Perturbation Evolution, from Inflation forward

$$\Phi'' + 3\mathcal{H}\Phi' + \left(2\frac{a''}{a} - \mathcal{H}^2\right)\Phi = 4\pi G a^2 \delta p$$

$$-k^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = 4\pi G a^2 \delta\rho$$

ignore anisotropic stress, so $\Phi = \Psi$

$$\Phi'' + 3\mathcal{H}(1+c_a^2)\Phi' + \left(2\frac{a''}{a} - \mathcal{H}^2 - 3c_a^2\mathcal{H}^2\right)\Phi + c_s^2 k^2 \Phi = 4\pi G a^2(\delta p - c_a^2 \delta\rho)$$

c_a^2 = adiabatic sound speed, $\dot{p}/\dot{\rho}$

For adiabatic perturbations, $\delta p = c_a^2 \delta\rho$

During radiation era $c_a^2 = \frac{1}{3}$

$\hookrightarrow$ Φ = constant for $k \ll \mathcal{H}$

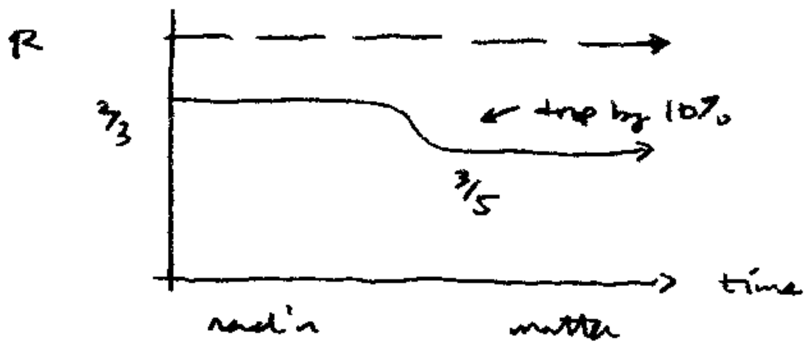
so $\mathcal{R} = \Phi + \frac{2}{3}\frac{\Phi}{1+\frac{1}{3}} = \frac{3}{2}\Phi$

matter era $c_a^2 = 0$

$\hookrightarrow$ Φ = constant for all k!

so $\mathcal{R} = \Phi + \frac{2}{3}\frac{\Phi}{1+0} = \frac{5}{3}\Phi$

Evolution



$$\Phi_m = \frac{9}{10}\Phi_r$$

For long wavelength modes during the radiation era

$$P_\Phi = \frac{4}{9} P_R$$

$$= \frac{8\pi}{9} \frac{H_*^2}{\epsilon M_p^2 k^3} \left(\frac{k}{a_* H_*} \right)^{n_s - 1}$$

and there is an additional $\left(\frac{9}{10}\right)^2$ factor in matter era.

In matter era, for all scales $\Phi =$ constant

so for $k \gg \mathcal{H}$, $\quad k^2 \Phi = -4\pi G a^2 \delta\rho$

so $\delta\rho \propto \frac{1}{a^2}$ or $\left.\frac{\delta\rho}{\rho}\right|_m \propto a$

Density contrast δ grows

Fluid Eq'ns of Motion

$$\delta' = -(1+w)(\theta - 3\phi') - 3\mathcal{H}\left(\frac{\delta p}{\delta\rho} - w\right)\delta$$

$$\theta' = -\mathcal{H}(1-3w)\theta - \frac{w'}{1+w}\theta + \frac{\delta p}{\delta \rho}\frac{k^2}{1+w}\delta - k^2\sigma + k^2\psi$$

Examine for matter: $w=0$, $\delta p = 0$

$$\delta' = -\theta + 3\phi'$$
$$\theta' = -\mathcal{H}\theta + k^2\psi$$

during radiation era, for $k \ll \mathcal{H}$

$\rightarrow$ θ decays, ϕ constant

so δ is constant

Examine for radiation: $w=\frac{1}{3}$, $\delta p = \frac{1}{3}\delta\rho$

$$\delta' = -\frac{4}{3}\theta + 4\phi'$$
$$\theta' = \frac{1}{4}k^2\delta + k^2\psi$$

