$$\rho_0 = nm \ , \tag{7.2.5}$$

with m the rest mass of the fluid particles.

[71] In thermodynamics the enthalpy H is defined as the sum of the internal energy U plus the pressure times the volume, $H = U + pV$. In other words, the enthalpy represents the total energy in the system capable of doing mechanical work. In relativity we also add the rest mass energy M to the definition of enthalpy, so that $H = M + U + pV = M(1+\epsilon) + pV$. The *specific enthalpy* is then defined as the enthalpy per unit mass: $h = H/M = 1 + \epsilon + p/\rho_0$.

A note about the interpretation of the fluid variables is important at this point. In the case of general relativity, the 3+1 evolution equations and the Hamiltonian constraint involve the energy density of matter as a source of the gravitational field. However, this energy density is assumed to be measured in the Eulerian reference frame (the one associated with the observers whose 4-velocity is normal to the spatial hypersurface), which will in general differ from the fluid's frame of reference (the Lagrangian frame). In order to avoid confusion we will from now on denote the energy density that appears in the ADM equations by $\rho_{\rm ADM}$. We can derive the relationship between the different energy densities by starting from the definition of $\rho_{\rm ADM}$:

$$\rho_{\rm ADM} := n^\mu n^\nu T_{\mu\nu} \ , \tag{7.2.6}$$

with n^μ the unit normal to the spacelike hypersurfaces. Substituting the above stress-energy tensor here and using the fact that $n_\mu n^\mu = -1$ we find

$$\rho_{\rm ADM} = \rho_0 h \left(u_\mu n^\mu\right)^2 - p = \rho_0 h W^2 - p \ , \tag{7.2.7}$$

where we have defined

$$W := -u^\mu n_\mu = u^0 \ . \tag{7.2.8}$$

The last equality follows from the fact that, in special relativity, we have $n_\mu = (-1, 0)$. Finally, using the fact that $u_\mu u^\mu = -1$ we obtain

$$W = \left(1 + \sum_i u^i\right)^{1/2} \ . \tag{7.2.9}$$

But this is nothing more than the Lorentz factor $1/\sqrt{1-v^2}$, since the standard three-dimensional speed of the fluid is given by

$$v^i = u^i/u^0 = \left(1 - v^2\right)^{1/2} \ u^i \ , \tag{7.2.10}$$

which can be shown to imply

$$W = 1/\sqrt{1-v^2} \ . \tag{7.2.11}$$

In the particular case when the local coordinates follow the fluid element we have $W = 1$, and the energy densities become equal:

$$\rho_{\rm ADM} = \rho_0 h W^2 - p = \rho_0 h - p = \rho_0(1+\epsilon) = \rho \ . \tag{7.2.12}$$

Notice, however, that if the flow is non-uniform we can not adapt the coordinate system to follow the fluid elements (the Lagrangian approach) without being forced to replace the Minkowski metric $\eta_{\mu\nu}$ with a more general metric $g_{\alpha\beta}$, since the fluid motion will in general deform the volume elements.

The state of the fluid at any given time is given in terms of the six variables $(\rho_0, \epsilon, p, v^i)$, which from now on will be called the *primitive variables*. The evolution equations for the fluid now follow from the conservation laws. We have in fact two sets of conservation laws, namely the conservation of particles and the conservation of energy-momentum:

$$\partial_\mu \left(\rho_0 u^\mu\right) = 0 \,, \tag{7.2.13}$$

$$\partial_\mu T^{\mu\nu} = 0 \,. \tag{7.2.14}$$

Notice that these conservation laws provide us with five equations. In order to close the system we therefore need an equation of state which can be assumed to be of the form

$$p = p\left(\rho_0, \epsilon\right) \,. \tag{7.2.15}$$

To proceed let us now introduce the quantity

$$D := \rho_0 W \,, \tag{7.2.16}$$

which is nothing more than the rest mass density as seen in the Eulerian frame. The conservation of particles now implies

$$\partial_t D + \partial_k \left(D v^k\right) = 0 \,. \tag{7.2.17}$$

This is known as the continuity equation and has exactly the same form as in the Newtonian case, but now D includes the relativistic correction coming from the Lorentz factor W. The continuity equation can be interpreted as an evolution equation for D.

