Quantum Gravity from the QFT perspective

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Partial support: CNPq, FAPEMIG

ICTP-SAIFR/IFT-UNESP – 1-5 April, 2019

Lecture 2.

Schwinger-DeWitt technique

One-loop divergences in curved spacetime.

Renormalization group.

**Form factors. High energy (UV) and low-energy (IR) limits.
Decoupling.**

Running of cosmological constant (CC) and Newton constant.

Schwinger-DeWitt technique

is the most useful method for practical 1-loop calculations.

Consider the typical form of the operator

$$\hat{\mathbf{H}} = \hat{1}\Box + \hat{\Pi} + \hat{1}m^2 .$$

It depends on the metric and maybe other external parameters (via $\hat{\Pi}$). The one-loop EA is given by the expression

$$\frac{i}{2}\,\mathrm{Tr}\,\ln\hat{\mathbf{H}} .$$

Let us perform variation with respect to the external parameters.

$$\frac{i}{2}\,\delta\,\mathrm{Tr}\,\ln\hat{\mathbf{H}} = \frac{i}{2}\,\mathrm{Tr}\,\hat{\mathbf{H}}^{-1}\,\delta\,\hat{\mathbf{H}} .$$

The Schwinger proper-time representation for the propagator

$$\hat{\mathbf{H}}^{-1} = \int_0^\infty ids\, e^{-is\,\hat{\mathbf{H}}} .$$

Then, we transform $\delta\,\hat{\mathbf{H}}\cdot\int_0^\infty ids\, e^{-is\,\hat{\mathbf{H}}} = \delta\int_0^\infty \frac{ds}{is}\, e^{-is\,\hat{\mathbf{H}}} .$

After all, we arrive at

$$\frac{i}{2}\,\mathrm{Tr}\,\log\hat{\mathbf{H}} = const - \frac{i}{2}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s}\,e^{-is\,\hat{\mathbf{H}}}\,,$$

where the constant term can be disregarded.

The next step is to introduce

$$\hat{U}(x,x';s) = e^{-is\,\hat{\mathbf{H}}}$$

$\hat{\mathbf{H}}$ **acts on the covariant** δ **-function and it proves useful to define**

$$\hat{U}_0(x,x';s) = \frac{\mathcal{D}^{1/2}(x,x')}{(4\pi i\,s)^{n/2}}\,\exp\left\{\frac{i\sigma(x,x')}{2s} - m^2 s\right\}.$$

$\sigma(x,x')$ **- geodesic distance between** x **and** x'**. It satisfies an identity** $2\sigma = (\nabla\sigma)^2 = \sigma^\mu\sigma_\mu$ **.**

$\mathcal{D}$ **is the Van Vleck-Morett determinant**

$$\mathcal{D}(x,x') = \det\left[-\frac{\partial^2\sigma(x,x')}{\partial x^\mu\,\partial x'^\nu}\right]\,,$$

which is a double tensor density, with respect to x **and** x'**.**

A useful representation for the evolution operator $\hat{U}(x,x';s)$ is

$$\hat{U}(x,x';s) \,=\, \hat{U}_0(x,x';s)\,\sum_{k=0}^{\infty}(is)^k\,\hat{a}_k(x,x')\,,$$

$\hat{a}_k(x,x')$ **are Schwinger-DeWitt coefficients.**

The evolution operator satisfies the equation

$$i\,\frac{\partial \hat{U}(x,x';s)}{\partial s} \,=\, -\,\hat{\mathbf{H}}\hat{U}(x,x';s)\,, \qquad U(x,x';0) = \delta(x,x')\,.$$

Using these relations one can construct the equation for the coefficients $\hat{a}_k(x,x')$:

$$\sigma^{\mu}\nabla_{\mu}\hat{a}_0 = 0\,,$$

$$(k+1)\hat{a}_{k+1}+\sigma^{\mu}\nabla_{\mu}\hat{a}_{k+1} \,=\, \Delta^{-1/2}\Box(\Delta^{1/2}\hat{a}_k)+\hat{\Pi}\hat{a}_k\,, \qquad k=1,2,3,\dots\,.$$

It is sufficient to know the coincidence limits

$$\lim_{x\to x'}\,\hat{a}_k(x,x')\,.$$

If we consider more general operator

$$S_2 \,=\, \hat{H} \,=\, \hat{1}\Box + 2\hat{h}^\mu \nabla_\mu + \hat{\Pi}\,,$$

the linear term can be indeed absorbed into the covariant derivative $\nabla_\mu \to \mathcal{D}_\mu = \nabla_\mu + \hat{h}_\mu$.

The commutator of the new covariant derivatives will be

$$\hat{S}_{\mu\nu} \,=\, \hat{R}_{\mu\nu} - (\nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu) - (\hat{h}_\nu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\nu)$$

and we arrive at

$$\hat{a}_1\Big| \,=\, \hat{a}_1(x,x) \,=\, \hat{P} \,=\, \hat{\Pi} + \frac{\hat{1}}{6} R - \nabla_\mu \hat{h}^\mu - \hat{h}_\mu \hat{h}^\mu\,.$$

and

$$\hat{a}_2\Big| \,=\, \hat{a}_2(x,x) \,=\, \frac{\hat{1}}{180}\,(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R)$$

$$+\,\frac{1}{2}\,\hat{P}^2 + \frac{1}{6}\,(\Box \hat{P}) + \frac{1}{12}\,\hat{S}^2_{\mu\nu}\,.$$

The great advantage of these expressions is their universality. They enable to analyze EA in various QFT models.

