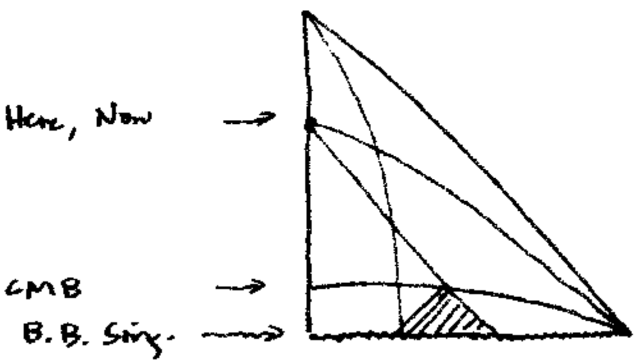
INFLATION

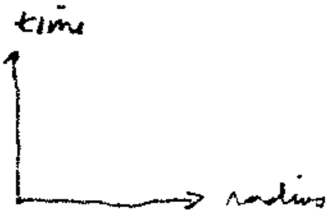
An epoch of exponential expansion
which solves horizon problem, etc
generates fluctuations of matter, metric.

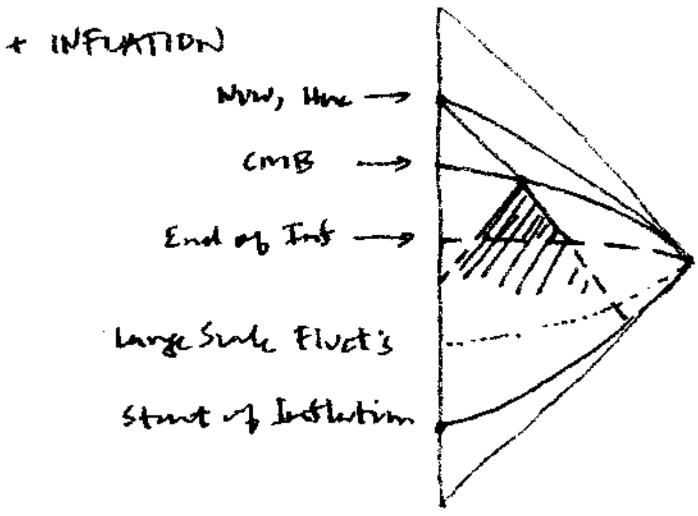
Sets the initial conditions for the HOT Big Bang

STD FRW spacetime $\quad ds^2 = a(\tau)^2\left[-d\tau^2 + d\vec{x}^2\right], \quad \tau > 0$

