1. Let ξ^α be a Killing vector. Show that the motion of a free particle conserves $u_\alpha \xi^\alpha$, using the geodesic equation in the form
$$u^\beta \nabla_\beta u^\alpha = 0.$$
For $\xi^\alpha = t^\alpha$ and for $\xi^\alpha = \phi^\alpha$, what is the physical meaning of the conserved quantities? Check for a particle in flat space (in special relativity) that your identification is right.

2. Dragging of inertial frames. The metric of a rotating star or black hole has the form
$$ds^2 = -e^{2\nu}(dt - \omega d\phi)^2 + e^{2\psi} d\phi^2 + e^{2\mu}(d\varpi^2 + dz^2).$$
Show that a particle with zero angular momentum has angular velocity ω. In particular, a particle dropped from rest at infinity, rotates with angular velocity ω in the direction of the star's rotation.

3. Using the fact that the metric is asymptotically flat, show that the asymptotic form of the equation for ω is
$$\nabla \cdot (r^2 \sin^2\theta \nabla\omega) = 0,$$
and check that the equation is satisfied by the asymptotic form $\omega = \frac{2J}{r^3}$, with J a constant (we will find that J is the angular momentum of the spacetime). Consider a spherical shell rotating with angular velocity Ω. For a nearly Newtonian shell, $\omega \ll \Omega$, and the metric coefficients in the equation for ω are again those of flat space, so that the equation for ω has the form
$$\nabla \cdot (r^2 \sin^2\theta \nabla\omega) = -16\pi\rho r \sin^2\theta\ \Omega,$$
Here $\rho = \frac{m}{4\pi r_0^2}\delta(r - r_0)$ from the requirement $\int \rho dV = m$. Find the spherically symmetric solution for ω.