In 4-dimensional space-time $\hat{a}_2\Big|$ logarithmic divergences, while $\hat{a}_1\Big|$ defines quadratic divergences.

The derivation of the “magic” coefficient

$$a_2 \quad \equiv \quad \mathrm{Tr}\ \hat{a}_2\Big|$$

is, in many cases, the most important thing.

The divergent part of EA, in the dimensional regularization, is

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{\epsilon}\int d^n x\sqrt{-g}\ \mathrm{tr}\,\hat{a}_2(x,x)\,, \quad \text{where} \quad \epsilon = (4\pi)^2(n-4)\,.$$

The last formula is a very powerful tool for deriving the divergences in the models of field theory in flat and curved space-times or even in Quantum Gravity.

Sometimes it has to be modified, for example the sign gets changed for a fermionic case.

In complicated cases we need the generalized Schwinger-DeWitt technique (Barvinsky & Vilkovisky, 1985).

Further coefficients $\hat{a}_k,\ k \geq 3$ correspond to the finite part.

They are given by the expressions like

$$\frac{1}{m^2}\,\mathcal{O}(R^3)\,, \qquad \frac{1}{m^2}\,R_{\mu\nu}\Box R^{\mu\nu}\,,\ldots \qquad (a_3 \text{ case})$$

and therefore contribute only to the finite part of EA.

Practical calculation of the coefficients $\hat{a}_k,\ k \geq 3$ is possible, despite rather difficult.

The $\hat{a}_3$ coefficient has been derived by Gilkey (1979) and by Avramidy (1986), who also derived $\hat{a}_4$ coefficient.
In 1989-1990 *I. Avramidy* **and** *A. Barvinsky & G.V. Vilkovisky* **derived important resummation of the Schwinger-DeWitt series.**
As an important application one can obtain, for massive theories, the exact one-loop form factors of the terms

$$R^2\,, \quad C^2\,, \quad F_{\mu\nu}^2\,, \quad (\nabla\phi)^2\,, \quad \phi^4\,.$$

E.Gorbar, I.Sh., G.de Berredo-Peixoto, B.Gonçalves, JHEP (2003); CQG (2005); PRD (2009).

In the EA $\Gamma[\Phi, g_{\mu\nu}]$ one can separate the part $\Gamma[g_{\mu\nu}]$ which doesn't depend on matter fields.

It corresponds to the Feynman diagrams, with the internal lines of matter fields and the external lines of the metric only.

$\Gamma[g_{\mu\nu}]$ is called the EA of vacuum. It is the most important part of EA, as far as gravitational applications are concerned.

Path integral representation of the vacuum EA

$$e^{i\Gamma_{vac}[g_{\mu\nu}]} = \int d\Phi \; e^{iS[\Phi;\, g_{\mu\nu}]}\,.$$

Here Φ is the set of all matter fields and gauge ghosts.
Γ_{vac} admits a loop expansion, at the tree level it is equal to S_{vac}.

Already at this level one can make some strong statements about possible and impossible form of quantum corrections.

The effective action $\Gamma[g_{\mu\nu}]$ is a well-defined diffeomorphism invariant quantity constructed from the metric $g_{\mu\nu}$.

As a consequence $\Gamma[g_{\mu\nu}]$ can not include odd powers of the metric derivatives.

Let us emphasize that this property is not related to the perturbative expansion and is valid independent on whether the effective action is a local functional of the metric.

Indeed, it is nonlocal, except the divergences.

This important property of effective action holds for any particular metric, including the cosmological one.

Let us consider the simplest part, that is divergences.

Consider one-loop divergences for the free fields, scalars, spinors and massless vectors in curved space-time.

Scalar field. N_s**-component case.**

$$\hat{H} \,=\, \delta^i_j \,\left(\Box - m_s^2 - \xi R\,\right)_x \,, \quad \textbf{where} \quad i,j = 1,2,...,N_s.$$

The identification with the general expression

$$\hat{H} \,=\, \hat{1}\Box + 2\hat{h}^\mu \nabla_\mu + \hat{\Pi} \quad \textbf{gives} \quad \hat{h}^\mu = 0\,, \quad \hat{\Pi} = -\delta^i_j\,(m_s^2 + \xi R).$$

Then, $\hat{S}_{\mu\nu} = 0$ **and** $\hat{P} \,=\, \delta^i_j \,\left[\left(\xi - \frac{1}{6}\right) R - m_s^2\right]\,.$

Finally,

$$\bar{\Gamma}^{(1)}_{div} = -\frac{N_s\;\mu^{n-4}}{\epsilon}\int d^n x\sqrt{-g}\Big\{\frac{1}{2}\,m_s^4 \,+\, m_s^2\Big(\xi - \frac{1}{6}\Big)R$$

$$+\,\frac{1}{2}\,\Big(\xi{-}\frac{1}{6}\Big)^2 R^2 + \frac{1}{180}\Big(R^2_{\mu\nu\alpha\beta}{-}R^2_{\alpha\beta}\Big){-}\frac{1}{6}\Big(\xi{-}\frac{1}{5}\Big)\Box R\Big\}\,.$$

For a complex scalar field, the divergent part of the EA is twice of the previous result. This is nothing but the overall factor N_s.