$a \propto \tau, \tau^2$







No inflation FRW: $ds^2 = a^2(\tau)\left[-d\tau^2 + d\vec{x}^{\,2}\right]$

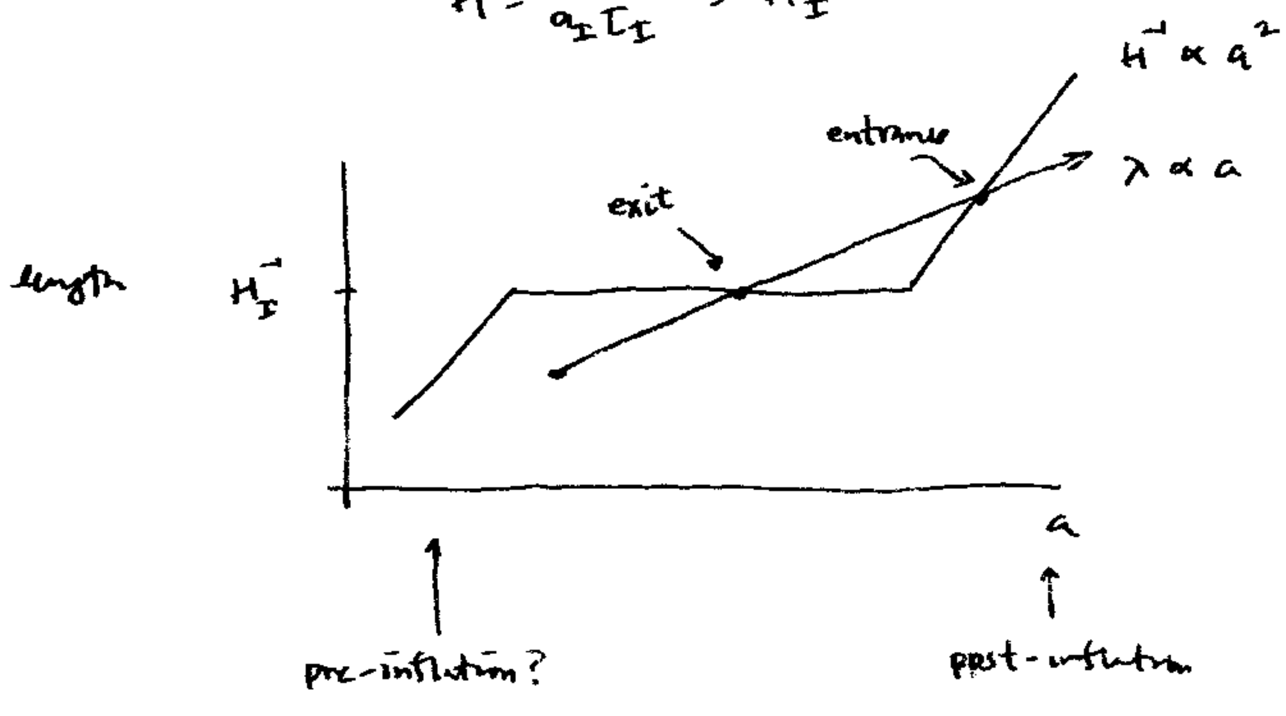
$a \propto \tau, \tau^2 \qquad 0 < \tau < \infty$

$$H = \frac{1}{a^2}\frac{da}{d\tau} = \frac{n}{a\tau} \qquad n = 1, 2$$

Yes inflation $\qquad a = a_I \dfrac{\tau_I}{2\tau_I - \tau}, \qquad -\infty < \tau < \tau_I$

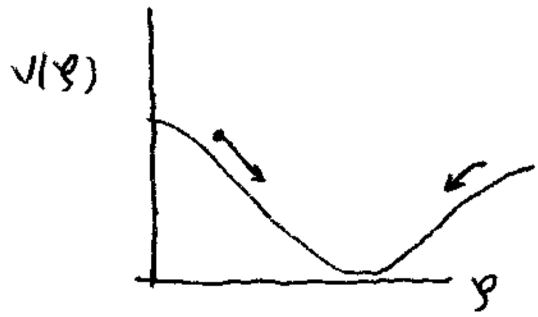
inflation extends the past conformal history

$$H = \frac{1}{a_I \tau_I} = H_I$$



"Horizon" exit and entrance refers to $\lambda = H^{-1}$

Inflation



An epoch of scalar field potential energy domination

$$S = \int d^4x \sqrt{g} \left[\frac{R}{16\pi G} - \frac{1}{2}(\partial \varphi)^2 - V(\varphi) \right]$$

$$\frac{3}{8\pi G} H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \qquad \dot{} = \frac{d}{dt}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$$

if V' is sufficiently small, whilst V is large
then $\varphi \sim$ constant and $\dot{\varphi}^2 \ll V$
so that $H \sim$ constant

SLOW ROLL PARAMETERS

$$\epsilon \equiv -\dot{H}/H^2, \quad \delta \equiv \ddot{H}/2H\dot{H}$$

Require $\epsilon, \delta \ll 1$ for inflation

Inflation

Scalar Field

$$T_{\mu\nu} = \partial_\mu \varphi \, \partial_\nu \varphi - g_{\mu\nu} \left(\tfrac{1}{2}(\partial\varphi)^2 + V \right)$$

$$\nabla^\mu T_{\mu\nu} = 0 \quad \rightarrow \quad \Box\varphi - V' = 0$$

in conformal time: $\quad \varphi'' + 2\mathcal{H}\varphi' + a^2 V' = 0$

in a weak gravitational field

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)d\bar{x}^2 \right]$$

$$\varphi \rightarrow \varphi + \delta\varphi(\tau, \bar{x})$$

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2 V''\delta\varphi = (3\Phi' + \Psi')\varphi' - 2a^2 V' \Psi$$

$$\delta T^0{}_0 = -\left(\frac{1}{a^2}\varphi'\delta\varphi' + V'\delta\varphi - \frac{1}{a^2}\varphi'^2\Psi \right)$$

$$\delta T^i{}_j = \left(\frac{1}{a^2}\varphi'\delta\varphi' - V'\delta\varphi - \frac{1}{a^2}\varphi'^2\Psi \right)\delta^i{}_j$$

$$\delta T^0{}_i = -\frac{1}{a^2}\varphi' \partial_i \delta\varphi$$

SCALAR FIELD HAS NO ANISOTROPIC STRESS : $\Psi = \Phi$.

