Lecture 8.3 - Dynamics of FLRW with General Relativity

In conformal - Cartesian coordinates, the FLRW metric is:

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & a^2(t) \delta_{ij} \end{pmatrix}$$

$$\Gamma^i_{ij} = (1 + k t^2)^{-1} \delta^i_{ij}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & a^{-2} \delta^{ij} \end{pmatrix}$$

$$K : \text{UNITS OF } [\text{LENGTH}]^{-2}$$

$$K > 0, K < 0, K = 0 ?$$

The connections for this metric were given previously:

$$\Gamma^0_{ij} = \dot{a} a \delta_{ij}$$

$$\Gamma^i_{0j} = \dot{a} \delta_i^j$$

$$\Gamma^i_{jk} = \frac{2K}{1 + K t^2} (\delta_{ij} \kappa - \delta_{jk} \kappa - \delta_{ki} \kappa^2)$$

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0$$
We can compute the Ricci tensor directly:

\[ R^m_{\ \nu \rho} = R_{\nu \rho} \]

\[ \Rightarrow R_{\alpha \alpha} = \frac{3 \dot{a}^2}{a} \]

\[ R_{\alpha i} = 0 \]

\[ R_{ij} = -(a \ddot{a} + 2 \dot{a}^2 + 2k) \eta_{ij} \]

\[ R = g^{\mu \nu} R_{\mu \nu} = -\frac{3 \ddot{a}^2}{a} - \frac{3 \dot{a}^2}{a^2} \left( \frac{a^4 + 2 \dot{a}^2 a^2 + 2k a^2}{a^2} \right) \]

\[ \Rightarrow R = -6 \ddot{a}^2 - 6 \dot{a}^2 - \frac{6k}{a^2} \]

The Einstein tensor is, therefore,

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \]

or:

\[ G_{\alpha \alpha} = -3 \left( \frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} \right) \]

\[ G_{\alpha i} = 0 \]

\[ G_{ij} = (2a \ddot{a} + \dot{a}^2 + k) \eta_{ij} \]

Einstein's equations:

\[ G_{\mu \nu} = -8 \pi G T_{\mu \nu} \]

What is this "matter"?
We saw earlier that, for a "fluid" in Minkowski spacetime, energy, momentum and the 4-velocity mix together to form the stress-energy tensor:

\[ T^{(m)}_{\mu\nu} = (\rho + p) U^\mu U^\nu + \phi^2 g_{\mu\nu} \]

\[ \text{CURVED SPACETIME} \]

\[ T^{\mu}_{\nu} = (\rho + p) U^\mu U^\nu + \phi^2 g^{\mu\nu} \]

The only "fluids" that can satisfy the Einstein field equations are those for which:

\[ T^{\mu}_{\nu} = 0 \Rightarrow U^\mu = 0 \Rightarrow U_\nu = -1 \]

\[ T^{0}_{0} = \rho = \rho(t) \]

\[ T^{ij} = \rho a^2 \delta^{ij} = \rho(t) a^2 \delta^{ij} \]

In principle, \( \rho(t) \) and \( p(t) \) are completely free functions of cosmic time; however, they might be related by the conservation of the stress-energy tensor:

\[ T^{\mu}_{\mu} = 0 \]
We have:

\[
T^\mu{}_{\nu, \lambda} = T^\mu{}_{\nu, \lambda} + \Gamma^\mu{}_{\nu \sigma} T^\sigma{}_{\lambda} + \Gamma^\mu{}_{\lambda \sigma} T^\sigma{}_{\nu}
\]

* For \( \nu = 0 \),

\[
T^\mu{}_{0, \lambda} = T^{\mu 0}{}_{,0} + \Gamma^\mu{}_{0 \sigma} T^{\sigma 0} + \delta_0^{\mu \nu} \delta_0^{\nu \lambda} = 3 \rho + 3 \frac{\dot{\alpha}}{\alpha} \rho + \dot{\alpha} \mathcal{R}^{ij} x^i x^j - \rho \dot{\rho}
\]

\[
= 3 \rho + 3 \frac{\dot{\alpha}}{\alpha} (\rho + \dot{\rho}) = 0
\]

\[
\Rightarrow \quad \rho + 3 \frac{\dot{\alpha}}{\alpha} (\rho + \dot{\rho}) = 0
\]

**Problem 74** — Show that the equation \( T^\mu{}_{\nu, \lambda} = 0 \) in fact vanishes identically.

Notice that the equation above, which is called the continuity equation, is equivalent to the statement that

\[
dE = - \dot{\phi} dV = \dot{\phi} dV \text{ in the work done by the system}
\]

\[
d(\phi V) = \dot{\phi} V + \phi V = - \dot{\phi} V
\]
\[ \dot{\rho} + (\rho + p) \dot{a} = 0 \]