In general, free fields give additional and independent contributions to the vacuum divergences.

In the $n=4$ conformal case $m_s=0,\ \xi=1/6$

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{360\epsilon}\int d^nx\sqrt{-g}\,\left\{3C^2 - E + 2\Box R\right\}.$$

Both classical action

$$S_0^c = \int d^4x\,\sqrt{-g}\left\{\,\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi + \frac{1}{12}\,R\varphi^2\,\right\}$$

and the log. divergence are conformal invariant

$$g_{\mu\nu}\ \longrightarrow\ g_{\mu\nu}\,e^{2\sigma(x)}\,,\qquad\quad \varphi\ \longrightarrow\ \varphi\,e^{-\sigma(x)}\,.$$

In the conformal scalar case the pole terms are conformal invariant or surface structures.

!! This result holds only in certain regularizations and may be violated in others.

Spinor field. **We meet another operator**

$$\hat{H} = i\,(\,\gamma^\alpha \nabla_\alpha - i\,m_f\,)\;.$$

The 1-loop EA is

$$\bar{\Gamma}^{(1)} = -\frac{i}{2}\,\mathrm{Tr}\,\log \hat{H}\,.$$

The sign change is due to the odd Grassmann parity of the fermion field, while Tr is taken in the usual "bosonic" way.

After some algebra we arrive at the following expression

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{\epsilon}\int d^n x\sqrt{-g}\,\Big\{\,\frac{m_f^2}{3}R - 2m_f^4 + \frac{1}{20}C^2(4) - \frac{11}{180}E + \frac{1}{30}\Box R\Big\}\,.$$

Again, in the conformal case $m_f = 0$ we meet only the conformal-invariant counterterms.

Vector field

In the massless case we do not need to distinguish Abelian and non-Abelian vectors, since only the free parts are important.

Consider a single Abelian vector. The action must be supplemented by the gauge fixing and ghost terms.

The 1-loop contribution to the vacuum EA

$$\bar{\Gamma}^{(1)} = \frac{i}{2}\,\mathrm{Tr}\,\log\hat{H} - i\,\mathrm{Tr}\,\log\hat{H}_{gh}\,,$$

$\hat{H}$ **and** $\hat{H}_{gh}$ **are bilinear forms of the field and ghost actions.**

The divergent part is

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{180\epsilon}\int d^4x\sqrt{-g}\,\left\{18(C^2 - \Box R) - 31\,E\right\}.$$

The divergences include conformal-invariant and surface terms.

Renormalization group equations

Renormalization group (RG) is one of the most efficient methods of Quantum Field Theory, also in Stat. Mechanics.

In QFT there are many versions of RG

- Perturbative RG based on the minimal subtraction scheme of renormalization ($\overline{\mathrm{MS}}$).

- Perturbative RG which is based on a more physical, e.g., momentum subtraction scheme of renormalization.

- Non-Perturbative RG based on the path integral integration over momenta beyond some cut-off (Wilson approach).

- Intermediate approach with the cut-off dependence for the Green functions by Polchinsky.

- The same in the EA formalism, by Wetterich, Morris, Percacci, Reuter et al.

- **Consider the standard $\overline{\text{MS}}$-based formalism of RG in curved space. Let us denote Φ the full set of matter fields**

$$\Phi = \varphi,\, \psi,\, A$$

and P the full set of parameters: couplings, masses, ξ and vacuum parameters.

The bare action $S_0[\Phi_0, P_0]$ depends on bare quantities, $S[\Phi, P]$ is the renormalized action.

Multiplicative renormalizability:

$$S_0[\Phi_0, P_0] \,=\, S[\Phi, P]\,,$$

(Φ_0, P_0) **and** (Φ, P) **are related by proper renormalization transformation. The generating functionals of the bare and renormalized Green functions are**

$$e^{iW_0[J_0]} \,=\, \int d\Phi_0\, e^{i(S_0[\Phi_0,P_0]+\Phi_0\cdot J_0)}\,,$$

$$e^{iW[J]} \,=\, \int d\Phi\, e^{i(S[\Phi,P]+\Phi\cdot J)}\,.$$

The transformation for matter fields is

$$\Phi_0 = \mu^{\frac{n-4}{2}} Z_1^{1/2} \Phi\,.$$

Make this change of variables and denote

$$J_0 = \mu^{\frac{4-n}{2}} Z_1^{-1/2} J\,.$$

Then

$$W_0[J_0] \,=\, W[J]\,.$$

Consequently, for the mean field we meet

$$\bar{\Phi}_0 \,=\, \frac{\delta W[J_0]}{\delta J_0} \,=\, \frac{\delta W[J]}{\delta J}\frac{\delta J}{\delta J_0} \,=\, \mu^{\frac{n-4}{2}} Z_1^{1/2} \bar{\Phi}\,.$$

Finally, for the effective action we find

$$\Gamma_0[\Phi_0, P_0] \,=\, W_0[J_0] - \bar{\Phi}_0 \cdot J_0 \,=\, W[J] - \bar{\Phi} \cdot J \,=\, \Gamma[\Phi, P]\,.$$

