The Boulware-Deser mode in Zwei-Dreiben gravity

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–Work in progress–
The papers say the Universe is accelerating!

Einsein himself thought that a cosmological constant was “like adding mass to gravity”. This is incorrect, but it makes some sense to explore to what extent $\Lambda$ can be replaced by a massive graviton.

I will not discuss the success/failure of this idea as a cosmological model.

I will discuss some features of massive gravity, specifically, the Bouware-Deser ghost.

To make it simpler we shall do it in three dimensions “Zwei-Dreiben gravity” (3d bigravity), as discussed recently by Bergshoef et al.
Fierz-Pauli theory

Adding mass to the graviton was started by Fierz and Pauli back in 1939! Let $h_{\mu\nu}$ a symmetric rank-2 tensor and

$$L(h_{\mu\nu}) = -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^\mu\nu \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h - \frac{1}{2} m^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2\right).$$

For $m^2 = 0$ this Lagrangian describes a massless graviton with 2 degrees of freedom. And it is equal to Einstein-Hilbert Lagrangian linearized around flat space $\eta_{\mu\nu}$.

For $m^2 \neq 0$, describes a massive graviton with 5 degrees of freedom

Two challenges:

- Find a covariant form for the mass term
- Find an interacting non-linear theory, which is unitary.
Covariant interacting action, Isham-Salam-Strathdee, 1971

Consider a theory with two metrics, coupled by a potential $U$

$$I(g, f) = \int \left( \sqrt{g} R(g) + \sqrt{f} R(f) - U(f, g) \right).$$

$$U(f, g) = \frac{1}{2} m^2 (g_{\mu\nu} - f_{\mu\nu})(g_{\alpha\beta} - f_{\alpha\beta})(f^{\mu\alpha} f^{\nu\beta} - f_{\mu\nu} f_{\alpha\beta})$$

Linearizing around the flat space (and diagonalizing) we obtain the Lagrangian,

$$-\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h +$$

$$-\frac{1}{2} \partial_\lambda k_{\mu\nu} \partial^\lambda k^{\mu\nu} + \partial_\mu k_{\nu\lambda} \partial^\nu k^{\mu\lambda} - \partial_\mu k^{\mu\nu} \partial_\nu k + \frac{1}{2} \partial_\lambda k \partial^\lambda k - m^2 (k_{\mu\nu} k^{\mu\nu} - k^2)$$

In words, this theory has two gravitons:

- $h_{\mu\nu}$ is massless carrying 2 degrees of freedom
- $k_{\mu\nu}$ is massive carrying 5 degrees of freedom

$\{7$ degrees of freedom$\}$
The Boulware-Deser Ghost. Non linear dynamics

Now count the number of degrees of freedom in the full theory:

‘Number of dynamical fields’ – ‘number of gauge symmetries’

Writing each metric in ADM form,

\[ ds^2_g = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \]
\[ ds^2_f = -M^2 dt^2 + f_{ij}(dx^i + M^i dt)(dx^j + M^j dt) \]

The counting goes as follows:

\( N, N_i; M, M_i \) : 4 Lagrange Multipliers/4 Auxiliary fields
\( \mathcal{H}, \mathcal{H}_i \) : 4 first class constraints (overall diffeomorphisms)
\( \{ g_{ij}, \pi^{kl} \}, \{ f_{ij}, p^{kl} \} \) : \( 6 \times 4 = 24 \) dynamical fields

The number of degrees of freedom is

\[ \frac{1}{2} (24 - 2 \times 4) = 8 = 2 + 5 + 1 \]

- The extra mode (Boulware Deser) appears at non-linear level
- This mode is a ghost (negative kinetic energy)
Massive gravity is indeed not free of trouble

1. It has ghosts; Boulware-Deser mode just described

2. van Dam-Veltman-Zakharov discontinuity
   - The limit $m^2 \to 0$ does not give back general relativity
     \[
     ds^2 \approx \left( 1 - \frac{2Me^{-mr}}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{Me^{-mr}}{r}} + r^2 d\Omega^2
     \]
     This problem might be solved by the Vainshtein mechanism (no-linearities)

3. Causality issues. There exists modes propagating faster than light (Osipov-Rubakov, 2008)
Getting rid of Boulware-Deser mode. A long history

\[ I(g, f) = \int \left( \sqrt{g} R(g) + \sqrt{f} R(f) - U(f, g) \right). \]

- The degrees of freedom count shown before applies to the generic situation, for an arbitrary \( U(g, f) \).
- Perhaps there exists particular \( U(f, g) \) with special properties such that the Boulware-Deser mode does not show up?

