Lecture II

NR…why

If we think hard enough we won’t need a computer

*With the right resources we can simulate situations we can't even begin to think through, and thereby provide us with **completely new and unexpected** things to think about [Choptuik]*

- Solve $G_{\mu\nu} = \kappa T_{\mu\nu}$ through simulations
  - nonlinear PDE system
  - *‘evolution’*
  - Initial and boundary data
  - *Singularities*
  - *Coordinate issues*
- Non-vacuum…
  - *Fluid?, shocks*, microphysics…
  - Initial and boundary data

+Computat Resources:

ALGORITHM ISSUES…

(too?) Many options!

- It isn’t a matter of “analytics” vs “brute force numerics”
  - (there is ‘brute force analytics’ and ‘elegant numerics’ as well)

- No need to lower standards: There is a control parameter. Richardson extrapolation → continuum limit can be obtained. Further, for smooth solutions, nothing magic about a technique (FD,FV,FE,Spectral) all can get the job done (some more easily than others)

- **Definition 16 (IDIOT)** *Anyone who publishes a calculation without checking it against an identical computation with smaller N OR without evaluating the residual of the pseudospectral approximation via finite differences is an IDIOT.*

J.P. Boyd: ``Chebyshev and Fourier Spectral Methods’’ The author apologizes to those who are annoyed at being told the obvious. However, I have spent an amazing amount of time persuading students to avoid the sins in this definition.

“A computation is a temptation that should be resisted as long as possible.”— J. P. Boyd, paraphrasing T. S. Eliot

- Some reminders: consider the equation $\phi_{,tt} = \phi_{,xx}$

→ Introduce $g = \phi_{,t}$; $f = \phi_{,x}$

→ System is now

$$\phi_{,t} = g$$
$$f_{,t} = g_{,x}$$
$$g_{,t} = f_{,x}$$

i.e. $\Longrightarrow$ $\begin{pmatrix} \phi \\ f \\ g \end{pmatrix}_{,t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ f \\ g \end{pmatrix}_{,x}$

Observation

i) All eigenvalues are real! $\lambda = 0$. $\lambda_{\pm} = \pm 1$

good sign!! (at least "weakly hyperbolic")

ii) Complete set of eigenvectors

great sign!! (strongly/symmetric hyperbolic)

With (i) + (ii) : $\rightarrow$ strong platform for numerical implementation!

a) At least locally a soln exists which is unique (with consistent ID!)

b) Not "overly sensitive" on initial data upon small changes.

c) "Bounded" eigenvalues? $\rightarrow$ good for "time stepping" requirements

d) Not depending on soln itself $\rightarrow$ good for smoothness of soln $\rightarrow$ discretization strategy is simple!!

Note If our system is given by $U \equiv \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_{,t} = M^i \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{pmatrix}_{,i}$

+ undifferentiated parts and:

i) Eigenvalues are real (✓ Needed this!)

ii) Complete set of eigenvectors

Then

(A) Can only concentrate on principal part to understand propagation of perturbations (locally)

(B) We indeed "just" have a bunch of "$u_{,t} = u_{,x}$" !!

i.e. $U_{,t} = M U_{,x}$ say $\exists\, T$ s.t $T^{-1} M T = D$

$$\Rightarrow \quad U_{,t} = T T^{-1} M T T^{-1} U_{,x}$$

$$\underbrace{T^{-1} U_{,t}} = T^{-1} T \left(T^{-1} M T\right) T^{-1} U_{,x}$$

$$(T^{-1}U)_{,t} - (T^{-1})_{,t} U = D\left\{(T^{-1}U)_{,x} - (T^{-1})_{,x} U\right\}$$

Define $V \equiv T^{-1} U$ $\leadsto$ $\underbrace{V_{,t} = D V_{,x}}$ + Rest

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{,t} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{,x} \quad !!$$

Further general solution of $v_{,t} = \lambda v_{,x}$ is $f(\lambda t + x)$!

$\Rightarrow$ sign of λ implies direction of motion of perturbations

$\Rightarrow$ tell us where to put boundary conditions & how!