S_0 **and** Γ_0 **are** 4 **- dimensional integrals,**
while S **and** Γ **are** n **- dimensional integrals.**

Γ **depends on the dimensional parameter** μ**, while** Γ_0 **does not depend on** μ **by construction.**

Therefore,

$$\Gamma_0[g_{\alpha\beta}, \Phi_0, P_0, 4] \,=\, \Gamma[g_{\alpha\beta}, \Phi, P, n, \mu]\,,$$

and we arrive at the differential equation

$$\mu\frac{d}{d\mu}\,\Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] \,=\, 0\,.$$

Taking into account the possible μ**- dependence of** P **and** Φ **we recast this equation into**

$$\Big\{\,\mu\frac{\partial}{\partial\mu} + \mu\frac{dP}{d\mu}\,\frac{\partial}{\partial P} + \int d^n x\,\mu\frac{d\Phi(x)}{d\mu}\,\frac{\delta}{\delta\Phi(x)}\,\Big\}\,\Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] \,=\, 0\,.$$

We define, as in flat space-time

$$\beta_P(n) = \mu \frac{dP}{d\mu}\,, \qquad \beta_P(4) = \beta_P$$

$$\gamma_\Phi(n) = \mu \frac{d\Phi}{d\mu}\,, \qquad \gamma_\Phi(4) = \gamma_\Phi\,.$$

Then, the RG equation is cast in the form

$$\left\{ \mu \frac{\partial}{\partial\mu} + \int_{x,n} \gamma_\Phi(n)\, \frac{\delta}{\delta\Phi} + \beta_P(n)\, \frac{\partial}{\partial P} \right\} \Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] = 0\,.$$

This is the general RG equation which can be used for different purposes, depending on the physical interpretation of μ.

Here $\int_{x,n} = \int d^n x \sqrt{-g}$ **and** $\int_x = \int_{x,4}\,.$

An example of RG equation.

The divergent part of the EA of vacuum for the theory with N_s scalars, N_f spinors and N_v vectors

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{n-4}\int d^n x\sqrt{-g}\,\{\beta_{EH} R + \beta_{CC} + \beta_W C^2 + \beta_E E + \beta_{R2} R^2 + \beta_d \Box R\}\,,$$

where $\beta_i = \frac{k_i}{(4\pi)^2}$ **and** $k_{CC} = \frac{1}{2} m_s^4 - 4m_f^4\,,$

$$k_{EH} = N_s m_s^2\Big(\xi - \frac{1}{6}\Big) + \frac{2N_f m_f^2}{3}\,, \qquad k_{R2} = \frac{N_s}{2}\Big(\xi - \frac{1}{6}\Big)^2\,,$$

$$w = k_W = \frac{N_s}{120} + \frac{N_f}{20} + \frac{N_v}{10}\,,$$

$$b = k_E = -\frac{N_s}{360} - \frac{11\,N_f}{360} - \frac{31\,N_v}{180}\,,$$

$$c = k_\Box = \frac{N_s}{180} + \frac{N_f}{30} - \frac{N_v}{10}\,.$$

Consider the Weyl-squared term.

$$\Delta S_W = \frac{\mu^{n-4}}{\epsilon}\int d^nx\sqrt{-g}\,wC^2\,, \quad w = \frac{N_s}{120} + \frac{N_f}{20} + \frac{N_v}{10}\,.$$

Renormalized action = to the bare one, $S_W(n) + \Delta S_W = S_W^0\,.$

Obviously, this means $a_1^0 = \mu^{n-4}\left(a_1 + \frac{w}{\epsilon}\right)\,.$ Taking

$$0 = \mu\,\frac{da_1^0}{d\mu} = \mu^{n-4}\left[(n-4)\left(a_1 + \frac{w}{\epsilon}\right) + \mu\,\frac{da_1}{d\mu}\right]$$

In this way we arrive at $\mu\,\frac{da_1}{d\mu} = -(n-4)a_1 - \frac{w}{(4\pi)^2}\,.$

or $\beta_W \,=\, \mu\,\frac{da_1}{d\mu}\Big|_{n=4} \,=\, -\,\frac{w}{(4\pi)^2}\,.$

For the coupling parameter $\lambda = -(2a_1)^{-1}$ **we have**

$$\mu\,\frac{d\lambda}{d\mu} \,=\, -\,\frac{w}{2\,(4\pi)^2}\,\lambda^2\,,$$

indicating asymptotic freedom, since in all cases $w > 0$.

In a similar way one can derive RG equations for $a_{2,3,4}$ and also for Λ and G , namely

$$\frac{da_3}{dt} = \mu\,\frac{da_3}{d\mu} = \frac{N_s}{2\,(4\pi)^2}\,\left(\xi - \frac{1}{6}\right)^2 ,$$

$$(4\pi)^2\,\frac{d}{dt}\,\left(\frac{\Lambda}{8\pi G}\right) = \frac{N_s m_s^4}{2} - 2N_f m_f^4 .$$

$$(4\pi)^2\,\mu\,\frac{d}{d\mu}\,\left(\frac{1}{16\pi G}\right) = \frac{N_s m_s^2}{2}\,\left(\xi - \frac{1}{6}\right) + \frac{N_f m_f^2}{3} .$$

These equations describe the short distance behavior of the corresponding effective charges.