Use Einstein Eq'ns to obtain

(A) $$\Phi'' + 2\left(\mathcal{H} - \frac{\varphi''}{\varphi'}\right)\Phi' + 2\left(\mathcal{H}' - \frac{\varphi''}{\varphi'}\mathcal{H}\right)\Phi - \nabla^2\Phi = 0$$

remark: in $k \to 0$ limit, the quantity

$$\zeta = \Phi + \frac{\delta\varphi}{\varphi'}\mathcal{H}$$

$$= \Phi + \frac{2}{3}\frac{\Phi' + \mathcal{H}\Phi}{\mathcal{H}(1+w)}$$

is constant, $\zeta' = 0$

(B) $$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2 V''\delta\varphi = 4\Phi'\varphi' - 2a^2\Phi V'$$

Define $R = \Phi + \frac{\delta\varphi}{\varphi'}\mathcal{H}$ to get

(C) $$R'' + 2\frac{z'}{z}R' - \nabla^2 R = 0 \qquad z = a\varphi'/\mathcal{H}$$

COMMENTS: ON SMALL SCALES DURING INFLATION, $-k\tau \to \infty$,
R-eq'n looks like Minkowski wave eq'n.

ON LARGE SCALES DURING INFLATION, $k \ll \mathcal{H}$,
R has constant and decaying solutions.
Long duration of inflation ensures decaying sol'ns
are negligible.

$\mathcal{R}$: $\quad \mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' - \nabla^2\mathcal{R} = 0$

or $\quad \frac{d^2}{d\tau^2}\mathcal{R} + 2\mathcal{H}(1+\epsilon+\delta)\frac{d}{d\tau}\mathcal{R} - \nabla^2\mathcal{R} = 0$

$\left\{ \begin{array}{l} \text{Weinberg} \\ \text{Peter \& Uzan} \end{array} \right.$

QFT of $\mathcal{R}$

set $q = z\mathcal{R}$

$$S_q = \int d\tau\, d^3x \; \frac{1}{2}\left[q'^2 - \partial_i q\, \partial^i q + \frac{z''}{z} q^2 \right]$$

$$\pi = \frac{\partial \mathcal{L}}{\partial q'} = q' \qquad \text{s.t.} \qquad [q, \pi] = i\,\delta(\bar{x} - \bar{y})$$

$$\mathcal{R}(\tau, \bar{x}) = \int \frac{d^3k}{(2\pi)^3}\left[\mathcal{R}_k(\tau)\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}(\vec{k}) + \text{H.C.} \right]$$

$$\langle \mathcal{R}(\tau,\bar{x})\, \mathcal{R}(\tau,\vec{y}) \rangle = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\bar{x}-\bar{y})}\, P_{\mathcal{R}}(k)$$

$$P_{\mathcal{R}}(k) = |\mathcal{R}_k|^2, \quad \Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} P_{\mathcal{R}} = A\left(\frac{k}{k_0}\right)^{n_s - 1}$$

ORIGIN OF INHOMOGENEITIES: AMPLIFICATION OF QUANTUM FLUCTUATIONS DURING INFLATION.