Example: $\phi_{,tt} = \phi_{,xx}$ $\Rightarrow$ $$\begin{pmatrix}\phi\\ f\\ g\end{pmatrix}_{,t} = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\phi\\ f\\ g\end{pmatrix}_{,x}$$

$\lambda = 0$ $\quad e_0 = (1,0,0)$ $\Rightarrow$ ϕ "doesn't propagate"

$\lambda_+ = 1$ $\quad e_+ = (0,1,1)$ $\Rightarrow$ $(f+g)$ propagates to left

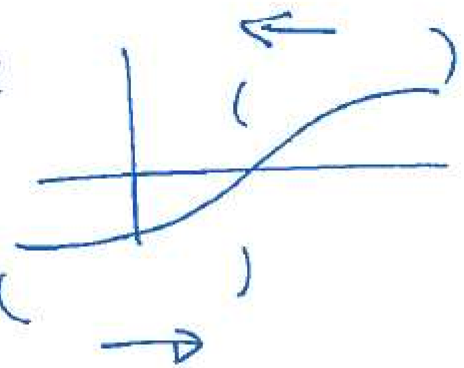
$\lambda_- = -1$ $\quad e_- = (0,1,-1)$ $\Rightarrow$ $(f-g)$ " " right

• What if $\lambda = \lambda(q)$? Eg. $\boxed{q_{,t} = q\, q_{,x}}$ (Burger's eqn)

Consider a "background" solution $q(t,x) = q^0(t,x)$ → Perturbations of it?
$q = q^0 + \delta q$

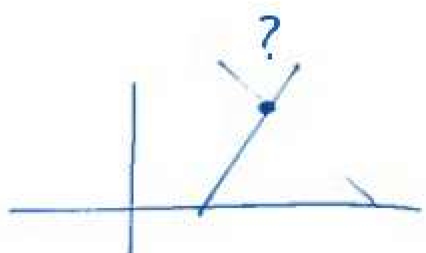
⇝ $\delta q_{,t} = q_0 \, \delta q_{,x}$ → soln $\delta q(x,t) = h(q_0 t + x)$ point to point

Eg



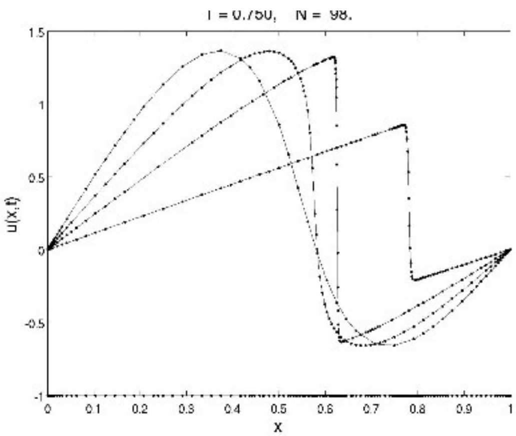
– Characteristics will cross! → Truly non linear eqn!

– past this point what is the value of solution?



Approach? — Consider weak solutions, i.e. $\iint \phi\,(q_{,t} + \partial_x f(q))\,dx\,dt = 0$

$$\iint \phi_{,t}\, q + f\,\partial_x \phi \; dx\,dt = \int \phi(x,0)\, q(x,0)\, dt$$

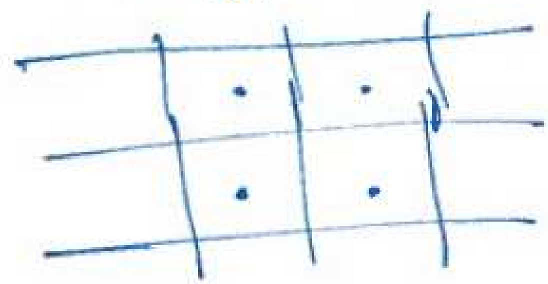


Unique solution picked through entropy considerations → Rankine-Hugoniot conditions

— Integration strategy? "Finite volume"

At their core, FV methods keep track of average quantities within a "cell", e.g.