However, it is not really clear how to apply them, e.g., to cosmology or to the black hole physics.

Short distance limit.

Perform a global rescaling of quantities according to their dimension

$$\Phi \to \Phi k^{-d_\Phi}, \quad P \to P k^{-d_P}, \quad \mu \to k\mu, \quad l \to k^{-1} l.$$

The effective action Γ does not change.

Since Γ does not depend on x^μ explicitly, one can replace $l \to l \times k^{-1}$ by the transformation of the metric $g_{\mu\nu} \to k^2 g_{\mu\nu}$.

Then, in addition to RG, we meet an identity

$$\Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] = \Gamma[k^2 g_{\alpha\beta}, k^{-d_\Phi}\Phi, k^{-d_P} P, n, k^{-1}\mu]\,,$$

whereas the curvatures transform as

$$R^2_{\mu\nu\alpha\beta} \sim k^{-4}, \quad R^2_{\alpha\beta} \sim k^{-4}, \quad R \sim k^{-4}\,.$$

Replace $k = e^{-t}$,

$$\frac{d}{dt}\,\Gamma[e^{2t}g_{\alpha\beta}, e^{-d_\Phi t}\Phi, e^{d_P t}P, n, e^{-t}\mu] = 0\,.$$

For $t = 0$ **we meet**

$$\Big\{\int d^nx\,\Big(2g_{\alpha\beta}\frac{\delta}{\delta\, g_{\alpha\beta}} - d_\Phi\,\frac{\delta}{\delta\Phi}\Big) - d_P\,\frac{\partial}{\partial P} - \mu\frac{\partial}{\partial\mu}\Big\}\Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] = 0.$$

Together with the RG equation it gives the solution

$$\Gamma[g_{\alpha\beta}e^{-2t}, \Phi, P, n, \mu] \;=\; \Gamma[g_{\alpha\beta}, \Phi(t), P(t), n, \mu]\,,$$

where $P(t)$ **and** $\Phi(t)$ **satisfy RG equations for "effective charges"**

$$\frac{d\Phi}{dt} \;=\; (\gamma_\Phi - d_\Phi)\Phi\,, \qquad\qquad \frac{dP}{dt} \;=\; \beta_P - Pd_P\,.$$

The limit $t \to \infty$ means, the limit of short distances and great curvatures.

It is equivalent to the standard rescaling of momenta in the flat-space QFT.

However, one has to be careful!

The time-dependence of the metric is very similar to the rescaling (we denote time as τ in order to avoid confusion)

$$g_{\alpha\beta} \to g_{\alpha\beta} \cdot e^{H\tau}, \qquad \text{where} \qquad H = const\,.$$

However, this situation does not correspond to the RG, because scalar curvature remains constant $R = -12H^2$.

For the most interesting physical applications we need some special scale-setting procedure, to associate μ with some physically relevant quantity (lecture IV - seminar).

What are terms in the EA which are behind the RG?

Consider the simplest case of QED and the one-loop diagram with electron loop and two photon tails (polarization operator).

Qualitatively, the effective action with the non-local form factor is similar to the expression

$$\Gamma^{(1)} \;\sim\; -\frac{1}{4e^2}\int d^4x\, F_{\alpha\beta}\Big\{1 \,+\, \beta\ln\Big(\frac{\Box - m_e^2}{\mu^2}\Big)\Big\}F^{\alpha\beta},$$

where β is a usual Minimal Subtraction beta function.

It is easy to see that the behavior of this form factor is very different is the UV and in the IR. In momentum Euclidean representation:

$$k^2 \gg m_e^2 \quad\Longrightarrow\quad \ln\Big(\frac{\Box - m_e^2}{\mu^2}\Big) \;\approx\; \ln\Big(\frac{k^2}{\mu^2}\Big),$$

$$k^2 \ll m_e^2 \quad\Longrightarrow\quad \ln\Big(\frac{\Box - m_e^2}{\mu^2}\Big) \;\approx\; \ln\Big(\frac{m_e^2}{\mu^2}\Big) \,+\, \mathcal{O}\Big(\frac{k^2}{\mu^2}\Big).$$

Decoupling at the classical level.

Consider propagator of massive field at very low energy

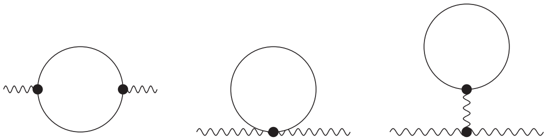
$$\frac{1}{k^2+m^2} = \frac{1}{m^2}\left(1 - \frac{k^2}{m^2} + \frac{k^4}{m^4} + \ldots\right).$$

In case of $k^2 \ll m^2$ there is no propagation of particle.

What about quantum theory, loop corrections?

Formally, in loops integration goes over all values of momenta.

Is it true that the effects of heavy fields always become irrelevant at low energies?



For simplicity, consider a fermion loop effect in QED.