After considerable work:

3. de Rham, Gabadadze (2010)
4. de Rham, Gabadadze, Tolley (2010): \( U \sim \text{Tr} \left( \sqrt{f^{\mu \nu}} g_{\nu \rho} \right) \)
5. Hassan, Rosen (2010),(2011)...

A special potential (apparently) does exist. It is best written in a first order formulation.
Hinterbichler-Rosen vielbein formulation (2011)

Let $e^a$ and $\ell^a$ two independent 1-form vielbeins. Let $R^{ab}$ and $Q^{ab}$ the associated 2-form curvatures. Consider the bigravity action (wedge $\wedge$ symbols omitted)

$$I = \int \epsilon_{abcd} \left( R^{ab} e^c e^d + \Lambda_1 e^a e^b e^c e^d + Q^{ab} \ell^c \ell^d + \Lambda_2 \ell^a \ell^b \ell^c \ell^d ight)$$

$$+ p_1 e^a e^b e^c \ell^d + p_2 e^a e^b \ell^c \ell^d + p_3 e^a \ell^b \ell^c \ell^d$$

- This is a nice, geometrical action (Lovelock spirit)
- The interaction is severely restricted. Only three parameters (at $d = 4$).
- Easily generalized to any dimension, and any number of vielbeins (multigravity)
- This bigravity action is claimed to have no Boulware-Deser ghost
- We shall critically check this assertion in three dimensions, where the canonical structure is simpler and well understood
Massive gravity in three dimensions. A long story too

1. Massive graviton. What graviton? $\int \sqrt{g} R$ propagates noting!

2. Topologically Massive Gravity, TMG (Deser et al):

$$\int \sqrt{g} R + \frac{1}{\mu} \left( w dw + \frac{2}{3} w^3 \right)$$

describes one helicity $\pm 2$, depending on sign of $\mu$.

3. New Massive Gravity, NMG (Bergshoeff, et al);

$$\int \sqrt{g} R + \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right)$$

describes two states $\pm 2$

4. 3d Bigravity (MB & Theisen)

$$\int \sqrt{g} R + \sqrt{f} R(f) - m^2 (g - f)^2$$

- linear theory: massless (0 states) plus a massive (two states) graviton;
- non-linear theory: three states, Boulware-Deser mode.
Hinterbichler-Rosen theory in three dimensions

Let $a^a$ and $\ell^b$ two 1-form dreibens in three dimensions

$$I[w, \pi, e, l] = \int \epsilon_{abc} \left( R^{ab} e^c + Q^{ab} \ell^c + p_1 e^a e^b \ell^c + p_2 \ell^a \ell^b e^c \right)$$

For simplicity we do not incorporate cosmological constants at each sector.

- This action was called Zwei-Dreiben gravity in Bergshoeff et al (2013)
- Note that only two allowed terms in the potential are allowed ($p_1$ and $p_2$)
- It is argued in Bergshoeff et al that the Boulware-Deser mode is absent
On general grounds, the Boulware-Deser mode is expected.

Performing a 2+1 decomposition of forms

\[ e^a_\mu \, dx^\mu = e^a_i \, dx^i + e^0_0 \, dt, \quad \ell^a_\mu \, dx^\mu = \ell^a_i \, dx^i + \ell^a_0 \, dt \]

(the same for \( \omega^a_\mu \, dx^\mu \), \( \pi^a_\mu \, dx^\mu \)) the action is “already” Hamiltonian

\[
I = \int \epsilon^{ij} \left( \frac{d\omega^a_i}{dt} e_{aj} + \frac{d\pi^a_i}{dt} \ell_{aj} \right) + \omega^a_0 \phi_{1a} + \pi^a_0 \phi_{2a} + e^a_0 G_{1a} + \ell^a_0 G_{2a}
\]

- \( \{ \omega^a_i, e^b_j \}, \{ \pi^a_i, \ell^b_j \} \) are 6 \times 4 = 24 canonical variables.
- \( \omega^a_0, \pi^a_0, e^a_0, \ell^a_0 \) are 3 \times 4 = 12 Lagrange Multipliers,
- \( \phi_{1a}, \phi_{2a}, G_{1a}, G_{2a} \) are 3 \times 4 = 12 constraints.