$$\bar{q}_i^n = \frac{1}{\text{Volume of cell}_i}\int_{\text{cell}_i} q(x,t_n)\,dx = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)$$



- At each time step, update is obtained through (approximations to) the flux through cell boundaries

- Thus, a <u>conservative form</u> of the equations is convenient (useful!)

e.g. $\partial_t u + a(u)\,\partial_x u = 0 \implies \partial_t u + \partial_x f = \overset{?}{0}$ a source

<u>Example</u> $\partial_t u + \partial_x\left(\frac{1}{2}u^2\right)$ is preferred to $\partial_t u + u u_{,x} = 0$

<u>Thms</u>: Lax Wendroff: If a method in conservative form <u>converges</u> it converges to the weak solution of the conservation law.

Hou LeFloch: If a method in non-conservative form converges, in the presence of a shock wave it converges to the wrong solution.

Finite-volume numerical methods

The integral form of the equation: $\partial_t \mathrm{u} + \partial_x \mathrm{f}(\mathrm{u}) = 0$

namely:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)\mathrm{d}x = f[u(x_{i-1/2},t)] - f[u(x_{i+1/2},t)]$$

suggests that we should study numerical methods in the form:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t_{n+1})\mathrm{d}x = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t_n)\mathrm{d}x + \frac{1}{\Delta x}\left(\int_{t_n}^{t_{n+1}} f[u(x_{i-1/2},t)]\mathrm{d}t - \int_{t_n}^{t_{n+1}} f[u(x_{i+1/2},t)]\mathrm{d}t\right)$$

or, in more compact form,

$$\overline{u}_{\mathrm{i}}^{\mathrm{n+1}} = \overline{u}_{\mathrm{i}}^{\mathrm{n}} - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right)$$

where

$$F_{i-1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f[u(x_{i-1/2},t)]\mathrm{d}t$$

Different numerical schemes differ in the prescription for computing the flux function *F*.

[a few slides from Baiotti]

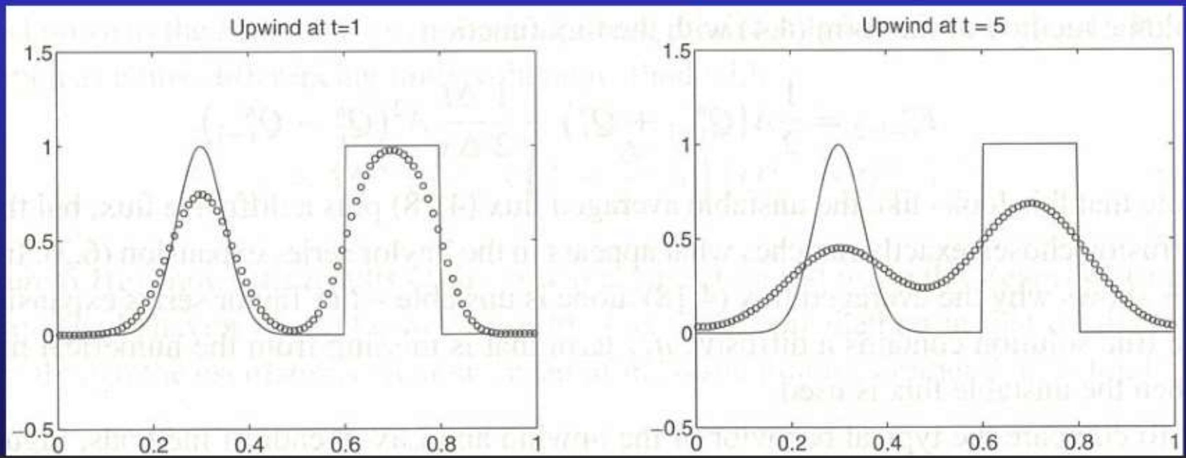
Discontinuities and numerical schemes

Since the occurrence of discontinuities is a fundamental property of the hydrodynamical equations, any numerical scheme must be able to handle them in a satisfactory way.

