Lecture 2

Relativistic Stars

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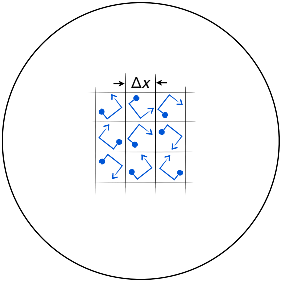
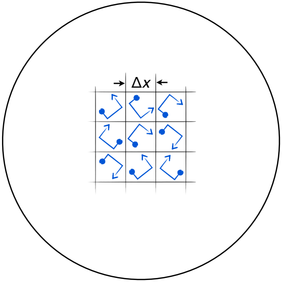
Equation of state of cold degenerate matter



- Non-relativistic electron degeneracy: low-mass white dwarf stars
- Relativistic electron degeneracy: high-mass white dwarf stars
- Relativistic neutron degeneracy: neutron stars

Equation of state of cold degenerate matter



- Each fermion lives in its own box of size $\Delta x \sim n_F^{-1/3}$ where n_F is fermion number-density
- $\Delta p \, \Delta x \sim \hbar$ and $p \sim \Delta p$ so $\boxed{p \sim \hbar n_F^{1/3}}$
- $P = \dfrac{\text{impulse}}{\text{area}} \sim \dfrac{p/\tau}{(\Delta x)^2}$ and $\tau \sim (\Delta x)/v$ so $\boxed{P \sim n_F p v}$

Non-relativistic degeneracy

Non-relativistic motion:

$$v = \frac{p}{m_\mathrm{F}} \sim \frac{\hbar n_\mathrm{F}^{1/3}}{m_\mathrm{F}}$$

where m_F is mass of fermion.
Let μ be number of baryons per degenerate fermion

- For electron-degeneracy, $\mu = A/Z \sim 2$
- For neutron-degeneracy, $\mu \sim 1$

Density is $\rho = \mu m_\mathrm{B} n_\mathrm{F}$ where m_B is the baryon mass
Equation of state is

$$P \sim n_\mathrm{F} p v \sim \frac{\hbar^2}{m_\mathrm{F}^2} n_\mathrm{F}^{5/3} \sim \frac{\hbar^2}{m_\mathrm{F}^2} \left(\frac{\rho}{\mu m_\mathrm{B}} \right)^{5/3}$$

Non-relativistic degeneracy

Exact calculation gives:

$$\boxed{P = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_F} \left(\frac{\rho}{\mu m_B} \right)^{5/3}}$$

A Polytropic equation of state with $\gamma = \frac{5}{3}$ or $n = \frac{3}{2}$
Recall: $M \sim R^{(3-n)/(1-n)} \sim R^{-3}$

- As matter is added, star *shrinks*
- Fermions move faster and eventually become relativistic
- White dwarfs: electron degeneracy

$$v = \frac{p}{m_e} \sim \frac{\hbar n_e^{1/3}}{m_e} \sim \frac{\hbar}{m_e} \left(\frac{\rho}{\mu m_B} \right)^{1/3}$$

at $\rho \sim 10^9\,\mathrm{kg\,m^{-3}}$, $v \sim c$

Relativistic degeneracy

Recall:

$$P \sim n_{\mathrm{F}} p v$$

and

$$p \sim \hbar n_{\mathrm{F}}^{1/3}$$

Fermions are relativistic: $v \sim c$ so

$$P \sim \hbar c n_{\mathrm{F}}^{4/3} \sim \hbar c \left(\frac{\rho}{\mu m_{\mathrm{B}}} \right)^{4/3}$$

Exact calculation gives

$$\boxed{P = \frac{(3\pi^2)^{1/3}}{4} \hbar c \left(\frac{\rho}{\mu m_{\mathrm{B}}} \right)^{4/3}}$$

Relativistic degeneracy

Relativistic degenerate matter has a polytropic equation of state with $\gamma = \frac{4}{3}$ or $n = 3$

- Recall: $M \sim R^{(3-n)/(1-n)}$ so mass is independent of radius
- Maximum mass that can be supported is

$$M = \frac{(3\pi^2)^{1/2}}{2} \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{(\mu m_\mathrm{B})^2} \left(-\xi_1^2 \left.\frac{d\theta}{d\xi}\right|_{\xi_1}\right)$$

- $-\xi_1^2 [d\theta/d\xi]_{\xi_1} = 2.018$ for an $n = 3$ polytrope, so

$$\boxed{M = 1.4\, M_\odot \times \left(\frac{2}{\mu}\right)^2}$$

Chadrasekhar mass

Chandrasekhar limit

Heuristic argument:

- Degenerate fermions have $p \sim \hbar n_{\mathrm{F}}^{1/3}$ where $n_{\mathrm{F}} \sim N_{\mathrm{F}}/R^3$
  N_{F} is number of degenerate fermions in star
- Total energy of relativistic degenerate gas (Fermi energy)

$$E_{\mathrm{F}} = N_{\mathrm{F}} pc \sim \frac{\hbar c N_{\mathrm{F}}^{4/3}}{R}$$

- Gravitational self binding energy

$$E_{\mathrm{G}} \sim -\frac{GM^2}{R} \sim -\frac{G N_{\mathrm{F}}^2 m_{\mathrm{B}}^2}{R}$$

- Total energy is $\quad E = E_{\mathrm{F}} + E_{\mathrm{G}} \sim \dfrac{\hbar c N_{\mathrm{F}}^{4/3}}{R} - \dfrac{G N_{\mathrm{F}}^2 m_{\mathrm{B}}^2}{R}$

Chandrasekhar limit

- Total energy is $E = E_F + E_G \sim \dfrac{\hbar c N_F^{4/3}}{R} - \dfrac{G N_F^2 m_B^2}{R}$
- Star will shrink/expand until E is a minimum
  But both terms $\sim 1/R$!
  - When $E > 0$, $R \uparrow$ until fermions become non-degenerate; then $E_F \sim 1/R^2$ and equilibrium is reached
  - When $E < 0$, $R \downarrow$ until $R \to 0$: star is unstable
- Stability requires $E > 0$:

$$\hbar c N_F^{4/3} > G N_F^2 m_B^2$$

$$N_{F,max} \sim \left(\frac{\hbar c}{G m_B^2}\right)^{3/2} \sim 2 \times 10^{57}$$

$$M_{max} = N_{F,max} m_B \sim 1.8\, M_{\odot} \quad \text{(close)}$$

Chandrasekhar limit

As $M \to M_{\text{max}}$, star shrinks and fermions become relativistic when

$$pc > m_F c^2$$
$$\frac{\hbar N_F^{1/3}}{R} > m_F c$$

For $N_F = N_{F,\text{max}} = (\hbar c / G m_B)^{3/2}$, instability occurs when

$$\begin{aligned} R < R_{\text{min}} &\sim \frac{\hbar}{m_F c} N_{F,\text{max}}^{1/3} \\ &\sim \frac{\hbar}{m_F c} \left(\frac{\hbar c}{G m_B} \right)^{1/2} \\ &\sim \begin{cases} 5000\,\text{km} & \text{white dwarfs: } m_F = m_e \\ 3\,\text{km} & \text{neutron stars: } m_F = m_B \end{cases} \end{aligned}$$

Maximum mass of a neutron star

What if the equation of state is not an $n = 3$ polytrope?
Requirements:

1. Density greater than nuclear density, $\rho > \rho_n \sim 4 \times 10^{17}\,\mathrm{kg\,m^{-3}}$
2. Radius greater than the Schwarzschild radius, $R > 2GM/c^2$

$$
\begin{aligned}
M &> \frac{4}{3}\pi R^3 \rho_n \\
&> \frac{4}{3}\pi \left(\frac{2GM}{c^2}\right)^3 \rho_n
\end{aligned}
$$

so

$$
\begin{aligned}
M &< \frac{c^3}{G}\sqrt{\frac{3}{32\pi}\frac{1}{G\rho_n}} \\
&\sim 7\,M_\odot
\end{aligned}
$$

Tolman-Oppenheimer-Volkoff (TOV) equation

In spherical symmetry, the static spacetime metric is

$$ds^2 = -e^{2\Phi(r)/c^2} c^2\, dt^2 + \frac{dr^2}{1 - \dfrac{2Gm(r)}{c^2 r}} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

where $\Phi(r)$ and $m(r)$ are some functions that depend on r alone

Matter is a perfect fluid,

$$T_{\hat{t}\hat{t}} = \rho c^2$$

$$T_{\hat{r}\hat{r}} = P$$

Tolman-Oppenheimer-Volkoff (TOV) equation

Einstein's field equations:

$$G_{\hat{t}\hat{t}} = \frac{8\pi G}{c^4} T_{\hat{t}\hat{t}} \Longrightarrow \boxed{m(r) = \int_0^r 4\pi r'^2 \, dr' \, \rho(r')}$$

$$G_{\hat{r}\hat{r}} = \frac{8\pi G}{c^4} T_{\hat{r}\hat{r}} \Longrightarrow \boxed{\frac{d\Phi}{dr} = \frac{Gm(r) + 4\pi G r^3 P/c^2}{r[r - 2Gm(r)/c^2]}}$$

$$(e_{\hat{r}})_\nu \nabla_\mu T^{\mu\nu} = 0 \Longrightarrow \boxed{\frac{dP}{dr} = -(\rho + P/c^2)\frac{Gm(r) + 4\pi G r^3 P/c^2}{r[r - 2Gm(r)/c^2]}}$$

Tolman-Oppenheimer-Volkov equation

Newtonian limit of TOV equation

The Newtonian limit has $P \ll \rho c^2$ and $Gm(r)/c^2 \ll r$:

$$
\begin{aligned}
m(r) &= \int_0^r 4\pi r'^2 \, dr' \, \rho(r') \\
\frac{d\Phi}{dr} &= \frac{Gm(r)}{r^2} \\
\frac{dP}{dr} &= -\frac{Gm(r)}{r^2} \, \rho
\end{aligned}
$$

These are the same equations as were used in Lecture 1.
(Φ is the Newtonian potential.)