Example : Power law Inflation

Start: $V(\varphi) = M^4 \exp(-\varphi/\Lambda)$

$$\rightarrow \varphi(t) = \Lambda \ln\left[\frac{8\pi\epsilon^2 M^4 t^2}{(3-\epsilon) M_p^2}\right] \quad , \quad M_p^2 = 1/G$$

$$H(t) = \frac{1}{\epsilon t} \quad , \quad a(t) = a_I \left(\frac{t}{t_I}\right)^{1/\epsilon}$$

$$\text{or} \qquad a(\tau) = a_I \left(\frac{\tau_I}{\tau}\right)^{\frac{1}{1-\epsilon}}$$

SLOW ROLL : $\epsilon = -\delta$

$$R_k'' + 2\mathcal{H} R_k' + k^2 R_k = 0$$

CHOOSE SOL'N S.T. $R_k R_k^{*\prime} - R_k' R_k^* = i\left(\frac{\mathcal{H}}{\varphi'}\right)^2$

AND $R_k \propto e^{-ik\tau}$ in $k\tau \to \infty$ LIMIT.

$$\rightarrow R_k(\tau) = C \frac{\mathcal{H}}{\sqrt{\epsilon} M_p} \frac{(-k\tau)^{3/2}}{k^{3/2}} H_\nu^{(1)}(-k\tau)$$

$$C = -\pi(1-\epsilon) e^{i\frac{\pi}{4}(1+2\nu)} \quad , \quad \nu = \frac{3}{2} + \frac{\epsilon}{1-\epsilon}$$

$$\rightarrow \text{SPECTRAL INDEX: } \quad n_s = 4 - 2\nu = 1 - \frac{2\epsilon}{1-\epsilon} \simeq 1 - 2\epsilon$$

Inflationary Power Spectrum

$$P_R = \frac{2\pi}{k^3} \frac{H_*^2}{\epsilon M_p^2} \left(\frac{k}{a_* H_*} \right)^{n_s - 1}$$

"$*$" indicates value at horizon crossing, $k = aH$

WMAP 7 (Komatsu et al, ApJ Supp. 192 18 2011)

$k_0 = 0.002\ \mathrm{Mpc}^{-1}$

$A = 2.43 (\pm 0.091) \times 10^{-9}$

$n_s = 0.967 \pm 0.014$

so $\epsilon = 0.0162 \pm 0.0067$

$H_* = 1.12 (\pm 0.23) \times 10^{-5}\ M_p$

TENSOR PERTURBATIONS

H_{ij} : $\quad H_{ij}'' + 2\mathcal{H} H_{ij}' - \nabla^2 H_{ij} = 8\pi G a^2 p \pi^T_{ij}$
$\hookrightarrow 0$

$H^i_i = 0, \quad \nabla^i H_{ij} = 0$

QFT of H

$$S = \frac{M_p^2}{64\pi} \int d\tau\, d^3x \quad a^2 \left[(H_{ij}')^2 - (\nabla_k H_{ij})^2 \right]$$

Define $H_{ij} = \sum_\lambda h_\lambda \epsilon^\lambda_{ij}$, $\quad q_\lambda = \sqrt{\frac{M_p^2}{32\pi}}\; a\, h_\lambda$

$$= \frac{1}{2} \sum_\lambda \int d\tau\, d^3x \left[(q_\lambda')^2 - (\nabla q_\lambda)^2 + \frac{a''}{a} (q_\lambda)^2 \right]$$

$$\pi_\lambda = \frac{\partial L}{\partial q_\lambda'} = q_\lambda' \quad \rightarrow \quad [q_\lambda(\bar{x},\tau), \pi_{\lambda'}(\bar{y},\tau)] = i\delta_{\lambda\lambda'} \delta(\bar{x}-\bar{y})$$

EOM: $\quad q_\lambda'' - \left(\frac{a''}{a} + \nabla^2 \right) q_\lambda = 0$

$$q_\lambda(\bar{x},\tau) = \int \frac{d^3k}{(2\pi)^3} \left[q_\lambda\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}(\vec{k}) + \text{H.C.} \right]$$

recipe: choose sol'n $q_\lambda(k) \propto e^{-ik\tau}$ as $-k\tau \to \infty$

$$P_T = 2 \times \frac{k^3}{2\pi^2} |h_\lambda|^2 , \qquad n_T = \frac{\partial \ln P_T}{\partial \ln k}$$

