Perturbative Gravity Expansions

John H. Schwarz
California Institute of Technology
Pasadena, CA 91125, USA

ICTP-SAIFR Quantum Gravity School
September 2, 2013

1 Introduction

I have been asked to describe the status of perturbative expansions of gravity theories. This is a fairly active and interesting subject. I will try to be brief, because I have a lot to say about string theory, which is the main subject of my lectures at this school. The motivation for studying perturbative gravity expansions is to ascertain whether standard methods of quantum field theory are applicable to theories based on actions that contain the Einstein–Hilbert action. It was realized long ago that classical Einstein gravity in four dimensions, reinterpreted as a quantum theory, is probably nonrenormalizable. At the time, this was not considered an important problem, because relativists were not interested in quantum mechanics and particle physicists were not interested in gravity. That has changed dramatically over the past 30 years.

The most straightforward possibility to consider for a theory of quantum gravity is the original one: gravity formulated as a quantum field theory, based on pointlike constituents, just like the other forces. Most of what we know about quantum field theory is based on perturbation expansions. So the first question is whether there is a sensible way of defining them for quantum gravity, and that is what Nathan has asked me to discuss. If we conclude that this is not possible, this does not mean that we are out of business. The real question ought to be the following: given that Einstein's theory of general relativity gives the correct effective theory of gravity at energies small compared to the Planck scale, what are the possibilities for a UV completion that is consistent with the dictates of quantum theory. I am convinced that superstring theory/M-theory provides a multitude of successful UV completions, one of which ought to be realistic, even though many details are not yet fully understood. It is reasonable to ask whether there are any alternative possibilities.

2 Power counting

Quantum field theories are usually formulated by starting with a Lagrangian formulation of a classical field theory and then "quantizing" it. Schematically, one is given an action $S[\phi] = \int L dx$ that is a functional of fields $\phi(x)$. (The Lagrangian density L is a function of the fields and their derivatives.) The classical theory is given by extremizing the action, whereas the quantum theory is formally given by Feynman path integrals. For example, the expectation value of an observable A, which is made out of the fields, is given by $\int A \exp(iS/\hbar) D\phi$. As you probably know, it is not easy to make sense of such formulas.

Perturbation theory is an attempt to describe the path integrals by an expansion in powers of $\hbar$. Usually, $\hbar$ multiplies a coupling constant g, so the expansion is equivalently viewed as one in g, and one loses nothing by setting $\hbar = 1$. These series never converge; the best one can hope for are asymptotic expansions. The first question then is whether the individual terms in the expansion are well-defined. There may be infrared divergences, but they can be dealt with, so the more important question is whether there are ultraviolet divergences. If there are, can they also be dealt with?

Given a classical Lagrangian, one can determine by elementary considerations the dimensions of the parameters that appear as coefficients of the various terms — coupling constants, masses, etc. If all such parameters are dimensionless or a positive power of a mass, then the theory is called "power-counting renormalizable". If this is the case, the next question is whether or not the theory is asymptotically free or finite, which is determined by the sign of a beta function. If that is also the case, then there is considerable evidence for the existence of a quantum field theory that is well-defined both perturbatively and nonperturbatively. QCD is the outstanding example of an asymptotically free theory, and $\mathcal{N} = 4$ super Yang–Mills theories are examples of superconformal (and hence finite) theories.

How about gravity? The Einstein–Hilbert action is schematically of the form

$$S_{\rm EH} \sim G^{-1} \int \sqrt{-g} R dx,$$

where G is Newton's constant and R is the curvature scalar. In D-dimensional spacetime dx has dimension $-D$, and R, which is quadratic in derivatives, has dimension 2. Therefore, G has dimension $2-D$. It follows that the Einstein–Hilbert action is power-counting nonrenormalizable for $D > 2$.[1]

3 Supergravity Theories

The fact that a field theory is power-counting nonrenormalizable does not prove that it is UV divergent. What it means is that the loop expansion is *generically* expected to give rise to UV divergences whose cancellation would require the introduction of operators of higher dimension than appear in the initial Lagrangian. It is conceivable, however, that such terms do not arise: there might be a miraculous cancellation, for example. In the case of pure Einstein gravity in four dimensions, all potential one-loop counterterms vanish on shell or are a total derivative. Thus, as noted by 't Hooft and Veltman in 1974, it is UV finite at one loop. The first potential problem for the pure gravity theory arises at two loops. (Coupling to matter typically gives nonrenormalizable UV divergences already at one loop.) Two of my students (Goroff and Sagnotti) carried out the two-loop gravity calculation in 1985. They found that the expected UV divergence does in fact occur. This established the absence of a miraculous cancellation and proved perturbative nonrenormalizability.[2]

One way one can hope to obtain a better UV behavior is to extend the theory so that there are additional symmetries that ensure the cancellations required to prevent the UV divergences. This is the situation in supergravity theories. The dimension of Newton's constant is unchanged, of course, so these theories are still power-counting nonrenormalizable for $D > 2$. However, specific candidate counterterms are incompatible with their symmetries. For example, $\mathcal{N} = 1$ supergravity in four dimensions, which contains a gravitino field, the gauge field for local supersymmetry, in addition to the gravity field, is two-loop finite. The first divergence allowed by symmetry is at three loops. As the amount of supersymmetry is increased, the situation improves further. For example, in the case of the maximally supersymmetric 4d gravity theory, $\mathcal{N} = 8$ supergravity, the first counterterm compatible

[1]For $D = 2$ the Einstein–Hilbert action is topological, since $\sqrt{-g}R$ is a total derivative. String world-sheet theories are two-dimensional quantum gravity theories.

