1. Suppose a relativistic gas has adiabatic index Γ, so that $P = K\rho^\Gamma$, when $s =$ constant. Use the second law of thermodynamics,

$$d\left(\frac{\epsilon}{\rho}\right) = -Pd\left(\frac{1}{\rho}\right) + \rho T ds,$$

to deduce

$$\epsilon = \rho + \frac{P}{\Gamma - 1}.$$

Thus, a relativistic polytrope with polytropic index n satisfies $\epsilon = \rho + nP$, where $\Gamma = 1 + 1/n$.

2. Rotating stars in the Newtonian approximation. The Euler equation is

$$(\partial_t + v^b\nabla_b)v_a + \frac{\nabla_a p}{\rho} + \nabla_a\Phi = 0,$$

with $\nabla^2\Phi = 4\pi\rho$. Show that the equation can be rewritten as

$$(\partial_t + \pounds_{\mathbf{v}})v_a + \frac{\nabla_a p}{\rho} + \nabla_a(\Phi - \frac{1}{2}v^2) = 0.$$

For a uniformly rotating star, $v^a = \Omega\phi^a$ (where $\boldsymbol{\phi} = x\hat{\mathbf{y}} - y\hat{\mathbf{x}}$); show that

$$\pounds_{\mathbf{v}} v_a = 0.$$

For a rotating star with a one-parameter equation of state $p = p(\rho)$, show that the Euler equation has the form

$$E_a \equiv \nabla_a(h - \frac{1}{2}v^2 + \Phi) + \varpi^2\Omega\nabla_a\Omega = 0,$$

where $\varpi^2 = \phi^a\phi_a = x^2 + y^a$ and the Newtonian enthalpy h is defined by $h = \int_0^p \frac{dp}{\rho}$. (The astrophysicists cylindrical coordinate is ϖ – script π – to distinguish it from ρ).
By taking the curl $\nabla_{[a}E_{b]} = 0$ of this equation, show that the angular velocity is stratified on cylinders: $\Omega = \Omega(\varpi)$.
Conclude that the Euler equation in for a uniformly rotating star has the first integral

$$h - \frac{1}{2}v^2 + \Phi = \mathcal{E},$$

where $\mathcal{E}$ is constant throughout the star.

3. Show that the Lie derivative of the metric has components

$$\pounds_\xi g_{\mu\nu} = \xi^\lambda\partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu\xi^\lambda + g_{\mu\lambda}\partial_\nu\xi^\lambda,$$

and (using the defining equation written in terms of covariant derivatives) show the relation,

$$\pounds_\xi g_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha.$$

A vector field ξ^α is a Killing vector if $\pounds_{\boldsymbol{\xi}} g_{\alpha\beta} = 0$ Note from the first of these equations, that if one chooses coordinates in which ξ^α is along one of the coordinates, so that $\boldsymbol{\xi} = \partial_\phi$, for example, then the components ξ^μ are constant and the statement that ξ^α is a Killing vector means that the components of the metric are independent of the coordinate (independent of ϕ in this example).

4. Lie derivatives and exterior derivatives (antisymmetric derivatives of antisymmetric tensors) commute. Check that this is true for a covector:

$$\pounds_{\mathbf{v}} \nabla_{[a}\sigma_{b]} = \nabla_{[a}\pounds_{\mathbf{v}}\sigma_{b]}$$

where $2\nabla_{[a}\sigma_{b]} = \nabla_a \sigma_b - \nabla_b \sigma_a$.