In the UV, the mass of quantum fermion is negligible, this simplifies the form factor, and we arrive at

$$\tilde{\beta}\, F^{\mu\nu}\, \ln\Big(\frac{\Box}{\mu^2}\Big)\, F_{\mu\nu}\,.$$

The momentum-subtraction β-function

$$\beta_e^1 \,=\, \frac{e^3}{6a^3\,(4\pi)^2}\,\Big[20a^3 - 48a + 3(a^2-4)^2\,\ln\Big(\frac{2+a}{2-a}\Big)\Big]\,,$$

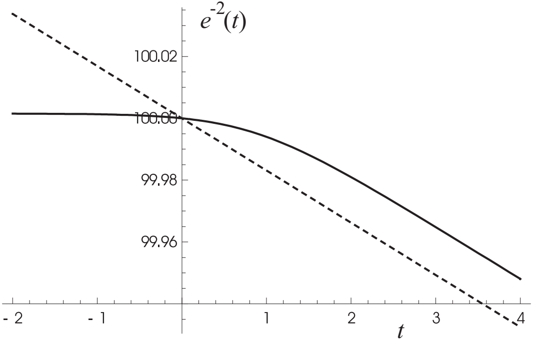
$$a^2 = \frac{4\Box}{\Box - 4m^2}\,.$$ **Special cases:**

UV limit $$p^2 \gg m^2 \;\Longrightarrow\; \beta_e^{1\ UV} \,=\, \frac{4\,e^3}{3\,(4\pi)^2} + \mathcal{O}\Big(\frac{m^2}{p^2}\Big)\,.$$

IR limit $$p^2 \ll m^2 \;\Longrightarrow\; \beta_e^{1\ IR} \,=\, \frac{e^3}{(4\pi)^2}\cdot\frac{4\,p^2}{15\,m^2} \,+\, \mathcal{O}\Big(\frac{p^4}{m^4}\Big)\,.$$

This is the standard form of the Appelquist and Carazzone decoupling theorem (PRD, 1977).

One can obtain the general expression which interpolates between the UV and IR limits.



These plots show the effective electron charge as a function of $\log(\mu/\mu_0)$ in the case of the MS-scheme, and for the momentum-subtraction scheme, with $\ln(p/\mu_0)$.

An interesting high-energy effect is a small apparent shift of the initial value of the effective charge.

Scalar contributions.

An example of finite (nonlocal) corrections (factor $1/64\pi^2$)

$$\mathcal{L}_{eff} \,=\, C_{\mu\nu\alpha\beta}\left[\frac{1}{60\epsilon} \,+\, \frac{8Y}{15a^4} + \frac{2}{45a^2} + \frac{1}{150}\right] C^{\mu\nu\alpha\beta}$$

$$+\lambda\phi^2\left[\frac{Y(a^2-4)}{12a^2} - \frac{1}{36} - \left(\frac{1}{2\epsilon} - Y\right)\left(\xi - \frac{1}{6}\right)\right] R \,+\, \dots,$$

where $$\frac{1}{\epsilon} = \frac{1}{2-\omega} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) - \gamma\,,$$

$$Y = 1 - \frac{1}{a}\ln\left(\frac{2+a}{2-a}\right)\,, \qquad a^2 = \frac{4\Box}{\Box - 4m^2}\,.$$

One can get a full form of the Appelquist and Carazzone theorem for gravity out of these expressions.

Gorbar & Sh. JHEP, 2003, hep-ph/0210388, 0303124; 0311190; Gorbar, Berredo-Peixoto & Sh. 2005, and others.

In the gravitational sector we meet Appelquist and Carazzone - like decoupling, but only in the higher derivative sectors.
In the perturbative approach, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we do not see running for the cosmological and inverse Newton constants.
Why do we get $\beta_\Lambda = \beta_{1/G} = 0$?

Momentum subtraction running corresponds to the insertion of, e.g., $\ln(\Box/\mu^2)$ formfactors into effective action.

Say, in QED:
$$-\frac{e^2}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^4}{3(4\pi)^2}\,F_{\mu\nu}\,\ln\Big(-\frac{\Box}{\mu^2}\Big)\,F^{\mu\nu}\,.$$

Similarly, one can insert formfactors into
$$C_{\mu\nu\alpha\beta}\,\ln\Big(-\frac{\Box}{\mu^2}\Big)\,C_{\mu\nu\alpha\beta}\,.$$

However, such insertion is impossible for Λ and for $1/G$, because $\Box\Lambda \equiv 0$ and $\Box R$ is a full derivative.

Further discussion:
Ed. Gorbar & I.Sh., JHEP (2003); J. Solà & I.Sh., PLB (2010).

Is it true that physical $\beta_\Lambda = \beta_{1/G} = 0$ **?**

Probably not. Perhaps the linearized gravity approach is simply not an appropriate tool for the CC and Einstein terms.

Let us use the covariance arguments. The EA can not include odd terms in metric derivatives. In the cosmological setting this means no $\mathcal{O}(H)$ **and also no** $\mathcal{O}(H^3)$ **terms, etc. Hence**

$$\rho_\Lambda(H) = \frac{\Lambda(H)}{16\pi G(H)} = \rho_\Lambda(H_0) + \frac{3\nu}{8\pi}\left(H^2 - H_0^2\right), \qquad \nu = const\,.$$

Then the conservation law for $G(H;\nu)$ **gives**

$$G(H;\nu) = \frac{G_0}{1 + \nu\,\ln\left(H^2/H_0^2\right)}, \quad \textbf{where} \quad G(H_0) = G_0 = \frac{1}{M_P^2}\,.$$

Here we used the identification

$$\mu \sim H \qquad \textbf{in the cosmological setting.}$$

The same $\rho_\Lambda(\mu)$ follows from the assumption of the Appelquist and Carazzone - like decoupling for CC.