For the conservation of momentum we first define the quantities

$$S^\mu := \rho_0 h W u^\mu \,. \tag{7.2.18}$$

Notice that since $u^i = W v^i$, the spatial components $S^i = \rho_0 h W^2 v^i$ are nothing more that the momentum density as seen in the Eulerian frame, with the correct Lorentz factors (the fact that the enthalpy h appears in the expression for S^i shows that in relativity the pressure contributes to the momentum density). In terms of S^μ, the mixed components of the stress-energy tensor become

$$T^\mu_\nu = \frac{S_\nu u^\mu}{W} + p\, \delta^\mu_\nu \,, \tag{7.2.19}$$

The conservation of momentum then takes the form

$$\begin{aligned} & \partial_\mu \left(S_i u^\mu / W + p\, \delta^\mu_i\right) = 0 \\ \Rightarrow \quad & \partial_t S_i + \partial_k \left(S_i v^k\right) + \partial_i p = 0 \,. \end{aligned} \tag{7.2.20}$$

These are the evolution equations for the momentum density and are known as the *Euler equations*. Notice that they have a structure similar to that of the

continuity equation, but now there is an extra term given by the gradient of the pressure. The momentum density can then change both because of the flow of momentum out of the volume element represented by the term $\partial_j \left(S_i v^j\right)$, and because of the existence of a force given by the gradient of the pressure $\partial_i p$. The Euler equations above have again exactly the same form as in the Newtonian case, but now the definition of the momentum density S_i includes the relativistic corrections.

We are still missing an evolution equation for the energy density. Such an equation can be obtained in a number of different ways. Experience has shown that it is in fact convenient to subtract the rest mass energy density in order to have higher accuracy, since for systems that are not too relativistic the rest mass can dominate the total energy density. However, there are several non-equivalent ways to do this. As a first approach, consider the internal energy density as measured in the Eulerian frame:

$$E = \rho_0 \epsilon W \ . \tag{7.2.21}$$

Notice that there is only one Lorentz factor W coming from the Lorentz contraction of the volume elements, since the specific internal energy ϵ can be considered a scalar (this is by definition the internal energy per particle in the fluid's frame). In order to derive an evolution equation for E we first notice that the conservation equations imply that

$$\partial_\mu \left(u_\nu T^{\mu\nu}\right) = T^{\mu\nu} \partial_\mu u_\nu \ . \tag{7.2.22}$$

Substituting here the expression for the stress-energy tensor, and remembering that $u_\mu u^\mu = -1$ implies $u_\mu \partial_\nu u^\mu = 0$, we find

$$\partial_\mu \left(u_\nu T^{\mu\nu}\right) = p \, \partial_\mu u^\mu \ . \tag{7.2.23}$$

On the other hand

$$u_\nu T^{\mu\nu} = -\rho_0 \left(1 + \epsilon\right) u^\mu \ , \tag{7.2.24}$$

and using now the conservation of particles, this implies

$$\partial_\mu \left(u_\nu T^{\mu\nu}\right) = -\partial_\mu \left(\rho_0 \epsilon u^\mu\right) \ . \tag{7.2.25}$$

Collecting results we obtain

$$\partial_\mu \left(\rho_0 \epsilon u^\mu\right) + p \, \partial_\mu u^\mu = 0 \ , \tag{7.2.26}$$

which can be rewritten as

$$\partial_t E + \partial_k \left(E v^k\right) + p \left[\partial_t W + \partial_k \left(W v^k\right)\right] = 0 \ . \tag{7.2.27}$$

This equation has been used successfully by Wilson and collaborators to evolve relativistic fluids (see *e.g.* [300]). However, as an evolution equation for E it has

one serious drawback, namely that it also involves the time derivative of W, so that it can not be written as a balance law, which in particular makes it impossible to use for analyzing the characteristic structure of the system.

Fortunately, there exists an alternative way of subtracting the rest mass energy from the system that does yield an equation in balance law form. We can simply decide to evolve instead the difference between the total energy density and the mass energy density as measured in the Eulerian frame:

$$\mathcal{E} := \rho_{\rm ADM} - \rho_0 W = \rho_{\rm ADM} - D = \rho_0 h W^2 - p - D \ . \tag{7.2.28}$$

Notice that the energies E and $\mathcal{E}$ differ since E does not include contributions from the kinetic energy while $\mathcal{E}$ does. To find the evolution equation for $\mathcal{E}$ we first notice that from the definition of S^μ we have $S^0 = \rho_0 h W^2$. The conservation of energy then takes the form

$$0 = \partial_\mu T^{0\mu} = \partial_\mu \left(S^0 u^\mu / W + p \eta^{0\mu} \right) \ , \tag{7.2.29}$$

which immediately yields

$$\partial_t S^0 + \partial_k \left(S^0 v^k \right) - \partial_t p = 0 \ . \tag{7.2.30}$$

Using now the evolution equation for D we finally find

$$\partial_t \mathcal{E} + \partial_k \left[(\mathcal{E} + p)\, v^k \right] = 0 \ , \tag{7.2.31}$$

where we used the fact that $S^0 = \mathcal{E} + D + p$.