Possible solutions to the discontinuity problem:

❖ **1st order accurate schemes**

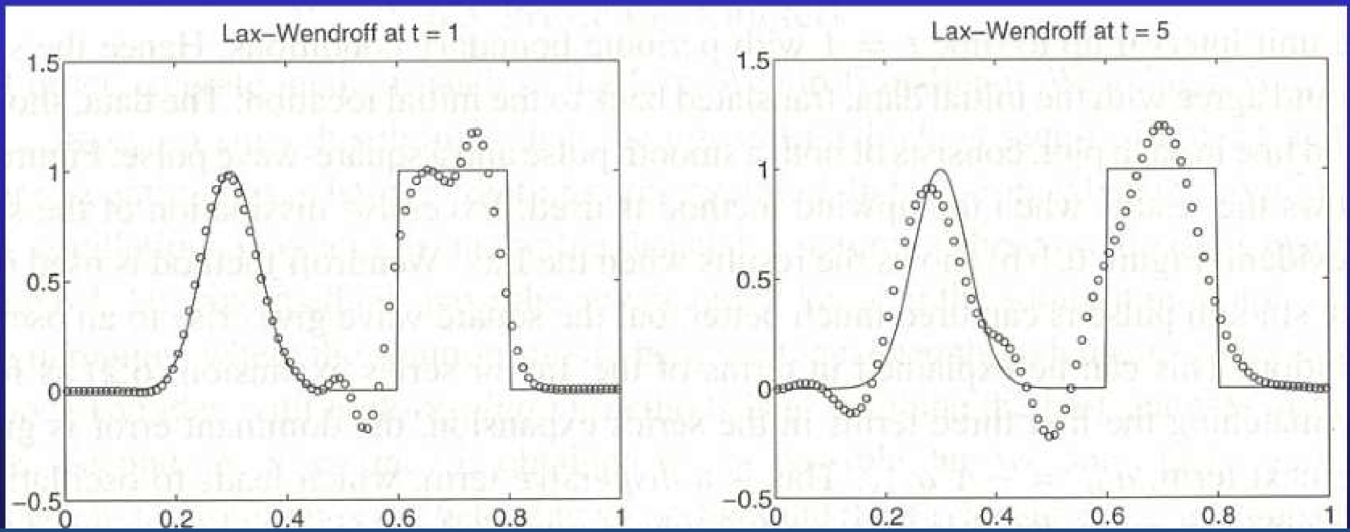
- generally fine, but very inaccurate across discontinuities (excessive diffusion); e.g. Lax-Friedrichs method $\overline{u}_{i}^{n+1} = \frac{1}{2}(\overline{u}_{i+1}^{n} + \overline{u}_{i-1}^{n}) - \frac{\Delta t}{2\Delta x}\left(F_{i+1} - F_{i-1}\right)$



Discontinuities and numerical schemes

Discontinuities and numerical schemes

- **2nd order accurate schemes**
  - more accurate, but generally introduce oscillations across discontinuities and are dispersive even on smooth data (especially for steep gradients), causing waves to move with a wrong group velocity (e.g. Lax-Wendroff method)



Discontinuities and numerical schemes

- **2^{nd} order accurate schemes with artificial viscosity**
  - mimic Nature, but problem-dependent and inaccurate for ultrarelativistic flows

- **Godunov methods**
  - discontinuities are not eliminated, rather they are exploited
  - based on the solution of Riemann problems
  - approximately second-order schemes can be derived
  - state of the art in relativistic hydrodynamics

Riemann problem

Definition: in general, for a hyperbolic system of equations, a Riemann problem is an initial-value problem with initial condition given by:

$$U(x,0) = \begin{cases} U_L & \text{if} \quad x < 0 \\ U_R & \text{if} \quad x > 0 \end{cases}$$

where U_L and U_R are two constant vectors representing the left and right state.