A.Babic, B.Guberina, R.Horvat, H.Štefančić, PRD 65 (2002); I.Sh., J.Solà, C.España-Bonet, P.Ruiz-Lapuente, PLB 574 (2003).

We know that for a single particle

$$\beta_\Lambda^{MS}(m) \sim m^4 \,,$$

hence the quadratic decoupling gives

$$\beta_\Lambda^{IR}(m) = \frac{\mu^2}{m^2}\,\beta_\Lambda^{MS}(m) \sim \mu^2 m^2 \,.$$

The total beta-function will be given by algebraic sum

$$\beta_\Lambda^{IR} = \sum k_i \mu^2 m_i^2 = \sigma M^2\,\mu^2 \propto \frac{3\nu}{8\pi}\,M_P^2\,H^2 \,.$$

This leads to the same result in the cosmological setting,

$$\rho_\Lambda(H) = \rho_\Lambda(H_0) + \frac{3\nu}{8\pi}\,M_p^2\left(H^2 - H_0^2\right).$$

One can obtain the same $G(\mu)$ in one more independent way.

I.Sh., J. Solà, JHEP (2002); C. Farina, I.Sh. et al, PRD (2011).

Consider $\overline{\mathrm{MS}}$-based renormalization group equation for $G(\mu)$:

$$\mu \frac{dG^{-1}}{d\mu} = \sum_{particles} A_{ij}\, m_i\, m_j = 2\nu\, M_P^2\,, \qquad G^{-1}(\mu_0) = G_0^{-1} = M_P^2\,.$$

Here the coefficients A_{ij} depend on the coupling constants, m_i are masses of all particles. In particular, at one loop,

$$\sum_{particles} A_{ij}\, m_i\, m_j = \sum_{fermions} \frac{m_f^2}{3(4\pi)^2} - \sum_{scalars} \frac{m_s^2}{(4\pi)^2}\Big(\xi_s - \frac{1}{6}\Big)\,.$$

One can rewrite it as

$$\mu \frac{d(G/G_0)}{d\mu} = -2\nu\,(G/G_0)^2 \quad \Longrightarrow \quad G(\mu) = \frac{G_0}{1+\nu\,\ln\big(\mu^2/\mu_0^2\big)}\,. \qquad (*)$$

It is the same formula which results from covariance and/or from AC-like quadratic decoupling for the CC plus conservation law. $(*)$ seems to be a unique possible form of a relevant $G(\mu)$.

All in all, it is not a surprise that the eq.

$$G(\mu) = \frac{G_0}{1 + \nu \ln\left(\mu^2/\mu_0^2\right)}\,.$$

emerges in different approaches to renorm. group in gravity:

- **Higher derivative quantum gravity.**

A. Salam & J. Strathdee, PRD (1978);
E.S. Fradkin & A. Tseytlin, NPB (1982).

- **Non-perturbative quantum gravity with (hipothetic) UV-stable fixed point.**

A. Bonanno & M. Reuter, PRD (2002).

- **Semiclassical gravity.**

B.L. Nelson & P. Panangaden, PRD (1982).

So, we arrived at the two relations:

$$\rho_\Lambda(H) = \rho_\Lambda(H_0) + \frac{3\nu}{8\pi} M_p^2 \left(\mu^2 - \mu_0^2\right) \tag{1}$$

and

$$G(\mu) = \frac{G_0}{1 + \nu \, \ln\left(\mu^2/\mu_0^2\right)} \,. \tag{2}$$

Remember the standard identification

$\mu \sim H$ **in the cosmological setting.**

A. Babic, B. Guberina, R. Horvat, H. Štefančić, PRD (2005).

Cosmological models based on the assumption of the standard AC-like decoupling for the cosmological constant:

- **Models with (1) and energy matter-vacuum exchange:**

I.Sh., J.Solà, Nucl.Phys. (PS), IRGA-2003;
I.Sh., J.Solà, C.España-Bonet, P.Ruiz-Lapuente, PLB (2003).

- - **Models with (1), (2) and without matter-vacuum exchange:**

I.Sh., J.Solà, H.Štefančić, JCAP (2005).

- **Models with constant $G \equiv G_0$ and permitted energy exchange between vacuum and matter sectors.**

For the equation of state $P = \alpha\rho$ the solution is analytical,

$$\rho(z\,;\nu) \;=\; \rho_0\,(1+z)^r\,,$$

$$\rho^\Lambda(z\,;\nu) \;=\; \rho_0^\Lambda \,+\, \frac{\nu}{1-\nu}\,\left[\,\rho(z\,;\nu) - \rho_0\,\right]\,,$$

The limits from density perturbations / LSS data: $|\nu| < 10^{-6}$.

Analog models:

Opher & Pelinson, PRD (2004); Wang & Meng, Cl.Q.Gr. 22 (2005).