Our set of evolution equations then becomes a system of conservation laws of the form[72]

$$\partial_t D + \partial_k \left(D v^k \right) = 0 \ , \tag{7.2.32}$$

$$\partial_t S_i + \partial_k \left(S_i v^k + p\, \delta_i^k \right) = 0 \ , \tag{7.2.33}$$

$$\partial_t \mathcal{E} + \partial_k \left[(\mathcal{E} + p)\, v^k \right] = 0 \ , \tag{7.2.34}$$

with the conserved quantities $(D, S_i, \mathcal{E})$ given in terms of the primitive quantities $(\rho_0, \epsilon, p, v^i)$ as

$$D = \rho_0 W \ , \qquad S_i = \rho_0 h W^2 v_i \ , \qquad \mathcal{E} = \rho_0 h W^2 - p - \rho_0 W \ . \tag{7.2.35}$$

Note that the Euler equations are frequently written for the speed v_i instead of the flux S_i and have the form (see *e.g.* [297])

$$\partial_t v_i + v^k \partial_k v_i = - \left[\partial_i p + v_i \partial_t p \right] / (\rho_0 h W^2) \ . \tag{7.2.36}$$

These equations can be easily derived by combining the evolution equations for D, S_i and $\mathcal{E}$. However, they are not as convenient as the evolution equations for

[72]The hydrodynamic equations in the conservative form given here were first derived by Marti, Ibañez, and Miralles at the University of Valencia in Spain and are often called the *Valencia formulation* of relativistic hydrodynamics [130, 198] (see also [129, 200]).

S_i since they are not written as conservation laws, and in particular involve the time derivative of the pressure.

There is another important consequence of the conservation equations. Consider the contraction

$$u_\mu \partial_\nu T^{\mu\nu} = 0 \, . \tag{7.2.37}$$

Substituting the expression for $T^{\mu\nu}$, and using again the fact that $u_\mu \partial_\nu u^\mu = 0$, we find

$$u^\mu \partial_\mu p - \partial_\nu \left(\rho_0 h u^\mu \right) = 0 \, . \tag{7.2.38}$$

This can be further simplified with the help of the equation for conservation of particles to

$$u^\mu \partial_\mu p - \rho_0 u^\mu \partial_\mu h = 0 \, , \tag{7.2.39}$$

and using now the expression for the specific enthalpy h we finally obtain

$$\frac{d\epsilon}{d\tau} + p \, \frac{d}{d\tau} \left(\frac{1}{\rho_0} \right) = 0 \, , \tag{7.2.40}$$

where $d/d\tau := u^\mu \partial_\mu$ is the derivative along the trajectory of the fluid elements. This equation is in fact nothing more than the local version of the first law of thermodynamics. To see this, consider a fluid element with rest mass M, internal energy U, and volume V. We then have in general that

$$\rho_0 = \frac{M}{V} \quad \Rightarrow \quad dV = M d \left(\frac{1}{\rho_0} \right) \, , \tag{7.2.41}$$

and similarly

$$\epsilon = \frac{U}{M} \quad \Rightarrow \quad dU = M d\epsilon \, . \tag{7.2.42}$$

The first law of thermodynamics then implies that

$$dQ = dU + p dV = M \left[d\epsilon + p \, d \left(\frac{1}{\rho_0} \right) \right] \, . \tag{7.2.43}$$

This shows that (7.2.40) is precisely the first law of thermodynamics for a fluid element for which $dQ = 0$ (this is to be expected since by definition a perfect fluid has no heat conduction). And since in general $dQ = T d\mathcal{S}$, with T the temperature and $\mathcal{S}$ the entropy of the fluid, we see that a perfect fluid behaves in such a way that entropy is preserved along flow lines.

Let us now go back and consider the relation between the primitive and conserved variables (7.2.35). In the Newtonian limit these relations reduce to $D = \rho_0$, $S_i = \rho_0 v_i$ and $\mathcal{E} = \rho_0(\epsilon + v^2/2)$, so that they are very easy to invert. In the relativistic case, however, inverting the relations becomes much more difficult since first W involves v^2, and also the pressure appears explicitly in the expression for

$\mathcal{E}$, so the equation of state is needed in order to recover the primitive variables. Unfortunately, the evolution equations for the conserved quantities involve the primitive variables directly, so that these must be recovered every time step. This requires that an algorithm for recovering such variables is implemented. Such an algorithm starts by choosing some trial value of the pressure p^* (for example the old value at the corresponding grid cell). Then, from the expressions for D and $\mathcal{E}$ we can recover the speed v^i in the following way

$$v_i(p^*) = \frac{S_i}{\rho_0 h W^2} = \frac{S_i}{\mathcal{E} + D + p^*} \,. \tag{7.2.44}$$