Incompressible star in General Relativity

Constant density: $\rho = \rho_c = \text{const}$ and $m(r) = \frac{4}{3}\pi r^3 \rho_c$
TOV equation can be integrated exactly:

$$P = \rho_c c^2 \frac{(1 - 2GM/c^2R)^{1/2} - (1 - 2GMr^2/c^2R^3)^{1/2}}{(1 - 2GMr^2/c^2R^3)^{1/2} - 3(1 - 2GM/c^2R)^{1/2}}$$

Central pressure is

$$P_c = \rho_c c^2 \frac{(1 - 2GM/c^2R)^{1/2} - 1}{1 - 3(1 - 2GM/c^2R)^{1/2}}$$

Note: $P_c \to \infty$ when $3(1 - 2GM/c^2R)^{1/2} = 1$ or

$$R = \frac{9}{4}\frac{GM}{c^2}$$

Maximum mass of a neutron star redux

Recompute maximum mass of neutron star

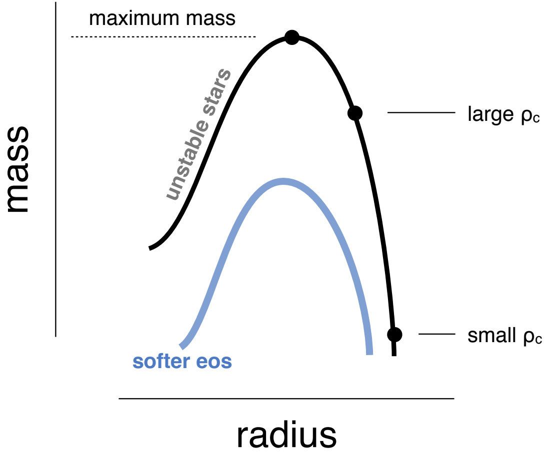
- Incompressible star with $\rho_c = \rho_n = 4 \times 10^{17}\,\mathrm{kg\,m^{-3}}$
- Central pressure must be finite so $R > \frac{9}{4} GM/c^2$

$$\begin{aligned} M &> \frac{4}{3}\pi R^3 \rho_n \\ &> \frac{4}{3}\pi \left(\frac{9}{4} \frac{GM}{c^2} \right)^3 \rho_n \end{aligned}$$

so

$$\begin{aligned} M &< \frac{4}{9} \frac{c^3}{G} \sqrt{\frac{1}{3\pi} \frac{1}{G\rho_n}} \\ &\sim 6\, M_\odot \end{aligned}$$

Mass-Radius curve



Mass-Radius curve

Simple numerical model: two-component polytrope

$$P = \begin{cases} K_0 \rho^{\Gamma_0} \text{ for } \rho < \rho_1 \\ K_1 \rho^{\Gamma_1} \text{ otherwise} \end{cases}$$

where low density region is $\rho < \rho_1 = 5 \times 10^{17}\,\mathrm{kg\,m}^3$

$$\Gamma_0 = \frac{5}{3} \quad \text{and} \quad K_0 = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar}{m_\mathrm{B}^{8/3}}$$

and continuity requires

$$K_1 = K_0 \rho_1^{\Gamma_0 - \Gamma_1}$$

Consider soft and stiff core equations of state:

$$\Gamma_1 = 2.5 \quad \text{(soft)} \qquad \text{and} \qquad \Gamma_1 = 3 \quad \text{(stiff)}$$

Mass-Radius curve: program tov.py

```python
import pylab, odeint
from scipy.constants import pi, G, c, hbar, m_n
Msun = 1.98892e30

# piecewise polytrope equation of state
Gamma0 = 5.0/3.0 # low densities: soft non-relativistic degeneracy pressure
K0 = (3.0*pi**2)**(2.0/3.0)*hbar**2/(5.0*m_n**(8.0/3.0))
Gamma1 = 3.0 # high densities: stiffer equation of state
rho1 = 5e17
P1 = K0*rho1**Gamma0
K1 = P1/rho1**Gamma1

def eos(rho):
    if rho < rho1: return K0*rho**Gamma0
    else: return K1*rho**Gamma1

def inveos(P):
    if P < P1: return (P/K0)**(1.0/Gamma0)
    else: return (P/K1)**(1.0/Gamma1)
```

to be continued...