δ for $k \ll \mathcal{H}$ is constant

COMPLEMENTARY VIEW: SYNCHRONOUS GAUGE

EE's:

$$k^2\eta - \frac{1}{2}\mathcal{H}h' = -4\pi G a^2 \sum_i \delta\rho_i$$

$$k^2\eta' = 4\pi G a^2(\rho+p)\theta$$

$$h'' + 2\mathcal{H}h' - 2k^2\eta = -24\pi G a^2 \delta p$$

$$(h''+6\eta'') + 2\mathcal{H}(h'+6\eta') - 2k^2\eta = -24\pi G a^2(\rho+p)\sigma$$

FLUID:

$$\delta' = -(1+w)(\theta + \tfrac{1}{2}h') - 3\mathcal{H}\left(\frac{\delta p}{\delta\rho} - w\right)\delta$$

$$\theta' = -\mathcal{H}(1-3w)\theta - \frac{w'}{1+w}\theta + \frac{\delta p}{\delta\rho}\frac{k^2}{1+w}\delta - k^2\sigma$$

cold dark matter as a pressureless fluid $(p = \delta p = \sigma = 0)$

$$\delta_c' = -(\theta_c + \tfrac{1}{2}h')$$

$$\theta_c' = -\mathcal{H}\theta_c$$

set $\theta_c = 0$ so that CDM defines the synchronous gauge frame.

$$\delta_c' = -\tfrac{1}{2}h'$$

COMBINE EE's: $h'' + \mathcal{H} h' = -8\pi G a^2 (\delta\rho + 3\delta p)$

$h' = -2\delta_c'$

$$\delta_c'' + \mathcal{H}\delta_c' = \frac{3}{2}\mathcal{H}^2 \left(\Omega_c(a)\,\delta_c + \sum_i \Omega_i(a)\,\delta_i\,(1+3w_i) \right)$$

general, independent of k.

During matter era, $\delta_r \Omega_r \ll \delta_c$

Include baryons (pressureless, too) w/ CDM

$$\delta_m'' + \mathcal{H}\delta_m' = \frac{3}{2}\mathcal{H}^2\,\Omega_m(a)\,\delta_m$$

while $\Omega_m(a) \simeq 1$, $\delta_m \propto a$ ✓

Upon dark energy influence

$\Omega_m(a) < 1$, $\mathcal{H}$ changes

δ_m grows slower than a ✓

LARGE SCALE STRUCTURE

$$P_R = \frac{2\pi^2}{k^3} A \left(\frac{k}{k_0}\right)^{n_s-1}$$

In radiation era, for modes $k \ll \mathcal{H}$

$$\phi = \frac{2}{3} R$$

$$P_\phi = \frac{4}{9} P_R = \frac{8\pi^2}{9k^3} A \left(\frac{k}{k_0}\right)^{n_s-1}$$

Evolve $\phi(k,a)$ to the present

TRANSFER FUNCTION: $T(k,a)$

$$T(k,a) = \frac{\phi(k,a)}{\phi(k,a_i)} \quad \leftarrow \text{s.t. } k \ll \mathcal{H}/a_i)$$

so "processed" power spectrum for ϕ is

$$P_\phi(k, a_{late}) = T(k, a_{late})^2 \frac{8\pi^2}{9k^3} A \left(\frac{k}{k_0}\right)^{n_s-1}$$

POWER SPECTRUM FOR ϕ AT PRESENT EPOCH (late)

