Power Law Inflation $\qquad V = M^4 \exp(-\varphi/\Lambda)$

$$\begin{cases} \ddot{\varphi} + 3H\dot{\varphi} + V' = 0 & (1) \\ 3H^2 = 8\pi G\left(\frac{1}{2}\dot{\varphi}^2 + V\right) & (2) \end{cases}$$

Ansatz: $\quad a(t) = a_0 \left(\frac{t}{t_0}\right)^p$

$$\varphi(t) = \varphi_0 \ln\left(\frac{t}{t_0}\right)$$

Plug in to (2):

$$3\frac{p^2}{t^2} = 8\pi G\left(\frac{1}{2}\frac{\varphi_0^2}{t^2} + M^4\left(\frac{t}{t_0}\right)^{-\varphi_0/\Lambda}\right)$$

Match powers of t: $\quad \varphi_0/\Lambda = 2$

$$\varphi_0 = 2\Lambda$$

Apply to (1) & (2)

$$\begin{cases} -2\Lambda + 6p\Lambda - M^4 t_0^2/\Lambda = 0 & (1) \\ 3p^2 = 8\pi G\left(\frac{1}{2}\varphi_0^2 + M^4 t_0^2\right) & (2) \end{cases}$$

$\Downarrow$

solve for p, t_0: $\quad p = 2K\Lambda^2, \quad t_0 = \frac{\sqrt{2}\,\Lambda}{M^2}\sqrt{6K\Lambda^2 - 1}$

where $K = 8\pi G$, $G = 1/M_p^2$.

Manipulate to show $\quad \epsilon = \frac{-\dot{H}}{H^2} = 1/16\pi G\Lambda^2$ ✓

and $\delta = \ddot{H}/2H\dot{H} = -\epsilon$

and finally $\quad \varphi(t) = 2\Lambda \ln\left(\frac{8\pi\epsilon^2 M^4 t^2}{(3-\epsilon)M_p^2}\right)$ ✓

POWER LAW INFLATION

$$R'' + 2\mathcal{H}(1+\epsilon+\delta)R' - \nabla^2 R = 0$$

$$\epsilon+\delta = 0$$

$$a(\tau) = a_I\,(\tau_I/\tau)^{\frac{1}{1-\epsilon}}$$

$$\mathcal{H} = \frac{-1}{\tau(1-\epsilon)} \qquad (\tau<0)$$

$$R_k'' + 2\mathcal{H}R_k' + k^2 R_k = 0$$

$$R_k = A(k)\,H(\tau)\,(-k\tau)^{3/2}\,H_\nu^{(1)}(-k\tau)$$
$$\quad + B(k)\,H(\tau)\,(-k\tau)^{3/2}\,H_\nu^{(2)}(-k\tau)$$

where $\nu = \frac{3}{2} + \frac{\epsilon}{1-\epsilon}$.

Since $x^{3/2}H_\nu^{(1)}(x) \propto e^{ix}\,x$ for $x \to \infty$ then to identify our solution with pure positive frequency at large-k/small-scales we set $B \to 0$.

$$R_k(\tau) = A(k)\,H(\tau)\,(-k\tau)^{3/2}\,H_\nu^{(1)}(k\tau).$$

QFT of R_k yields Wronskian condition

$$R_k R_k^{*\prime} - R_k^* R_k' = i\left(\frac{H}{\varphi'}\right)^2$$

Since $H = \frac{1}{\epsilon t} = \frac{-1}{\tau(1-\epsilon)\, a(\tau)}$

and $\varphi(t) = \Lambda \ln\left(\frac{8\pi\epsilon^2 M^4 t^2}{(3-\epsilon) M_p^2}\right)$

$$\left(\frac{H}{\varphi'}\right)^2 = 4\pi \, \frac{2\nu-1}{2\nu-3} \, \frac{(\tau/\tau_I)^{2\nu-1}}{a_I^{\,2} M_p^{\,2}}$$

which leads to $|A(k)|^2 = \left(\frac{2\pi}{M_p}\right)^2 k^{-3} / (2\nu-1)(2\nu-3)$

so $A = \frac{2\pi}{M_p} k^{-3/2} \left((2\nu-1)(2\nu-3)\right)^{-1/2} e^{i\theta}$

a phase

$$R_k(\tau) = \frac{2\pi}{M_p} \cdot \frac{1-\epsilon}{2\sqrt{\epsilon}} \cdot e^{i\theta} \, \frac{H(\tau)(-k\tau)^{3/2}}{k^{3/2}} \, H_\nu^{(1)}(-k\tau) \quad ✓$$