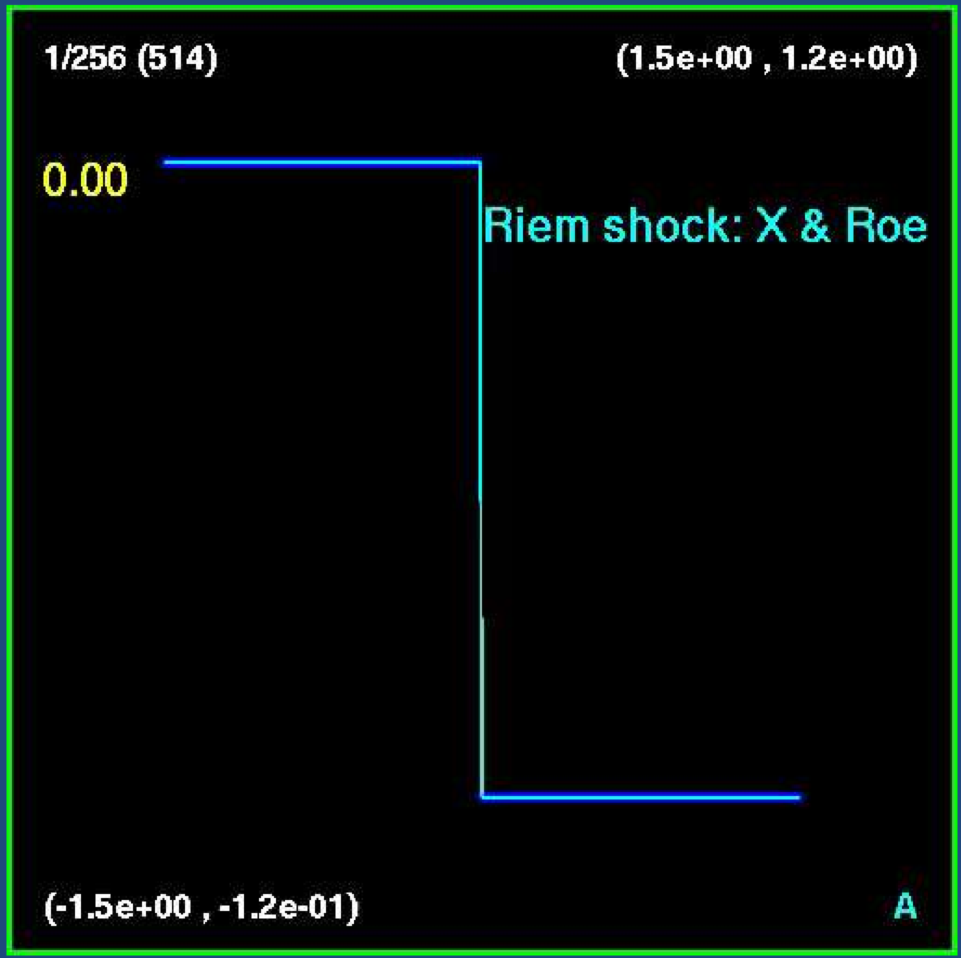
For hydrodynamics, a (physical) Riemann problem is the evolution of a fluid initially composed of two states with different and constant values of velocity, pressure and density.

- Back to Burger's equation. $q_{,t} + \frac{1}{2}(q_{,x})^2$=0
- Asssume q(0,x)>0 → pulse moves to right, right?
- Riemann problem q(0,x<0) = qL ; q(0,x>0) = qR