POWER LAW INFLATION

$$n_T = 3-2\nu \simeq -2\epsilon$$

$$P_T = 16\epsilon P_R = -8 n_T P_R$$

ϵ CONTROLS RELATIONSHIP BETWEEN R & H

IN GENERAL, FOR SINGLE FIELD, SLOW ROLL INFLATION

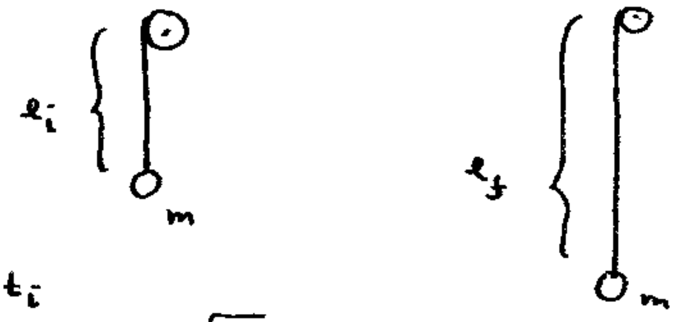
$$\frac{P_T}{P_R} = 16\epsilon = -8 n_T$$

$$n_s = 1 + 2\delta - 4\epsilon$$
$$n_T = -2\epsilon$$

NOTE: INDICES MAY "RUN"

$$\alpha_s = \frac{d n_s}{d \ln k} \quad , \quad \alpha_T = \frac{d n_T}{d \ln k}$$

AMPLIFICATION OF A QUANTUM OSCILLATOR (Allen, 1996)



@ t_i $\omega_i = \sqrt{\dfrac{g}{\ell_i}}$

@ t_f $\omega_f = \sqrt{\dfrac{g}{\ell_f}}$

QM: At t_i, $E_i = \frac{1}{2}\hbar\omega_i$

At t_f IF $T = t_f - t_i \gg \dfrac{2\pi}{\omega_f}$

THEN $E_f = \frac{1}{2}\hbar\omega_f$

IF $T \ll 2\pi/\omega_i$

THEN $E_f = \frac{1}{4}\hbar\omega_i = \hbar\omega_f\left(N + \frac{1}{2}\right)$

$N = \dfrac{1}{4}\dfrac{\omega_i}{\omega_f} - \dfrac{1}{2} \gg 1$

HIGH OCCUPATION NUMBER

Calculation: Use ground state wave function for harmonic oscillator of frequency ω_i to evaluate energy of harmonic oscillator with frequency $\omega_f \ll \omega_i$, as a SUDDEN change in the oscillator.

Quantum to Classical Peter & Uzan

Light scalar during inflation

$$\ddot{\sigma}_k + 3H\dot{\sigma}_k + \frac{k^2}{a^2}\sigma_k = 0$$

or in conformal time, where $u_k = a\sigma_k$, $u_k'' + (k^2 - \frac{a''}{a})u_k = 0$

$$u_k = A(k)\, e^{-ik\tau}\left(1 + \frac{1}{ik\tau}\right) + B(k)\, e^{ik\tau}\left(1 - \frac{1}{ik\tau}\right)$$

In order to match Minkowski on small scales, $B \to 0$, so

$$\sigma_k = \frac{H\tau}{\sqrt{2k}}\left(1 + \frac{1}{ik\tau}\right) e^{-ik\tau}$$

But on large scales $\sigma_k \simeq \frac{H}{\sqrt{2k^3}}$ s.t.

$$\hat{\sigma} = \int \frac{d^3k}{(2\pi)^3}\left[\sigma_k e^{i\vec{k}\cdot\vec{x}}\, \hat{a}_{\vec{k}} + \text{H.C}\right]$$

$$\simeq \int \frac{d^3k}{(2\pi)^3}\left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{-\vec{k}}\right) e^{i\vec{k}\cdot\vec{x}} \frac{H}{\sqrt{2k^3}}$$

QUANTUM OPERATORS OF σ COMMUTE:

σ BEHAVES LIKE A CLASSICAL, STOCHASTIC FIELD

WITH GAUSSIAN STATISTICS

Quality of Inflationary Perturbations

single field inflation produces adiabatic perturbs

$$\frac{\delta p}{\delta \rho} = \frac{\dot{p}}{\dot{\rho}}$$

Adiabatic pert'ns transfer to radiation, matter

radi'n : $\delta p = \frac{1}{3} \delta \rho$

matter : $\delta p = 0$

$$\frac{\delta_r}{1+w_r} = \frac{\delta_m}{1+w_m} \rightarrow \frac{3}{4}\delta_r = \delta_m$$