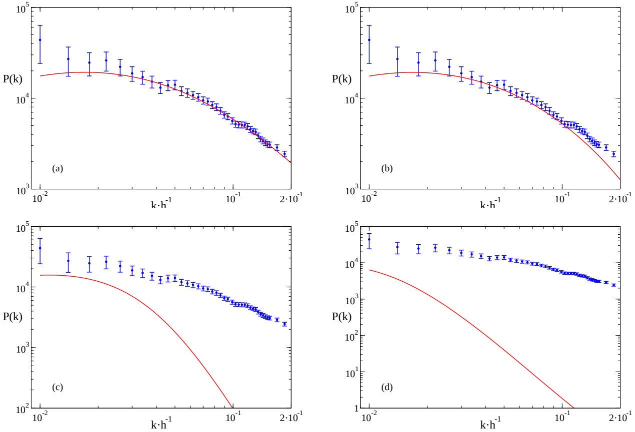
Direct analysis of cosmic perturbations:

J. Fabris, I.Sh., J. Solà, JCAP 0702 (2007).

Given the Harrison-Zeldovich initial spectrum, the power spectrum today can be obtained by integrating the eqs. for perturbations.

Initial data based on $w(z)$ **from** *J.M. Bardeen et al, Astr.J. (1986).*

Results of numerical analysis for the ● model:



The ν-dependent power spectrum vs the LSS data from the 2dfFGRS. The ordinate axis represents $P(k) = |\delta_m(k)|^2$ where $\delta_m(k)$ is the solution at $z = 0$. $\quad \nu = 10^{-8},\ 10^{-6},\ 10^{-4},\ 10^{-3}$. In all cases $\Omega^0_B, \Omega^0_{DM}, \Omega^0_\Lambda = 0.04, 0.21, 0.75$.

●● **Models with variable $G = G(H)$ but without energy exchange between vacuum and matter sectors.**

Theoretically this looks much better!

$$\rho_\Lambda(H) \,=\, \rho_\Lambda(H_0) \,+\, \frac{3\nu}{8\pi}\, M_p^2 \left(H^2 - H_0^2\right).$$

By using the energy-momentum tensor conservation we find

$$G(H;\nu) = \frac{G_0}{1 + \nu\, \ln\left(H^2/H_0^2\right)}\,, \quad \textbf{where} \quad G(H_0) = \frac{1}{M_P^2}\,.$$

These relations exactly correspond to the RG approach discussed above, with $\mu = H$.

I.Sh., J.Solà, H.Štefančić, JCAP (2005).

General situation

We can not prove that there is a relevant IR running of cosmological and/or Newton constants.

And we can not disprove it too.

We can instead use phenomenological approach and try to check what can be the consequences of such a running at the level of the universe or in the astrophysical domain.

D. Rodrigues, P. Letelier & I.Sh., JCAP (2010).

Another possibility is to model such a running with some covariant terms, e.g.,

$$R\,\frac{1}{\Box^2}\,R\,,\qquad R_{\alpha\beta}\,\frac{1}{\Box^2}\,R^{\alpha\beta}\,,\qquad R\,\frac{1}{\Box}\,R\,,\qquad R_{\alpha\beta}\,\frac{1}{\Box}\,R^{\alpha\beta}\,.$$

Gorbar & Sh. JHEP, 2003, hep-ph/0210388.
and many other papers after that.

Conclusions

- **The renormalization program is a full success of we are interesting in getting free of divergences.**

- **Perturbative Renormalization Group is formulated without difficulties within Minimal Subtraction scheme.**

- **Unfortunately the problems start right at the point when we need to calculate finite part of EA. For, example, there is no unique interpretation of μ or $t = \ln(\mu/\mu_0)$ for the case of inflation and, in fact, in many other cases.**

- **The question of whether CC can be variable is, to some extent, reduced to existing-nonexisting paradigm.**

Exercises and references

1. About the Schwinger-DeWitt technique the useful reading is

Refs.: [1.1] B.S. DeWitt, *Dynamical Theory of Groups and Fields,* (Gordon and Breach, 1965).
[1.2] B.S. DeWitt, *The Global Approach to Quantum Field Theory,* (Clarendon Press, Oxford. Vol. 2 - 2003).
[1.3] I.G. Avramidi, hep-th-9410140; *Heat kernel and quantum gravity.* (Springer-Verlag, 2000).
[1.4] D.V. Vassilevich, Phys. Rept. 388 (2003) 279, hep-th/0306138.
[1.5] A.O. Barvinsky and G.A. Vilkovisky, *The generalized Schwinger-DeWitt technique in gauge theories and quantum gravity,* Phys. Rep. 119 (1985) 1.

2. Many examples of using Schwinger-DeWitt technique, both basic and advanced, can be found in

[2.1] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective Action in Quantum Gravity.* (IOP Publishing, 1992). [BOS].

3. Definition of renormalization group equations in curved space. The use of renormalization group equations for deriving effective potential and other sectors of effective action.

Refs.: [3.1] BOS.
[3.2] I.L. Buchbinder, On Renormalization Group Equations in Curved Space-Time. Theor. Math. Phys. 61 (1984) 393.

4. Discuss the renormalization group equations for the effective action. Does parameter μ have physical sense? How can we use μ?

Refs.: [4.1] arXiv:0910.4925.
[4.2] BOS, arXive: 1107.2262.

5. Write down the renormalization-group corrected classical action. Use the fact that the overall dependence on μ should cancel. What changes when we go from flat to curved space?