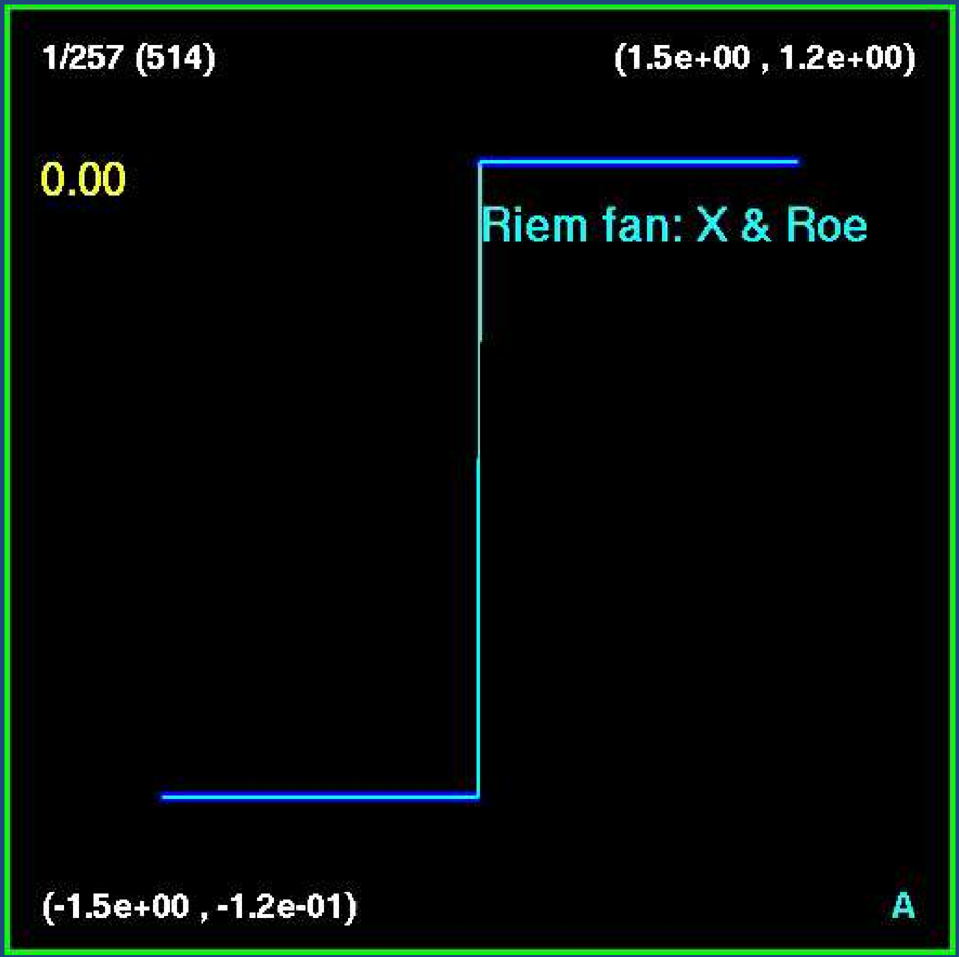
- IF, qL>qR, 'left' state moves onto 'right' state. *Shock* . Multivalued solution at x = st = (qL+qR)/2

- If, qL < qR, left state moves slower than right state. 'rarefraction' wave.