Having found v_i we then compute W as

$$W(p^*) = \frac{1}{\sqrt{1 - v^2(p^*)}} \,. \tag{7.2.45}$$

This allows us to find the density ρ_0 as

$$\rho_0(p^*) = \frac{D}{W(p^*)} \,. \tag{7.2.46}$$

Finally, we find the specific internal energy ϵ through the definition of h

$$\begin{aligned} \epsilon(p^*) &= \frac{\mathcal{E} + \rho_0 W\,(1 - W) + p^*\left(1 - W^2\right)}{\rho_0 W^2} \\ &= \frac{\mathcal{E} + D\,(1 - W(p^*)) + p^*\left(1 - W^2(p^*)\right)}{D W(p^*)} \end{aligned} \tag{7.2.47}$$

Of course, the chosen value of p^* will almost certainly not satisfy the equation of state $p = p(\rho_0, \epsilon)$, so we must now evaluate the residual $r(p^*)$ defined as

$$r(p^*) := p(\rho_0(p^*), \epsilon(p^*)) - p^* \,, \tag{7.2.48}$$

and change the value of p^* until this residual vanishes. This can typically be accomplished by standard non-linear root-finding techniques (*e.g.* one-dimensional Newton–Raphson). For some simple equations of state, such as that of an ideal gas discussed in Section 7.5, the whole procedure can in fact be done analytically and involves finding the physically admissible root of a high order polynomial (a fourth order polynomial in the case of an ideal gas). However, this is typically more computationally expensive than using the non-linear root finder directly.

7.3 General relativistic hydrodynamics

The generalization of the evolution equations (7.2.32)–(7.2.34) to the case of a non-trivial gravitational field is rather straightforward. We again start from the stress-energy tensor for a perfect fluid, but now for an arbitrary metric $g_{\mu\nu}$:

$$T_{\mu\nu} = \rho_0 h u_\mu u_\nu + p\, g_{\mu\nu} \,, \tag{7.3.1}$$

where as before u^μ is the 4-velocity of the fluid elements, ρ_0 is the rest mass energy density measured in the fluid's rest frame, p is the pressure and h is the specific enthalpy

$$h := 1 + \epsilon + \frac{p}{\rho_0} \,, \tag{7.3.2}$$

with ϵ the specific internal energy. The evolution equations for the fluid again follow from the conservation laws, which however now take the form

$$\nabla_\mu \left(\rho_0 u^\mu\right) = 0 \,, \tag{7.3.3}$$

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{7.3.4}$$

Using the fact that the divergence of a vector can be written in general as

$$\nabla_\mu \xi^\mu = \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\, \xi^\mu\right) \,, \tag{7.3.5}$$

with g the determinant of the metric tensor $g_{\mu\nu}$, we can immediately rewrite the conservation of particles as

$$\partial_\mu \left(\sqrt{-g}\, \rho_0 u^\mu\right) = 0 \,, \tag{7.3.6}$$

and the conservation of energy and momentum as

$$\partial_\mu \left(\sqrt{-g}\, T^\mu_\nu\right) = \sqrt{-g}\, \Gamma^\alpha_{\mu\nu} T^\mu_\alpha \,, \tag{7.3.7}$$

with $\Gamma^\alpha_{\mu\nu}$ the Christoffel symbols associated with the metric $g_{\mu\nu}$.

We now assume that we are using a standard 3+1 decomposition of spacetime, in which case we find

$$g = -\alpha^2 \gamma \quad \Rightarrow \quad \sqrt{-g} = \alpha\sqrt{\gamma} \,, \tag{7.3.8}$$

with α the lapse function and γ the determinant of the spatial metric γ_{ij}.

Just as we did in special relativity, let us define the scalar parameter W as

$$W := -u^\mu n_\mu \,, \tag{7.3.9}$$

with n^μ the unit normal to the spatial hypersurfaces. In this case we have $n_\mu = (-\alpha, 0)$, so that

$$W = \alpha u^0 \,. \tag{7.3.10}$$

Define now

$$v^i := \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha} \,, \tag{7.3.11}$$

with β^i the shift vector. With this definition v^i is precisely the speed of the fluid elements as seen by the Eulerian observers. To see this notice that u^i/u^0 is the coordinate speed of the fluid elements, so we first need to add the shift to go to the Eulerian reference frame and then divide by the lapse to use the proper time

of the Eulerian observers instead of coordinate time. Notice also that since u^μ is a 4-vector while v^i is only a 3-vector, when we lower the indices we have

$$\begin{aligned} u_i &= g_{i\mu}u^\mu = \beta_i u^0 + \gamma_{ik}u^k \\ &= \beta_i u^0 + \gamma_{ik}u^0\left(\alpha v^k - \beta^k\right) = \alpha u^0 v_i = W v_i \ . \end{aligned} \tag{7.3.12}$$

