Homework exercises: LQG and black holes

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- Quasilocal first law using test particles

  a Using that the spacetime geometry and maxwell fields representing the most general (physically relevant) stationary BH solution—the Kerr-Newman solution (see R.M. Wald GR, page 313) has killing fields ξ (stationarity) and ψ (axy-symmetry); therefore

$$\mathscr{L}_\xi g_{ab} = \mathscr{L}_\psi g_{ab} = 0 \tag{1}$$
$$\mathscr{L}_\xi A_a = \mathscr{L}_\psi A_a = 0 \tag{2}$$

  Show that the following two quantities are conserved along the trajectory of a unit mass test particle and charge q with four-velocity w^a

$$\mathcal{E} = -(\xi^a w_a + q\xi^a A_a) \tag{3}$$
$$L = -(\psi^a w_a + q\psi^a A_a) \tag{4}$$

  Show that these can be interpreted as total energy per unit mass and axial component of angular momentum for certain inertial observers placed at infinity (which ones?).

  b Show that the local surface gravity $\bar{\kappa} = \kappa/(||\chi||)$ defined for the special family of local observers previously introduced is universal in the leading order for proper distance $\ell \ll 1$ (i.e. independent of the BH parameters); more precisely show that

$$\bar{\kappa} = \frac{1}{\ell}(1 + curvature\ \ corrections) \tag{5}$$

  Interpret in terms of Rindler geometry. In what limit can one say that the near horizon geometry is Rindler?

  c With the above ingredients reproduce the argument leading to the local first law and the local energy formula, given in the previous pages, in all detail.

  d The quasi-local energy and the Komar-like energy formula. Show that the local energy can be written in terms of the Komar like integral

$$E = -\frac{1}{8\pi}\int_H \epsilon_{abcd}\nabla^c u^d \tag{6}$$

  where u^a is the four velocity of the local stationary observers defined in previous pages.

- Quasilocal first law using fields and Einsteins equations

  a Define the energy momentum current $J_a = T_{ab}\chi^b$ and show that it is conserved, where $T_{ab} = T^{(0)}_{ab} + \delta T_{ab}$ (i.e. a background term plus a perturbation representing a small amount of matter infalling into and otherwise stationary Kerr-Newman black hole).

  b Recalling that at the horizon the Killing generator χ satisfies $\chi^a\nabla_a\chi_b = \kappa\chi_b$ (where κ is the surface gravity), and that the Killing generator vanishes at the bifurcate horizon, show that there exist an affine generator k^a (i.e. $k^a\nabla_a k_b = 0$) such that

$$\chi^a = \kappa V k^a, \tag{7}$$

  with V the affine parameter associated to k^a and singled out by the property that $V = 0$ at the bifrcate horizon.

c Gauss law is subtle when applied to null surfaces (see Wald GR pages 432-434). Show that the flux of J^a across the horizon takes the form

$$F_{horizon} = \int_H dV dS^2 T_{ab} \chi^a k^b \tag{8}$$

where V is the affine parameter of point (b), and dS^2 is the area element of the spheres $V = constant$.

d Use Gauss law in the region limited by the horizon and the world-sheet of local observers (see previous slide) combined with Raychaudhuri equation (Wald 9.2.32) and Einsteins equations to prove the quasilocal first law

$$\delta E = \frac{\bar{\kappa}}{8\pi} \delta A. \tag{9}$$

HINT: use the correct boundary condition for the expansion of k^a at $V = \infty$.

e Notice that the derivation of the local first law does not need the normalization of the Killing field χ at infinity. Therefore, the local first law is valid in a more general context than asymptotically flat spacetimes. Indeed no asymptotic conditions are necessary for its validity due to its intrinsically quasilocal nature.

- Quantum geometry

a Show that

$$\Sigma^i \delta\Gamma^i = d(e_i \wedge \delta e^i), \tag{10}$$

where Γ^i is the spin connection satisfying first Cartan structure equation

$$de^i + \epsilon^{ijk} \Gamma^j \wedge e^k \tag{11}$$

b From the differential equation defining holonomies (see notes of the second lecture) prove the following properties of holonomies.

i) The holonomy associated to an oriented path is independent of its parametrization.

ii) The holonomy along the product of two oriented paths that can be multiplied is the suitable product of holonomies.

iii) Under a gauge transformation the holonomy h_e along the path e is mapped to $g_t h_e g_s^{-1}$ where g_t is the value of the gauge transformation at the target of the path and g_s is the value of the gauge transformation at the source.

iv) Let $\phi : M \to M$ be a diffeomorphism then

$$h_{\phi(e)}[A] = h_e[\phi^* A] \tag{12}$$

- BH entropy

a In this exercise we will compute BH entropy in the simplest LQG scenario. We assume punctures of the horizon (area quantum excitations) are distinguishable. In the microcanonical ensemble we must count how many states there are such that the following constraint is satisfied (according to the form of the area spectrum in LQG)

$$C_1 : \sum_j \sqrt{j(j+1)}\, s_j = \frac{A}{8\pi \ell_g^2}, \tag{13}$$

Ignoring global constraints (due to Chern-Simons formulation) show that the number of states $d[\{s_j\}]$ associated with a configuration $\{s_j\}$ (where s_j denotes the number of punctures with spin j) is

$$d[\{s_j\}] = \Big(\sum_k s_k\Big)! \prod_j \frac{(2j+1)^{s_j}}{s_j!}. \tag{14}$$

b Look for the configuration that maximizes the entropy $\log(d[\{s_j\}])$ subject to the above constraint.

c Show that—using Stirling's approximation—the dominant configuration

$$\frac{s_j}{N} = (2j+1)e^{-\lambda\sqrt{j(j+1)}}, \tag{15}$$

where is a solution of

$$1 = \sum_j (2j+1)e^{-\lambda\sqrt{j(j+1)}}. \tag{16}$$

d Show that the entropy (defined as the value of $\log(d[\{s_j\}])$ on the dominant configuration) is

$$S = \frac{\gamma_0}{\gamma}\frac{A}{4\ell_p^2} \tag{17}$$

where $\gamma_0 = \lambda/(2\pi)$.

e Show that the previous result is in conflict with the local first law (and hence with the usual first law) unless $\gamma = \gamma_0$.

f Redo the exercise by imposing an additional constraint

$$C_2: \sum_j s_j = N.$$

and show by computing $S(A, N)$ that the conflict with the first law disappears and all values of γ are allowed. Obtain an expression of the entropy as a function of the area alone using the equation of state.