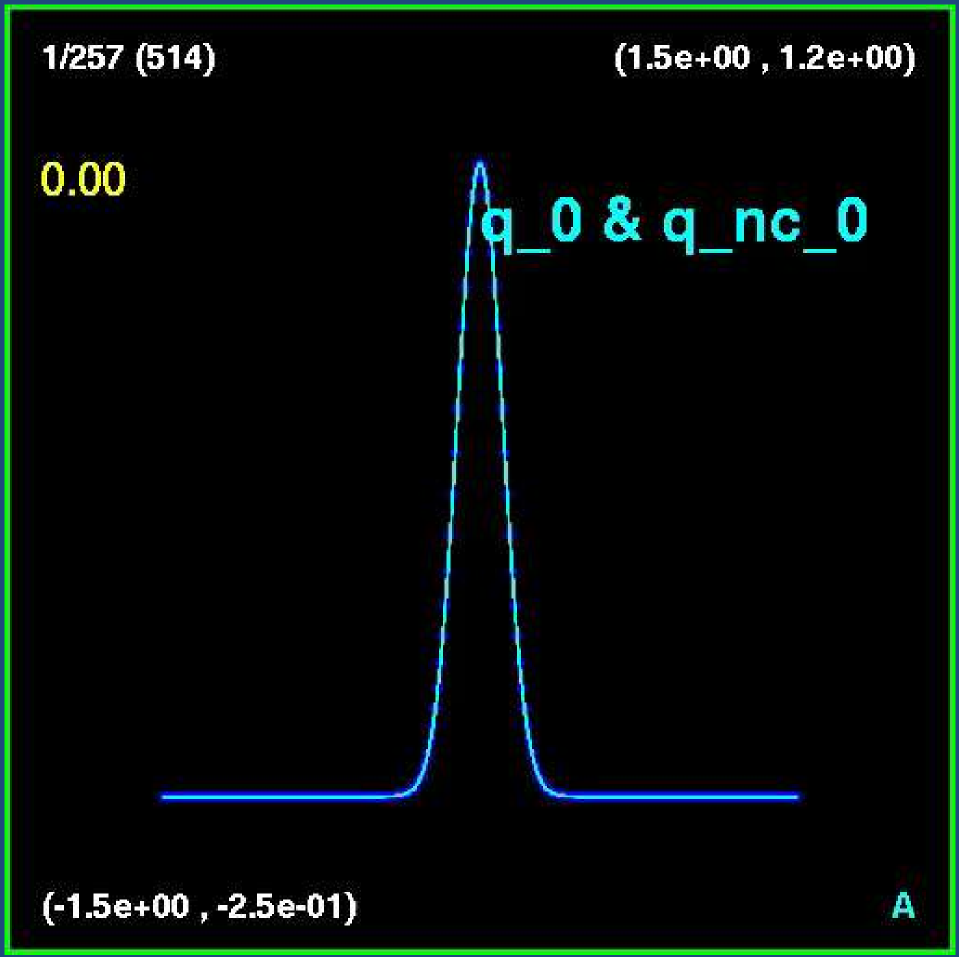
shock



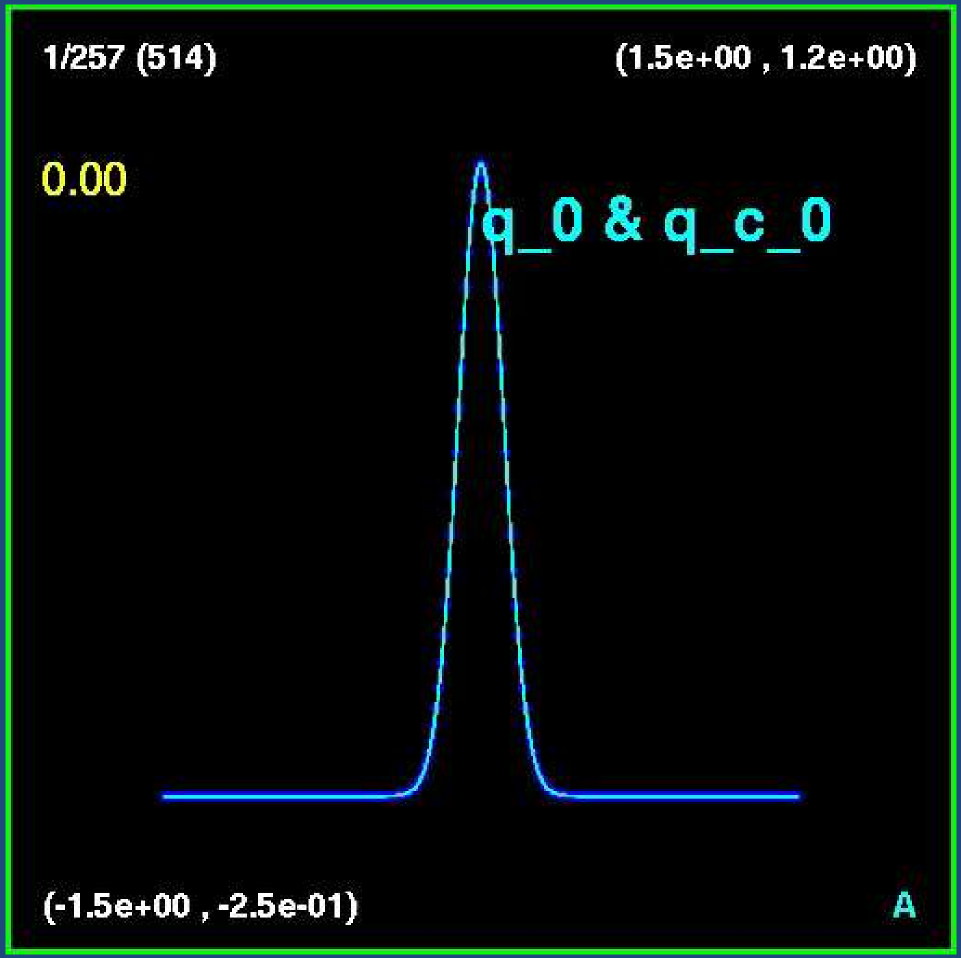
rarefraction



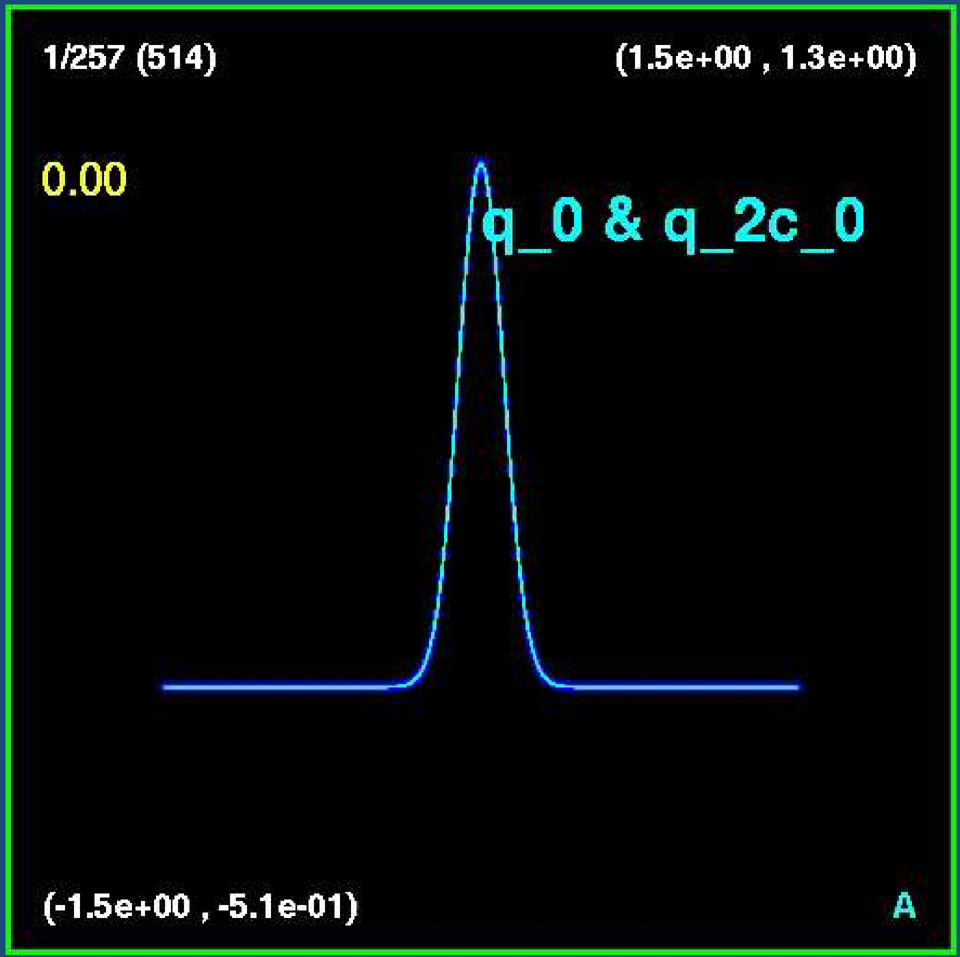
Gaussian profile. Upwind-non conservative



Gaussian profile. Upwind, conservative, 1st order



Gaussian profile, upwind, 2nd order



High resolution shock capturing methods

- Solve full problem as a series of Riemann problems between cells

- Enforce total variation diminishing, or 'essentially non-oscillatory' property from a step to the next → remove spurious under/overshoots

- Exploit characteristic structure

- Lots of options… check Leveque's book!