Using again the fact that $u_\mu u^\mu = -1$, we find that W takes the simple form

$$W = 1/\sqrt{1-v^2} \ , \tag{7.3.13}$$

where now $v^2 := \gamma_{ij}v^i v^j$, *i.e.* W is again the Lorentz factor as seen by the Eulerian observers. Define again D as the rest mass density measured by the Eulerian observers

$$D := \rho_0 W \ , \tag{7.3.14}$$

we can then rewrite the conservation of particles as

$$\partial_t\left(\sqrt{\gamma}\, D\right) + \partial_k\left[\sqrt{\gamma}\, D\left(\alpha v^k - \beta^k\right)\right] = 0 \ . \tag{7.3.15}$$

This is again a conservation law for D, but in contrast to the case of special relativity it involves the lapse α, the shift vector β^k, and the determinant of the spatial metric γ (compare with equation (7.2.32)).

For the conservation of momentum we again introduce the quantities

$$S^\mu := \rho_0 h W u^\mu \ , \tag{7.3.16}$$

and rewrite the stress-energy tensor as

$$T^\mu_\nu = \frac{u^\mu S_\nu}{W} + p\,\delta^\mu_\nu \ . \tag{7.3.17}$$

The conservation of momentum then becomes

$$\partial_t\left(\sqrt{\gamma}S_i\right) + \partial_k\left\{\sqrt{\gamma}\left[S_i\left(\alpha v^k - \beta^k\right) + \alpha p\,\delta^k_i\right]\right\} = \alpha\sqrt{\gamma}\,\Gamma^\mu_{\nu i}T^\nu_\mu \ , \tag{7.3.18}$$

where $S_i = \rho_0 h W u_i = \rho_0 h W^2 v_i$. The last equations are the general relativistic version of the Euler equations. When we compare them with their special relativistic version (7.2.33) we see that in the case of general relativity, apart from the lapse and shift factors that correct for the motion of the Eulerian observers, we don't have strict conservation of momentum anymore since there is a source term on the right hand side that represents the gravitational forces.

Finally, for the conservation of energy we again start by defining

$$\mathcal{E} = \rho_0 h W^2 - p - D \ . \tag{7.3.19}$$

The conservation of energy then takes the form

$$\partial_\mu\left(\alpha\sqrt{\gamma}\,T^{0\mu}\right) = -\alpha\sqrt{\gamma}\,\Gamma^0_{\mu\nu}T^{\mu\nu} \ . \tag{7.3.20}$$

It is in fact convenient to rewrite the term on the left hand side as

$$\partial_\mu \left(\alpha\sqrt{\gamma}\, T^{0\mu}\right) = \frac{1}{\alpha}\partial_\mu \left(\alpha^2\sqrt{\gamma}\, T^{0\mu}\right) - \sqrt{\gamma}\, T^{0\mu}\partial_\mu \alpha \,, \tag{7.3.21}$$

so that the conservation of energy becomes

$$\partial_\mu \left(\alpha^2\sqrt{\gamma}\, T^{0\mu}\right) = \alpha^2\sqrt{\gamma}\left(T^{0\mu}\partial_\mu \ln\alpha - \Gamma^0_{\mu\nu}T^{\mu\nu}\right) \,. \tag{7.3.22}$$

Using now the expression for $T^{0\mu}$ we obtain, after some algebra,

$$\begin{aligned}\partial_\mu \left(\alpha^2\sqrt{\gamma}\, T^{0\mu}\right) &= \partial_t \left[\sqrt{\gamma}\left(\rho_0 h W^2 - p\right)\right] \\ &\quad + \partial_k \left\{\sqrt{\gamma}\left[\rho_0 h W^2\left(\alpha v^k - \beta^k\right) + p\beta^k\right]\right\} \,,\end{aligned} \tag{7.3.23}$$

and using the evolution equation for D we find that the final expression for the conservation of energy takes the form

$$\begin{aligned}&\partial_t \left(\sqrt{\gamma}\,\mathcal{E}\right) + \partial_k \left\{\sqrt{\gamma}\left[\mathcal{E}\left(\alpha v^k - \beta^k\right) + \alpha p v^k\right]\right\} \\ &\qquad = \alpha^2\sqrt{\gamma}\left(T^{0\mu}\partial_\mu \ln\alpha \,-\, \Gamma^0_{\mu\nu}T^{\mu\nu}\right) \,.\end{aligned}$$