Since the 3D physical volume \( V \sim a^3 \), we have that

\[ \frac{\ddot{V}}{V} = 3H \]

\[ \frac{\dot{a}}{a} = H \]

\[ \Rightarrow \rho + 3H(\rho + p) = 0 \]

**The Friedman Equations**

\[ G_{\mu \nu} = -8\pi G T_{\mu \nu} \Rightarrow \]

\[ 3\left( \frac{H^2}{a^2} \right) = 8\pi G \rho \quad (\dagger) \]

\[ -H^2 - 2\dot{\frac{a}{a}} - \frac{k}{a^2} = 8\pi G \rho \quad (\star \star) \]

It is also useful to write (and easier to remember):

\[ (\star) + 3(\star \star) \Rightarrow \frac{\ddot{a}}{a} = -4\pi G (\rho + 3p) \]

\[ \text{"Acceleration" } \quad \text{"Force"} \]

**These equations are functions of time only!**

(Notice that \( \rho_{\text{UV}}, \rho_{\text{EM}} \) and \( g_{\mu \nu} \) are not functions of time only.)
These equations are particular to Einstein’s General Relativity. In theories of gravity when, for instance,
\[ S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} R \rightarrow \frac{1}{16\pi G} \int d^4x \sqrt{g} \tilde{f}(R) \]
the Friedmann equations are modified.

We can write: \( \tilde{f}(R) = R + \mathcal{R}(R) \)

\( \mathcal{R} = \text{small corrections to General Relativity} \)

\( \mathcal{R}(R) \rightarrow +2\Lambda \)

Modified Einstein Equations:

\[ G_{\mu\nu} + \frac{\partial \tilde{f}_R}{\partial R} g_{\mu\nu} + \left( \frac{\partial \tilde{f}_R}{\partial R} - \frac{\mathcal{R}}{2} \right) g_{\mu\nu} - \nabla_\mu \nabla_\nu \frac{\partial \tilde{f}_R}{\partial R} = -8\pi G T_{\mu\nu} \]

\( \frac{\partial \tilde{f}_R}{\partial R} \) is here a scalar field.

For \( \kappa = 0 \), the 0-0 EFE becomes:

\[ H^2 = \tilde{f}_R (H^2 + H'') - \frac{1}{6} \tilde{f} + H^2 \tilde{f}_{RR} \tilde{R} = \frac{8\pi G \rho}{3} \]

\[ H' = \frac{dH}{d\eta} = \frac{1}{H} \frac{d\dot{H}}{d\eta} + \frac{H}{\dot{H} \rho} \]
Problem 15

Consider the metric:

\[ ds^2 = -N^2(t) dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \]

\[ \delta_{ij} = (1 + k \pi^2)^2 \delta_{ij} \]

(a) Compute the connections, Ricci tensor, and Ricci scalar.

(b) Find the modified Friedman equation. 0.0 for the theory with

\[ S_{\text{mod}} = \int d^n x \sqrt{g} \left( \frac{R + f(R)}{16 \pi G} \right) \]

by varying this action with respect to \( N \), and then taking \( N \to 1 \).

(c) By imposing the continuity equation for matter (the LHS of the modified Friedman equation), find the modified i-i Friedman equation.
Matter

Usually we assume that, almost always, the number density of particles is conserved, for each individual species (protons, e, H, He, Ne, ...).

\[ n(t) = \frac{N(t)}{V} \propto a^{-3} \]

The 4-momentum of the particles is

\[ p^\mu = (E, \vec{p}) \]

\[ p^\mu p_\mu = -m^2 \]

\[ \Rightarrow \text{(Flat spacetime)} \quad E^2 = \vec{p}^2 + m^2 \]

The energy density of a given species is:

\[ \rho_s = m_s \bar{E}_s = \text{average energy of } S \text{ particles} \]

The pressure (density) is:

\[ p_s^i = m_s \bar{E}^i_s \bar{n}^i_s \quad \text{(summed over } i) \]

If we assume that the distribution of this species is homogeneous and isotropic, then

\[ \sum_i p_s^i = m_s \sum_i \bar{E}^i_s \bar{n}^i_s = 3 \rho_s \]

\[ \Rightarrow \rho_s = \frac{1}{3} m_s \bar{E}_s \bar{n}_s \]
Interesting limits:

(a) \( m \gg m_0, \quad E \approx m \quad \Rightarrow \quad T_m = m, \quad M \\
\Rightarrow \quad p_m = \frac{1}{3} m \cdot m \cdot \frac{E^2}{m^2} \ll m \)

In this case, \( A_m = m \cdot a \cdot a^3 \) ("Dust")

"Pseudoless matter"

(b) Relativistic limit ("ultra-relativistic")

\( m \rightarrow 0, \quad E \approx m_0, \quad \overrightarrow{q} \cdot \overrightarrow{v} \approx |p| \)

\( \Rightarrow \quad p_r = m E_r \)