Density contrast δ

Late times, small scales $\quad K \gg \mathcal{H}$

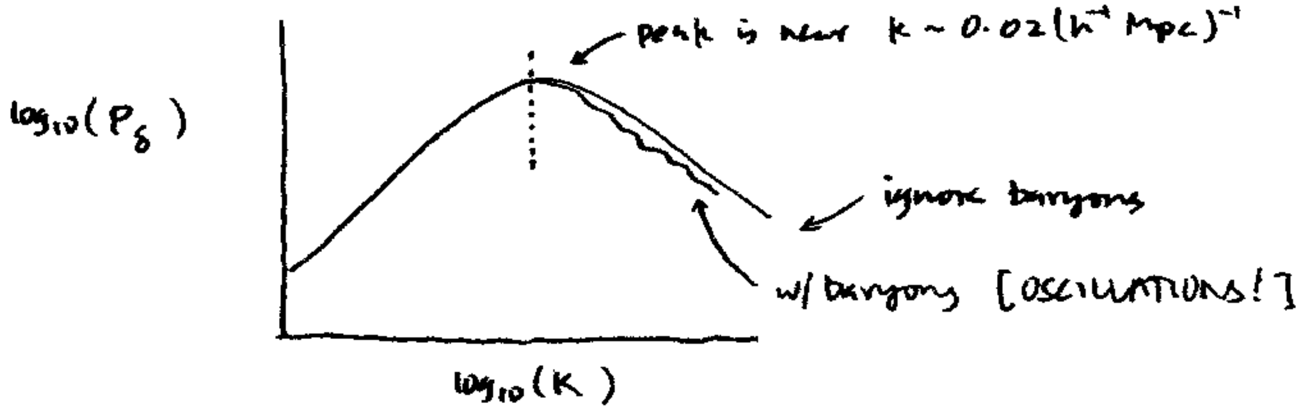
$$K^2 \Phi = -4\pi G a^2 \delta\rho$$

$$\hookrightarrow \quad \Phi = -\frac{3}{2}\frac{\mathcal{H}^2}{K^2}\Omega_m(a)\,\delta$$

if only matter (c, b) fluctuate on these scales

$$\delta = -\frac{2}{3}\frac{K^2}{\mathcal{H}^2}\frac{1}{\Omega_m(a)}\Phi$$

$$P_\delta = \frac{4}{9}\left(\frac{K}{\mathcal{H}}\right)^4 \frac{1}{\Omega_m(a)^2} P_\Phi \quad \checkmark$$



[see Dodelson]

or since $\delta_{SYNC} = \delta_{CONF}$ for $K \gg \mathcal{H}$

USE GROWTH FACTOR $D_\delta(a)$

$$P_\delta = \frac{4}{9}\left(\frac{K}{H_0}\right)^4 \frac{1}{[\Omega_m^0]^2} P_\Phi(K, a_0) \times \left[D_\delta\, a\right]^2$$

MASS FLUCTUATION EXCESS

$$\frac{\delta M}{M} = \frac{\delta \rho}{\rho}$$

rms $\langle (\frac{\delta M}{M})^2 \rangle$ on $r = 8\,h^{-1}$ Mpc scales is close to 1.

STEP 1: SMOOTH DENSITY FIELD ON SCALE r

WINDOW: $$W(r) = \frac{\Theta(r - r)}{\frac{4}{3}\pi r^3} \quad \text{s.t.} \quad \int W\, d^3x = 1$$

STEP 2: APPLY!

$$\sigma_r^2 = \langle (\frac{\delta M}{M})^2 \rangle_r$$

$$= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}\, W(r_1)\, W(r_2)\, P_\delta(k)\, d^3r_1\, d^3r_2$$

SINCE $$\int W(r)\, e^{i\vec{k}\cdot\vec{r}}\, d^3r = 3\, \frac{j_1(kr)}{kr}$$

$$= \frac{1}{2\pi^2} \int k^2\, dk\, P_\delta(k) \left[\frac{3\, j_1(kr)}{kr} \right]^2$$

STEP 3: TEST YOUR POWER SPECTRUM P_δ

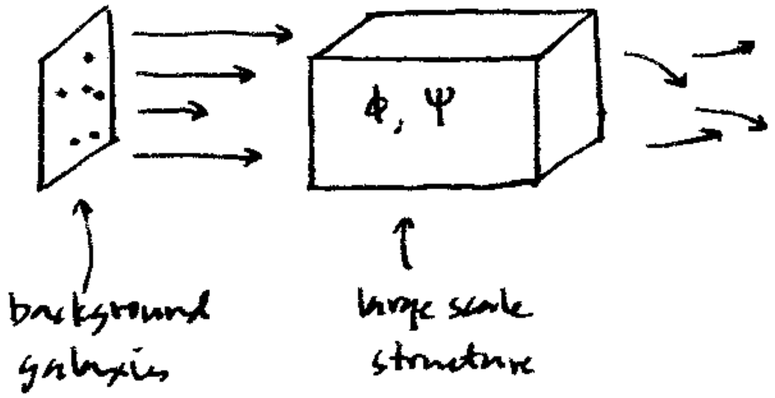
$$\sigma_r \simeq 0.8 \quad \text{for } r = 8\,h^{-1}\,\text{Mpc}.$$