The final set of evolution equations is then

$$\partial_t \left(\sqrt{\gamma}\, D\right) + \partial_k \left[\sqrt{\gamma}\, D\left(\alpha v^k - \beta^k\right)\right] = 0 \,, \tag{7.3.24}$$

$$\partial_t \left(\sqrt{\gamma}\, S_i\right) + \partial_k \left\{\sqrt{\gamma}\left[S_i\left(\alpha v^k - \beta^k\right) + \alpha p \delta^k_i\right]\right\} = \alpha\sqrt{\gamma}\,\Gamma^\mu_{\nu i}T^\nu_\mu \,, \tag{7.3.25}$$

$$\begin{aligned}\partial_t \left(\sqrt{\gamma}\,\mathcal{E}\right) + \partial_k \left\{\sqrt{\gamma}\left[\mathcal{E}\left(\alpha v^k - \beta^k\right) + \alpha p v^k\right]\right\} &= \alpha^2\sqrt{\gamma}\left(T^{0\mu}\partial_\mu \ln\alpha\right. \\ &\quad \left. - \,\Gamma^0_{\mu\nu}T^{\mu\nu}\right) \,,\end{aligned} \tag{7.3.26}$$

where the conserved quantities $(D, S_i, \mathcal{E})$ and primitive variables $(\rho_0, \epsilon, p, v^i)$ are related through

$$D = \rho_0 W \,, \qquad S_i = \rho_0 h W^2 v_i \,, \qquad \mathcal{E} = \rho_0 h W^2 - p - D \,. \tag{7.3.27}$$

Notice that the system of equations (7.3.24)–(7.3.26) reduces to the special relativistic counterpart (7.2.32)-(7.2.34) when we take $\alpha = 1$, $\beta^i = 0$ and $\gamma_{ij} = \delta_{ij}$, in which case $\Gamma^\alpha_{\mu\nu} = 0$ and the system is truly conservative. The presence of a non-trivial gravitational field implies that there is no longer true conservation of energy and momentum, but the equations are still in the form of balance laws: $\partial_t u + \partial_k F^k(u) = s(u)$.

Before concluding this Section, it is important to write down the relation between the quantities $(D, S_i, \mathcal{E})$ and the matter terms measured by the Eulerian observers that appear in the ADM equations, namely the energy density $\rho_{\rm ADM}$, the momentum density $j^i{}_{\rm ADM}$ and the stress tensor $S^{ij}{}_{\rm ADM}$. Using the expression for $T^{\mu\nu}$ we find

$$\rho_{\rm ADM} := n^\mu n^\nu T_{\mu\mu} = \rho_0 h W^2 - p = \mathcal{E} + D \,, \tag{7.3.28}$$

$$j^i{}_{\rm ADM} := -n^\mu P^{\nu i} T_{\mu\nu} = \rho_0 h W^2 v^i = S^i \,, \tag{7.3.29}$$

$$S^{ij}{}_{\rm ADM} := P^{\mu i} P^{\nu j} T_{\mu\nu} = \rho_0 h W^2 v^i v^j + \gamma^{ij} p \,, \tag{7.3.30}$$

where $P^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$ is the standard projection operator onto the spatial hypersurfaces.

7.4 3+1 form of the hydrodynamic equations

The general relativistic hydrodynamic evolution equations (7.3.24)–(7.3.26) derived in the previous Section have been written as a set of balance laws, which has some advantages from a numerical point of view. However, from the perspective of the 3+1 formulation of general relativity they are not written in the most convenient form as they are not manifestly 3-covariant. In this Section we will rewrite these equations as tensor equations in 3+1 language.

Let us start from the evolution equation for the rest mass density D. Since this is just the particle density as seen by the Eulerian observers, times the rest mass of the individual particles, it is in fact a scalar in 3+1 terms. The original evolution equation has the form

$$\partial_t \left(\sqrt{\gamma}\, D\right) + \partial_k \left[\sqrt{\gamma}\, D \left(\alpha v^k - \beta^k\right)\right] = 0 \,. \tag{7.4.1}$$

Notice first that, for any three-dimensional vector w^i we have

$$\frac{1}{\sqrt{\gamma}}\, \partial_k \left(\sqrt{\gamma} w^k\right) = D_k w^k \,, \tag{7.4.2}$$

with D_k the covariant derivative associated with the spatial metric γ_{ij}. This implies that

$$\frac{1}{\sqrt{\gamma}}\, \partial_k \left[\sqrt{\gamma}\, D \left(\alpha v^k - \beta^k\right)\right] = D_k \left(\alpha D v^k\right) - \left(D_k \beta^k\right) D - \beta^k \partial_k D \,. \tag{7.4.3}$$

On the other hand

$$\frac{1}{\sqrt{\gamma}}\, \partial_t \left(\sqrt{\gamma}\, D\right) = \partial_t D + \frac{D}{2}\, \partial_t \ln \gamma \,. \tag{7.4.4}$$

