Charged and Extremal Black Holes

For Reisner-Nordstrom black holes the metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

with

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$$

a) Find the two solutions for $f(r) = 0$. The outer horizon is located at the largest solution r_H.

b) Find the energy of a test mass m located near the horizon as measured by an observer at infinity. Derive the force that this observer needs to apply to keep the mass fixed at the location of the horizon. Show that this force can be written as

$$F = mg$$

and express g in $f'(r_H)$. The quantity g is called the surface acceleration.

c) Introduce the 'tortoise' coordinate r^* defined by

$$dr^* = \frac{dr}{f(r)}$$

Next, approximate $F(r)$ by $F(r) \sim F'(r_H)(r - r_H)$ to find the form of the metric near the horizon.

d) Show that in terms of the coordinates

$$U = C \exp \tfrac{1}{2} f'(r_H)(r^* + t) \qquad V = C \exp \tfrac{1}{2} f'(r_H)(r^* - t)$$

is regular at the horizon. Verify that for an appropriately chosen constant C the metric near the horizon becomes

$$ds^2 \sim dUdV + r^2(U,V)d\Omega^2$$

e) The analytic continuation to Euclidean signature is regular only for a certain periodicity of the time t. Find this period and argue that this implies that the black hole has a temperature equal to

$$T = \frac{f'(r_H)}{4\pi}$$

and express your result in the surface acceleration. This is the general expression for the Hawking temperature.

f) Derive the Bekenstein-Hawking formula

$$S = \frac{A}{4G}$$

by showing that it obeys the 1st law of thermodynamics

$$dM = TdS - \Phi dQ$$

where Φ is the standard Coulomb potential. Hint: use the defining equation $f(r_H) = 0$ for the horizon and demand that it remains valid after a combined variation of r_H, M en Q.

The Unruh temperature in de Sitter space

De Sitter space is described by the metric

$$ds^2 = -(1 - v^2)dt^2 + \frac{dr^2}{1 - v^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $v = H_0 r$.

a) Determine the force on a particle with mass m that is kept fixed at position r as measured by an observer O at the origin $r = 0$.

b) Determine the surface acceleration at the cosmological horizon.

c) The Hawking temperature is the temperature of the radiation that comes from the horizon as measured by the observer O. It is given by

$$T = \frac{g}{2\pi}$$

Find the temperature as measured by a static observer O' at radius r in de Sitter space.

d) Express your result in terms of the acceleration a of this observer O' in his own frame, and the surface acceleration a_0 at the horizon. Show that

$$T = \frac{\sqrt{a^2 + a_0^2}}{2\pi}$$

Cardy's formula

In string theory one often makes use of conformal field theory (CFT) techniques. One of these involves the calculation of the number of quantum states as a function of the energy E measured in units of the radius R of the worldsheet circle.

$$L_0 = ER$$

The number of quantum states can be deduced from the partition function

$$Z(\tau) = \mathrm{tr}\left(e^{2\pi i\tau(L_0 - c/24)}\right)$$

Here c denotes the central charge of the CFT and is a measure for the number of degrees of freedom. The parameter τ takes values in the upper half of the complex plane, and may be identified as the so-called complex modulus of the two dimensional torus obtained my modding out the complex plane by the lattice generated by the complex numbers 1 and τ.

$$\mathbf{T}^2 = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$$

The fact that one can make this identification implies that the partition function $Z(\tau)$ will have special properties under the modular group defined by

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

with a, b, c, and d integers satisfying $ad - bc = 1$.

a) Explain why this is the case.

We will now assume that $Z(\tau)$ is modular invariant. This means in particular that $Z(\tau) = Z(-1/\tau)$. We want to use this fact to determine the number of quantum states d_N with energy $L_0 = N$ for large values of N.

b) Express $Z(\tau)$ as a series in terms of d_N, and derive conversely an expression for d_N as a Fourier integral (or inverse Laplace transform) of $Z(\tau)$.

c) Argue that for large N the integral is dominated by the behavior of $Z(\tau)$ in the limit $\tau \to 0$.

d) Now use the modular invariance of $Z(\tau)$ to express its leading behavior for $\tau \to 0$ in term of the central charge c. Use the fact that the ground state of the CFT is unique.

e) Approximate $Z(\tau)$ by its leading behavior, and evaluate the resulting expression by making use of a saddle point approximation. What is the value for τ in the saddle point?

f) Show that

$$\log d_N = 2\pi\sqrt{\frac{c}{6}(N - \frac{c}{24})}$$