Problem set – Lecture on Inflation [Raul Abramo]

1. Compute the particle horizon at the time of decoupling ($t \simeq 380.000$ y, $z \simeq 1100$, $a \simeq 10^{-3}$), assuming that the scale factor evolves as a power-law as $a \sim t^{1/2}$.

Answer: $d_{pH} \simeq 200\ Kpc$.

2. Show that the power-law inflation model indeed yields a FRW background model whose scale factor that evolves as $a \sim t^p$. As you may recall, the power-law scenario has an inflaton potential $V(\phi) = M^4 e^{-\phi/s}$, where M and s are mass scales. Given that potential, solve the coupled equations:

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} &= 0 \\ 3H^2 &= \frac{8\pi}{M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) . \end{aligned}$$

Find p in terms of the scale s. *(Hint: up to a transient solution, the scalar field behaves as a logarithm.)*

3. Find the *exact* solutions that describe the perturbation modes v_k for the power-law inflation scenario. You will need to solve the mode equation:

$$v_k'' + (k^2 + \mu^2) v_k = 0 \ ,$$

where primes denote derivatives with respect to *conformal time*, $d\eta = dt/a(t)$, and $\mu^2 = -z''/z$, with $z = a^2\phi'/a'$. *[Hint: the solutions are basically Bessel functions.]*

Next, find the "Bogolyubov mapping" between the UV modes and the IR modes, and write the coefficient for the "growing mode" in the IR regime, D_k, in terms of the coefficients A_k and B_k that denote the positive- and negative-frequency modes of the UV regime. Namely:

$$\begin{aligned} UV\ [k \to \infty] : \quad v_k &= A_k\, e^{+ik\eta} + B_k\, e^{-ik\eta} \\ IR\ [k \to 0] : \quad v_k &= C_k\, z(\eta) + D_k\, z(\eta) \int^{\eta} \frac{d\eta'}{z^2(\eta')} . \end{aligned}$$

I will spare you the trouble of looking up the asymptotic behavior of Bessel functions. Here it is, to lowest order in the argument:

$$\begin{aligned} J_n(x \to 0) &\simeq \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n \\ J_n(x \to \infty) &\simeq \sqrt{\frac{2}{\pi\, x}}\ \cos\left(x - \frac{1+2n\pi}{4}\right) \\ Y_n(x \to 0) &\simeq -\frac{\Gamma(n)}{\pi}\left(\frac{2}{x}\right)^n \\ Y_n(x \to \infty) &\simeq \sqrt{\frac{2}{\pi\, x}}\ \sin\left(x - \frac{1+2n\pi}{4}\right) . \end{aligned}$$