Now, from the ADM evolution equations for the spatial metric (2.3.12), we can easily find that

$$\partial_t \ln \gamma = -2\alpha K + 2 D_k \beta^k \,, \tag{7.4.5}$$

with K the trace of the extrinsic curvature K_{ij}. Collecting results we find that the evolution equation for D in 3+1 language takes the final form

$$\partial_t D - \beta^k \partial_k D + D_k \left(\alpha D v^k\right) = \alpha K D \,. \tag{7.4.6}$$

This equation is clearly a scalar equation. The different terms are easy to interpret: The shift appears only in the advection term, as it should, since the only role of the shift is to move the coordinate lines. The last term on the left hand side shows that the change in D along the normal lines is given essentially by the divergence of the flux of particles. Finally, the source term shows that the density of particles D can also change because of an overall change in the spatial volume elements. For example, in the case of cosmology the so-called cosmological fluid is co-moving with the Eulerian observers so that $v^i = \beta^i = 0$, but the density of

particles still becomes smaller with time because of the overall expansion of the Universe ($K < 0$).

It is also interesting to note that in the last equation the shift appears as an advection term that is not in flux-conservative form (the shift is outside the spatial derivatives). The flux conservative form (7.3.24) comes about because as we bring the shift vector into the spatial derivative we pick up a term with the divergence of the shift. This term is canceled by a corresponding term coming from the time derivative of the volume element. This shows that, quite generally, advection terms on the shift can be transformed into flux conservative type terms by bringing a $\sqrt{\gamma}$ factor into the time derivative.

Consider next the evolution equation for $\mathcal{E}$. Again, since this is by definition the energy density measured by the Eulerian observers minus the rest mass density, $\mathcal{E} = \rho_{\rm ADM} - D = \rho_0 h W^2 - p - D$, it is clearly a scalar quantity in 3+1 terms. Its original evolution equation is

$$\partial_t \left(\sqrt{\gamma}\,\mathcal{E}\right) + \partial_k \left\{\sqrt{\gamma}\left[\mathcal{E}\left(\alpha v^k - \beta^k\right) + \alpha p v^k\right]\right\} = \alpha^2\sqrt{\gamma}\left(T^{0\mu}\partial_\mu \ln\alpha - \Gamma^0_{\mu\nu}T^{\mu\nu}\right) .$$

Let us first look at the source term. Using the expression for the stress-energy tensor (7.3.1), the definition of v^i (7.3.11), and the expressions for the 4-Christoffel symbols in 3+1 language found in Appendix B, it is not difficult to show that

$$\begin{aligned}\alpha^2\left(T^{0\mu}\partial_\mu \ln\alpha - \Gamma^0_{\mu\nu}T^{\mu\nu}\right) &= \rho_0 h W^2\left(\alpha v^m v^n K_{mn} - v^m\partial_m\alpha\right) + \alpha p K \\ &= \left(\mathcal{E} + p + D\right)\left(\alpha v^m v^n K_{mn} - v^m\partial_m\alpha\right) + \alpha p K\ . \end{aligned} \tag{7.4.7}$$

We can now rewrite the left hand side of the evolution equation for $\mathcal{E}$ in exactly the same way as the evolution equation for D. We then find the following evolution equation for $\mathcal{E}$ in 3+1 form

$$\begin{aligned}\partial_t\mathcal{E} - \beta^k\partial_k\mathcal{E} + D_k\left[\alpha v^k\left(\mathcal{E}+p\right)\right] &= \left(\mathcal{E}+p+D\right)\left(\alpha v^m v^n K_{mn} - v^m\partial_m\alpha\right) \\ &\quad + \alpha K\left(\mathcal{E}+p\right)\ . \end{aligned} \tag{7.4.8}$$

The last term on the right hand side is interesting. Assume that we have a fluid that is co-moving with the Eulerian observers so that $v^k = \beta^k = 0$, we then find that $\partial_t\mathcal{E} = \alpha K(\mathcal{E}+p)$. This shows that the internal energy density changes both as a reflection of a simple change in the volume elements ($\alpha K\mathcal{E}$), and because of the existence of a non-zero pressure ($\alpha p K$). But we know that $\alpha K = -\partial_t \ln\sqrt{\gamma}$, so that $\alpha p K = -p\,\partial_t \ln\sqrt{\gamma}$, which is nothing more than the work done by the fluid as space expands. That is, the term $\alpha p K$ in the source term is there in accordance with the first law of thermodynamics.

