We found that the curl of the Euler equation for a barotropic star is conservation of circulation:

$$(\partial_t + \pounds_{\mathbf{v}})(\nabla_a v_b - \nabla_b v_a) = 0.$$

1. Consider a mode of a uniformly rotating star. The mode has t-ϕ behavior $e^{i(m\phi+\omega t)}$. Show that an observer riding on the star (rotating with angular velocity Ω) sees a frequency $\omega_r = \omega + m\Omega$ and note that, for any quantity Q with this t-ϕ dependence, $(\partial_t + \pounds_{\mathbf{v}})Q = i\omega_r Q$, where $\mathbf{v} = \boldsymbol{\Omega}\phi^{\mathbf{a}}$. That is, the time derivative computed by a corotating observer is $\partial_t + \pounds_{\mathbf{v}}$.
(To relate the two frequencies, use the fact that a point at constant angular coordinate ϕ_r measured by a rotating observer has coordinate $\phi = \phi_r + \Omega t$ measured by an inertial observer.)

2. Consider linear perturbations with perturbed velocity $\delta\mathbf{v}$. Show that $\delta\mathbf{v}$ satisfies

$$(\partial_t + \pounds_{\mathbf{v}})(\nabla_a \delta v_b - \nabla_b \delta v_a) + \pounds_{\delta\mathbf{v}}(\nabla_a v_b - \nabla_b v_a) = 0$$

and write the $\theta - \phi$ component of this equation for a perturbation of a uniformly rotating star ($v^a = \Omega\phi^a$).

3. Axial modes (r-modes) of rotating stars have (for slow rotation) velocity fields with angular dependence $\mathbf{r} \times Y_{\ell\ell}$, and the mode with $\ell = 2$ is the dominant unstable mode (because gravitational radiation is weaker for larger values of ℓ):

$$\delta\mathbf{v} = U(r)\mathbf{r} \times \boldsymbol{\nabla} Y_{22} e^{-i\omega t}$$

Using the fact that the mode has $t - \phi$ dependence $e^{i(m\phi+\omega t)}$ and the relation $\nabla^2 Y_{lm} = -l(l+1)Y_{lm}$, show that the $\theta - \phi$ component of the equation you just wrote down implies the mode has frequency

$$\omega_r = \frac{2}{3}\Omega, \qquad \omega = -\frac{4}{3}\Omega.$$

The opposite signs mean that a mode that travels backward relative to the star is dragged forward by the star's rotation and is therefore unstable.