\( p_r = \frac{1}{3} m \cdot \frac{E_r^2}{m} \approx \frac{1}{3} m_0^2 E_r \)

Note that, in an expanding universe, \( E_r \sim \frac{1}{a} \)

\( \Rightarrow \quad "radiation" : \quad \begin{cases} 
A_r \propto a^{-4} \\
\rho_r = \frac{1}{3} \rho_r 
\end{cases} \)

Problem #16

Verify that the expressions for "dust" and "radiation" are consistent with the continuity equation.
(c) Cosmological constant ($\Lambda$)

Consider a type of matter/energy for which

$$\rho_\Lambda = \text{constant} = \frac{\Lambda}{8\pi G}$$

By the continuity equation, $\dot{\rho} + 3H(\rho + P) = 0$

$$\Rightarrow \quad \dot{\rho}_\Lambda = -\rho_\Lambda$$

**Negative pressure**

Example of negative pressure: bag model of QCD

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\begin{align*}
\text{gluons, } F_{\mu\nu} x^\nu \\
\Rightarrow \phi &\leq -\frac{1}{3} \rho
\end{align*}
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(d) General expression

$$\phi = W \rho$$

$W$: Equation of State

$\rho$ constant, or function of time.

$$\begin{align*}
W_m &= 0 \quad \text{(dark matter)} \\
W_r &= \frac{1}{3} \quad \text{(radiation)} \\
W_\Lambda &= -1 \quad (\Lambda)
\end{align*}$$
Problem #17

In the CPL parametrization, $W = W(a)$, with:

$$\frac{W_{\text{CPL}}}{W} = W_0 + W_a (a_0 - a) \quad [a_0 \to 1 \text{ in CPL}]$$

Find the functional form of $P(a)$ given this equation of state.
The density parameter

The Friedmann equation reads:

\[ \frac{3}{8\pi G} \left( H^2 + \frac{k}{a^2} \right) = \rho = \rho_1 + \rho_k + \ldots \]

or,

\[ \frac{3H^2}{8\pi G} = \rho - \frac{3k}{8\pi G a^2} \]

Let's define \( \rho_c(t) = \frac{3H^2}{8\pi G} \rightarrow \text{Today, } \rho_c^0 \approx 10^{-29} \text{ g/cm}^3 \text{ or } \rho_c^0 \approx 6 \text{ protons/m}^3 \text{ of all matter} \]

\[ \Rightarrow 1 = \frac{\rho}{\rho_c} + \frac{\rho_k}{\rho_c} \quad \Rightarrow 1 = \frac{\rho_1}{\rho_c} + \frac{\rho_2}{\rho_c} + \ldots + \frac{\rho_k}{\rho_c} \]

Density parameter:

\[ \Omega_s = \frac{\rho_s}{\rho_c} \quad \Omega_s^0 = \frac{\rho_{c0}(a=a_0)}{\rho_c(a=a_0)} \]

\[ \Rightarrow \text{Friedman equation: } \sum \Omega_i = 1 - \Omega_k \]

\[ \left\{ \begin{array}{l} \Omega_m + \Omega_\Lambda + \Omega_1 + \Omega_w + \Omega_k = 1 \\ H^2 = H_0^2 \left( \Omega_m a^{-3} + \Omega_\Lambda a^{-4} + \Omega_1 + \Omega_w a^2 + \ldots \right) \end{array} \right. \]
Spatial curvature rewritten

\[ \rho_m = -\frac{3k}{8\pi G a^2} \quad \Omega_k = \frac{A_k}{P_c} \]

⇒ Flat space ⇒ \( \Omega_k = 0 \), \( \sum \Omega_s = 1 \)

"Closed" space ⇒ \( \Omega_k < 0 \), \( \sum \Omega_s > 1 \)

"Open" space ⇒ \( \Omega_k > 0 \), \( \sum \Omega_s < 1 \)

The CMB indicates that (*with BAO + Ho*)

\[ \Omega_0 = \sum \Omega_s^0 \approx 1.01 \pm 0.01 \]

⇒ \( -0.08 \leq \sum \Omega_s^0 \leq 0.01 \) (WMAP7 years + BAO + H0)
Problem 18

Let's say that there is only matter, some spatial curvature, and also some type of matter \( \chi \) such that its equation of state is \( \omega_{\chi} = -\frac{2}{3} \).

(a) Compute the luminosity-distance as a function of redshift in the limit \( z \ll 1 \), to second order in \( z \) (we derive \( d_L(z) \) if \( k \neq 0 \)).

(b) An astronomical observation tells you that the deceleration parameter \( q_0 \) is:

\[ -0.4 \leq q_0 \leq -0.2 \]

Another observation tells you that:

\[ -0.05 \leq \omega_{\chi} \leq 0.05 \]

How do these observations limit the possible values of \( \Omega_{m0} \) and \( \Omega_{\chi0} \)?

Try to make a plot such as:

![Plot](image)