LENSING

$$\theta = \frac{4GM}{bc^2} \quad \text{in GR}$$

$$= 2(1+\gamma)\frac{GM}{bc^2} \quad \text{in PPN gravity}$$

in cosmology



MAGNIFICATION
SHEAR DISTORTION

idealize

$$\frac{d^2}{dr^2} x^i = -\varphi_{,i}$$

$$\varphi = \Phi + \Psi$$

small displacements: $x^i = r\,\theta^i$

$$\theta^i = \theta^i_0 + \frac{1}{r}\int_0^r dr' \,(r'-r)\,\varphi_{,i}(\vec{x}(r'))$$

light ray deviation

$$\Delta\theta^i = \Delta\theta^i_0 + \Delta\theta^j_0 \int_0^r dr' \,\frac{(r'-r)}{r}\, r' \,\varphi_{,ij}(\vec{x}(r'))$$

light ray deviation or distortion tensor

$$D_{ij} = \int_0^{r_s} dr' \, \frac{r'-r_s}{r_s} \, r' \, \Phi_{,ij} \;\Rightarrow\; -\begin{pmatrix} \kappa+\gamma_1 & \gamma_2 \\ \gamma_2 & \kappa-\gamma_1 \end{pmatrix}$$

CONVERGENCE

$$\kappa = \frac{1}{2}\int_0^{r_s} dr' \left(1-\frac{r'}{r_s}\right) r' \, \Phi_{,i}{}^{i}$$

SHEAR

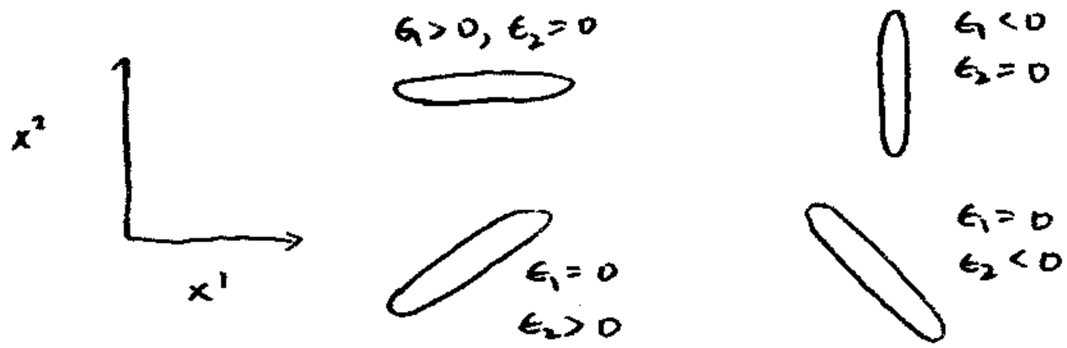
$$\gamma_1 = \frac{1}{2}\int_0^{r_s} dr' \left(1-\frac{r'}{r_s}\right) r' \left(\Phi_{,11}-\Phi_{,22}\right)$$

$$\gamma_2 = \int_0^{r_s} dr' \left(1-\frac{r'}{r_s}\right) r' \, \Phi_{,12}$$

In the limit of weak distortions, ELLIPTICITY = SHEAR x 2

$$\epsilon_i = 2\gamma_i$$

so a round source image gets smaller, brighter (converges) and stretched (shear)



[DODELSON]

WEAK LENSING OF LARGE SCALE STRUCTURE

DESCRIBE SOURCE POPULATION (galaxies)

$$W(r) \quad \text{s.t.} \quad \int_0^\infty W(r)\,dr = 1$$

DISTORTION:

$$D_{ij} = \int_0^\infty dr\; \varphi_{,ij}(\vec{x}(r))\; g(r)$$

$$g(r) = r\int_r^\infty dr' \left(\frac{r}{r'}-1\right) W(r')$$

ELLIPTICITY, CONVERGENCE CORRELATIONS

relate to δ, ϕ CORRELATIONS

OBSERVATIONS OF WL: POWERFUL TEST OF COSMOLOGY AND GRAVITY!

[Refregier, Ann. Rev. Astron. Astrophys 41 645 (2003)]

WL OF CMB [Lewis & Challinor, Phys. Rept. 429 1 (2006)]

FIRST DETECTIONS!

MORE ON THE WAY!