There are 6 gauge symmetries: 3 diffs + 3 Lorentz transformations. So, the 12 constraints split into 6 first class + 6 second class

\[
\frac{1}{2} (24 - 2 \times 6 - 6) = 3 = 2 + 1
\]

The Boulware-Deser mode is still around..!!
Checking out the details. I.

Could there be other (secondary) constraints? No.

\[ I[q^i, p_j, \lambda^\alpha] = \int dt \left( p_i \dot{q}^i - \lambda^\alpha \phi_\alpha(q, p) \right) \]

Consistency of constraints with time evolution imply,

\[ 0 = \frac{d\phi_\alpha}{dt} = \frac{\partial\phi_\alpha}{\partial q^i} \dot{q}^i + \frac{\partial\phi_\alpha}{\partial p_i} \dot{p}_i = [\phi_\alpha, \phi_\beta] \lambda^\beta = 0 \quad (\ast) \]

Despite being algebraic, these equations are not constraints!

- If \([\phi_\alpha, \phi_\beta] = C_\beta^\gamma \phi_\gamma\), the constraints are preserved. There is a gauge symmetry and \(\lambda^\alpha\) are arbitrary. Eq (\ast) imposes nothing.

- If \([\phi_\alpha, \phi_\beta]\) is invertible, then Eq. (\ast) implies \(\lambda^\beta = 0\). No gauge symmetry. End of algorithm

- If \([\phi_\alpha, \phi_\beta]\) has a some non-zero eigenvalues, some Lagrange multipliers are fixed, some are arbitrary. End of algorithm

For this family of actions, \(\frac{d\phi_\alpha}{dt} = 0\) never yields secondary constraints. Only conditions on the Lagrange multipliers, if any.
Bifurcations

To be fair, let us consider the consistency equation \([\phi_\alpha, \phi_\beta] \lambda^\beta = 0\) again and look at two toy models, say,

Model A

\[
\begin{pmatrix}
0 & c_1 \\
-c_1 & 0
\end{pmatrix}
\begin{pmatrix}
\lambda^1 \\
\lambda^2
\end{pmatrix} = 0.
\]

In this model, \(c_1 \neq 0\) is a constant and assumed different from zero. The only solution to the consistency condition is

\[
\lambda^1 = 0, \quad \lambda^2 = 0.
\]

Model B

\[
\begin{pmatrix}
0 & p \\
-p & 0
\end{pmatrix}
\begin{pmatrix}
\lambda^1 \\
\lambda^2
\end{pmatrix} = 0.
\]

- **Branch I:** For generic values of \(p\), the matrix is invertible and implies \(\lambda^1 = \lambda^2 = 0\).
- **Branch II:** Interpret this as an equation for \(p\) and impose a secondary constraint \(p = 0\). (Implies \(\dot{p} = 0\), and so on.)

A constraint system may have bifurcations, branches with different number of degrees of freedom.
Bifurcations and the Boulware-Deser mode

- The Boulware-Deser mode is present in 3d massive gravity.

- It can be hidden away, by choosing a branch with extra constraints (Bergshoeff et al, 2013). In other words, it can be set to zero by an initial condition.

- Nothing can prevent it to reappear under a generic perturbations
Checking out the details II. Consistency algorithm for Zwei-Dreiben gravity

\[ I[w, \pi, e, l] = \int \epsilon_{abc} \left( R^{ab} e^c + Q^{ab} \ell^c + p_1 e^a e^b \ell^c + p_2 \ell^a \ell^b e^c \right) \]

The equations of motion are,

\[ R^{ab} = -2p_1 e^a \ell^b - p_2 \ell^a \ell^b, \quad D e^a = 0, \]
\[ Q^{ab} = -p_1 e^a e^b - 2p_2 e^a \ell^b, \quad \nabla \ell^a = 0. \]

Using Cartan equations one finds integrability algebraic relations like

\[ (p_1 e^a + p_2 \ell^a) e_b \ell^b = 0 \iff [\phi_\alpha, \phi_\beta] \lambda^\beta = 0 \]

(plus others). These are exactly equal to the constraint consistency conditions. They can be solved by:

- Impose further constraints as \( e_a \ell^a = 0 \) (Bergoshoeff et al (2013)) \( \Rightarrow \) **2 degrees of freedom.**

- Interpret as equations for the Lagrange multipliers \( e_0^a, \ell_0^b, \ldots \) (MB & Pino (2013)) \( \Rightarrow \) **3 degrees of freedom.**
Is it “natural” to impose $e_a \ell^a = 0$?