[2]Their result was confirmed many years later by others.

with the symmetry is believed to arise at seven loops. Nevertheless, as I will discuss, this theory might be UV finite to all orders in the perturbation expansion.

Even though there has been enormous progress in recent years in explicitly constructing multiloop amplitudes, a seven-loop supergravity calculation is not yet feasible. Fortunately, there are other ways of developing our intuition and testing our understanding. One way is by extending the analysis to arbitrary number of spatial dimensions, even a number that is not an integer. (The number of time dimensions is always one.) Another way is to study analogous issues for supersymmetric Yang–Mills theories. The super Yang–Mills results may be relevant to the supergravity problem if one can relate the two problems. Bern, Carrasco, and Johansson (BCJ) proposed a specific way of doing this (arXiv:1004.0476), which has been tested in several cases, but not proved in general.

4 Dimensional Reduction

Maximally supersymmetric Yang–Mills (MSYM) theories have 16 conserved supercharges. In 4d these are four Majorana (or Weyl) spinors of the $\mathcal{N} = 4$ theory, whereas in ten dimensions they belong to a single Majorana–Weyl spinor. Dimensional reduction to D dimensions is achieved by compactifying the ten-dimensional theory on a $(10 - D)$-dimensional torus and keeping only the zero Fourier modes on the torus. The formulas for amplitudes computed in this way can be analytically continued in D. For each number of loops L there is a maximum dimension $D(L)$ below which the amplitudes are UV finite. There is a completely analogous construction for maximally supersymmetric supergravity (MSG) theories. For example, one can toroidally compactify 11-dimensional supergravity.

In 1982 Brink, Green, and I showed that both MSYM and MSG are finite at one loop for $D < 8$. The way we did this was to explicitly compute corresponding one-loop four-particle superstring theory amplitudes, which are UV finite, and then to deduce the field theory amplitudes by evaluating the limit of the amplitudes in which the string excitations decouple. This means sending the string mass scale to infinity. The resulting amplitudes, analytically continued in dimension, were shown to be UV finite for $D < 8$ and to have poles at $D = 8$.

For each theory it is an interesting problem to determine the dimension $D(L)$, which is the onset of the L-loop UV divergence. In the case of MSYM there is a plausible argument that the answer for $L \geq 2$ should be $D(L) = 4 + 6/L$. (This formula is not applicable for $L = 1$, since $D(1) = 8$.) This formula came into question for a while, when Douglas speculated that 5d MSYM might actually be UV finite. The motivation involved its relationship to a certain superconformal theory in six dimensions. Remarkably, Bern, Douglas et al. (arXiv:1210.7709) settled the question by showing by explicit calculation that 5d MSYM has a UV divergence at six loops, exactly as predicted by the formula for $D(L)$.

The BCJ conjecture, if true, would imply that $D(L)$ should be the same for MSYM and MSG. In fact, in a series of papers, Bern and collaborators have verified the formula $D(L) = 4 + 6/L$ for MSG for $L = 2, 3, 4$. If that continues to be the case for all L, perhaps because the BCJ conjecture is correct, this would mean that $\mathcal{N} = 8$ supergravity in four dimensions is perturbatively UV finite!

5 Conclusion

There is a reasonable possibility that $\mathcal{N} = 8$ supergravity is perturbatively finite to all orders, even though it is power-counting nonrenormalizable. This would be "miraculous" inasmuch as there exists an operator, consistent with all known symmetries, that would ordinarily lead one to expect a seven-loop divergence. A miraculous cancellation of its coefficient would be required. Additional miraculous cancellations would be required for $L > 7$. If perturbative finiteness turns out to be the case, would it mean that $\mathcal{N} = 8$ supergravity is a consistent quantum theory? Could it be an alternative to superstring theory for a fundamental unified theory? Concerning the second question, it should be noted that there is no known plausible scenario for relating this theory to the standard model.

In the case of QCD we know that the quantum theory exists nonperturbatively, at least by the standards of rigor in theoretical physics. The same would not be the case for perturbatively finite $\mathcal{N} = 8$ supergravity. Whether or not it is perturbatively finite, one can ask for its (nonperturbative) UV completion. Only one such UV completion is known at the present time, and it is given by toroidally compactified superstring theory/M-theory. From this point of view, it seems to me that whether or not $\mathcal{N} = 8$ supergravity is perturbatively finite isn't very important.