Mass-Radius curve: program tov.py

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```
def tov(y, r):
    """ Tolman-Oppenheimer-Volkov equations. """
    P, m = y[0], y[1]
    rho = inveos(P)
    dPdr = -G*(rho + P/c**2)*(m + 4.0*pi*r**3*P/c**2)
    dPdr = dPdr/(r*(r - 2.0*G*m/c**2))
    dmdr = 4.0*pi*r**2*rho
    return pylab.array([dPdr, dmdr])
```

to be continued...

Mass-Radius curve: program tov.py

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```
def tovsolve(rhoc):
    """ Solves TOV equations given a central density. """
    r = pylab.arange(10.0, 20000.0, dr)
    m = pylab.zeros_like(r)
    P = pylab.zeros_like(r)
    m[0] = 4.0*pi*r[0]**3*rhoc
    P[0] = eos(rhoc)
    y = pylab.array([P[0], m[0]])
    i = 0 # integrate until density drops below zero
    while P[i] > 0.0 and i < len(r) - 1:
        dr = r[i+1] - r[i]
        y = odeint.rk4(tov, y, r[i], dr)
        P[i+1] = y[0]
        m[i+1] = y[1]
        i = i + 1
    # return mass and radius of star
    return m[i-1]/Msun, r[i-1]/1000.0
```

to be continued...

Mass-Radius curve: program tov.py

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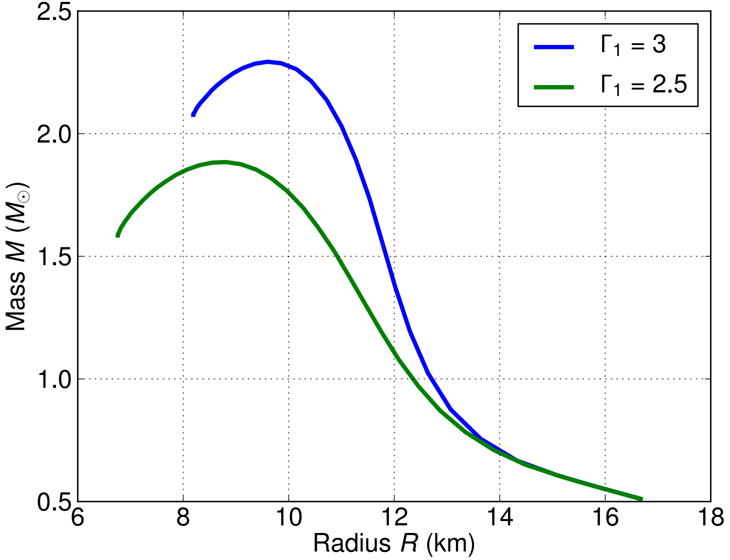
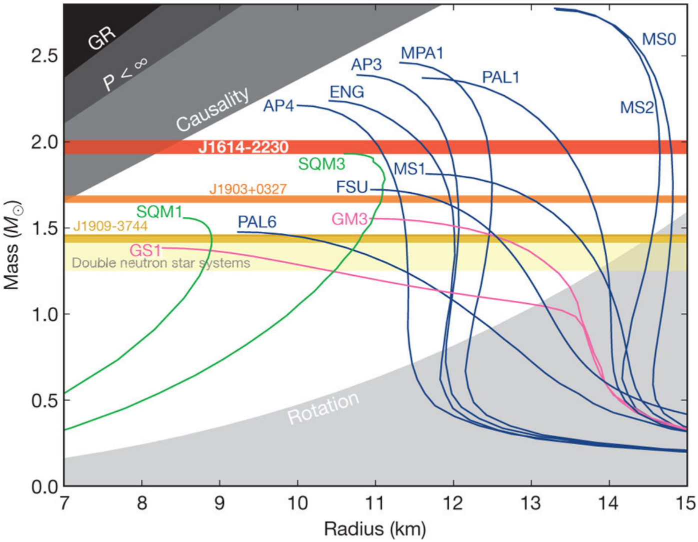
```
# plot mass-radius curve
rhoc = pylab.logspace(17.5,20) # logspace range of central densities
M = pylab.zeros_like(rhoc)
R = pylab.zeros_like(rhoc)
for i in range(len(rhoc)):
    M[i], R[i] = tovsolve(rhoc[i])

pylab.plot(R, M)
pylab.xlabel('Radius (km)')
pylab.ylabel('Mass (solar)')
pylab.grid()
pylab.show()
```

Mass-Radius curve



Mass-Radius curve



Demorest et al., *Nature* **467**, 1081 (2010)