Finally, let us consider the evolution equation for the momentum density S_i. Since we have $S_i = \rho_0 h W^2 v_i$, then we can consider S_i a vector with respect to the 3-geometry. Its original evolution equation is

$$\partial_t\left(\sqrt{\gamma}S_i\right) + \partial_k\left\{\sqrt{\gamma}\left[S_i\left(\alpha v^k - \beta^k\right) + \alpha p\,\delta^k_i\right]\right\} = \alpha\sqrt{\gamma}\,\Gamma^\mu_{\nu i}T^\nu_\mu\ , \tag{7.4.9}$$

From the expression for the stress-energy tensor, the right hand side of this equation can easily be shown to be

$$\alpha\sqrt{\gamma}\,\Gamma^{\mu}_{\nu i}T^{\nu}_{\mu} = p\,\partial_i\left(\alpha\sqrt{\gamma}\right) + \alpha\sqrt{\gamma}\,\frac{\rho_0 h}{2}\,u^{\mu}u^{\nu}\partial_i g_{\mu\nu}\ . \tag{7.4.10}$$

Substituting now the components of the 4-metric $g_{\mu\nu}$ in terms of 3+1 quantities we find, after some algebra, that

$$\frac{\alpha}{2}\,u^{\mu}u^{\nu}\partial_i g_{\mu\nu} = W^2\left[-\partial_i\alpha + v^k D_i\beta_k + {}^{(3)}\Gamma^m_{ik}\,\frac{u^k v_m}{u^0}\right]\ , \tag{7.4.11}$$

where ${}^{(3)}\Gamma^m_{ik}$ are the Christoffel symbols associated with the 3-geometry. On the other hand we have

$$\frac{1}{\sqrt{\gamma}}\,\partial_t\left(\sqrt{\gamma}S_i\right) = \partial_t S_i - \alpha K S_i + S_i D_k\beta^k\ , \tag{7.4.12}$$

and

$$\begin{aligned}\frac{1}{\sqrt{\gamma}}\,\partial_k\left[\sqrt{\gamma}S_i\left(\alpha v^k - \beta^k\right)\right] &= D_k\left[\alpha S_i\left(v^k-\beta^k\right)\right] \\ &\quad + S_m\left(\alpha v^n - \beta^n\right){}^{(3)}\Gamma^m_{in}\ .\end{aligned} \tag{7.4.13}$$

Collecting results we find that the evolution equation for S_i becomes

$$\begin{aligned}&\partial_t S_i - \alpha K S_i + D_k\left(\alpha S_i v^k\right) - \beta^k D_k S_i + S_m\left(\alpha v^k - \beta^k\right){}^{(3)}\Gamma^m_{ik} + \frac{1}{\sqrt{\gamma}}\,\partial_i\left(\alpha\sqrt{\gamma}\,p\right) \\ &\quad = \frac{p}{\sqrt{\gamma}}\,\partial_i\left(\alpha\sqrt{\gamma}\right) + \rho_0 h W^2\,\left[-\partial_i\alpha + v^k D_i\beta_k + {}^{(3)}\Gamma^m_{ik}\frac{u^k v_m}{u^0}\right],\end{aligned} \tag{7.4.14}$$

which can be simplified to

$$\partial_t S_i - \pounds_{\vec{\beta}} S_i + D_k\left(\alpha S_i v^k\right) + \partial_i\left(\alpha p\right) = -\left(\mathcal{E}+D\right)\partial_i\alpha + \alpha K S_i\ . \tag{7.4.15}$$

Notice that the shift vector again only appears in the Lie derivative term, as expected.

The full set of hydrodynamic equations in 3+1 form can then be written as

$$\partial_t D - \beta^k\partial_k D + D_k\left(\alpha D v^k\right) = \alpha K D\ , \tag{7.4.16}$$

$$\partial_t S^i - \pounds_{\vec{\beta}} S^i + D_k\left[\alpha\left(S^i v^k + \gamma^{ik}p\right)\right] = -\left(\mathcal{E}+D\right)D^i\alpha + \alpha K S^i\ , \tag{7.4.17}$$

$$\begin{aligned}\partial_t\mathcal{E} - \beta^k\partial_k\mathcal{E} + D_k\left[\alpha v^k\left(\mathcal{E}+p\right)\right] &= \left(\mathcal{E}+p+D\right)\left(\alpha v^m v^n K_{mn} - v^m\partial_m\alpha\right) \\ &\quad + \alpha K\left(\mathcal{E}+p\right)\ .\end{aligned} \tag{7.4.18}$$

The above equations are now manifestly 3-covariant when we consider $(D, \mathcal{E}, p)$ as scalars and S_i as a 3-vector.[73] The 3+1 equations just derived can also be used

[73] These 3+1 hydrodynamic equations have also been derived previously by Salgado using a somewhat different notation in [248].