The constraint $e_a \ell^a$ imposed by Bergoshoeff et al is consistent with but not a consequence of the equations of motion. The following field,

$$e^a_{\mu} = \begin{pmatrix} r & 0 & 0 \\ 0 & \frac{\sqrt{p_2}}{\sqrt{2p_1}} \frac{1}{r} & 0 \\ 0 & 0 & r \end{pmatrix},$$

$$\ell^a_{\mu} = \begin{pmatrix} -\frac{p_1}{p_2} r & 0 & 0 \\ 0 & c & \frac{r}{p_2} \sqrt{2c^2 p_1 p_2^2 - p_1^2} \\ 0 & 0 & -\frac{p_1}{p_2} r \end{pmatrix}. $$

solves all equations of motion (de Sitter space) and yet

$$e_a \ell^a = -\frac{\sqrt{2c^2 p_1 p_2^2 - p_1^2}}{\sqrt{2p_2 p_1}} \neq 0$$

is not zero. Imposing $e_a \ell^a = 0$ does kill interesting solutions.
Checking details III. Maximum rank

\[ [\phi_1(\xi), \phi_1(\chi)] = -\epsilon^{ab}_{\ c} \xi_a \chi_b D e^c, \quad [\phi_1(\xi), \phi_2(\chi)] = 0 \]
\[ [\phi_2(\xi), \phi_2(\chi)] = -\epsilon^{ab}_{\ c} \xi_a \chi_b \nabla l^c \]
\[ [G_1(\xi), G_1(\chi)] = 2p_1\epsilon_{abc} \xi^a \chi^b D l^c \]
\[ [G_2(\xi), G_2(\chi)] = 2p_2\epsilon_{abc} \xi^a \chi^b \nabla e^c \]
\[ [G_1(\xi), G_2(\chi)] = -2\epsilon_{abc} (D \xi^a \chi^b + \xi^a \nabla \chi^b) (p_1 e^c + p_2 l^c) \]
\[ [G_1(\xi), \phi_1(\chi)] = \epsilon_{bc} \xi^b \chi_a R^c - 2p_1\epsilon_{abc} \epsilon^{ad}_{\ e} \xi^b \chi_d l^c e^e \]
\[ [G_2(\xi), \phi_2(\chi)] = \epsilon_{bc} \xi^b \chi_a Q^c - 2p_2\epsilon_{abc} \epsilon^{ad}_{\ e} \xi^b \chi_d e^c l^e \]
\[ [G_1(\xi), \phi_2(\chi)] = -2\epsilon_{abc} \epsilon^{ad}_{\ e} \xi^b \chi_d (p_1 e^c + p_2 l^c) l^e \]
\[ [G_2(\xi), \phi_1(\chi)] = -2\epsilon_{abc} \epsilon^{ad}_{\ e} \xi^b \chi_d (p_2 l^c + p_1 e^c) e^e \]

This is a 12 × 12 matrix.

- Evaluating on generic solution \( e_a \ell^a \neq 0 \rightarrow \text{rank} = 6 \), as expected. Confirms 3 degrees of freedom.
- Evaluating on solutions with \( e_a \ell^a = 0 \rightarrow \text{rank} = 4 \); 2 degrees of freedom. (No hidden symmetry! Further constraints arise from \( \frac{d}{dt} e_a \ell^a = 0 \).)
Conclusions

- Massive gravity in its first order formulation is an attractive theory.

- Just like Lovelock choice avoids ghosts in higher curvature gravity, one could have expected that massive gravity would be unitary. Apparently it is not.

- In four dimensions the calculation is more complicated because the spin connection $\omega^{ab}$ does not have the same number of components as the vielbein $e^a$. Work in progress.