MHD turbulence in stars, discs, & galaxies

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(...just google for Pencil Code)
Comments on yesterday’s exercise

- Saturation slower for smaller \( \eta \) (\( \rightarrow \) fluxes)
- Higher resolution
  - Larger growth rate
  - Similar saturation
  - Theoretical prediction ok

Results (all for \( \text{nu}=2e-2, \text{force}=0.07 \)):

<table>
<thead>
<tr>
<th>Res</th>
<th>2e-3</th>
<th>lambda</th>
<th>tsat</th>
<th>ampl</th>
<th>brms</th>
<th>urms</th>
<th>bfrms</th>
<th>b2kf_k1</th>
<th>Run</th>
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<td>2e-3</td>
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<td>850</td>
<td>0.0295</td>
<td>0.199</td>
<td>0.123</td>
<td>0.100</td>
<td>0.0311</td>
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Simulations of the solar dynamo?

• Tremendous stratification
  – Not only density, also scale height change
• Near-surface shear layer (NSSL) not resolved
• Contours of $\Omega$ cylindrical, not spoke-like
• (i) $R_m$ dependence (catastrophic quenching)
  – Field is bi-helical: to confirm for solar wind
• (ii) Location: bottom of CZ or distributed
  – Formation of active regions near surface
**Standard dynamo wave**

**Differential rotation**  
(faster inside)

**Cyclonic convection;**  
Buoyant flux tubes  
→ $\alpha$-effect

**Equatorward migration**  

**New loop**
ASH code: anelastic spherical harmonics

Brown et al. (2011)
Ghizaru, Charbonneau, Racine, ...

- Cycle now common!
- Activity from bottom of CZ
- but at high latitudes

Racine et al. (2011)
• Started in Sept. 2001 with Wolfgang Dobler
• High order (6th order in space, 3rd order in time)
• Cache & memory efficient
• MPI, can run PacxMPI (across countries!)
• Maintained/developed by ~80 people (SVN)
• Automatic validation (over night or any time)
• 0.0013 μs/pt/step at 1024^3, 2048 procs
• http://pencil-code.googlecode.com

• Isotropic turbulence
  – MHD, passive scl, CR
• Stratified layers
  – Convection, radiation
• Shearing box
  – MRI, dust, interstellar
  – Self-gravity
• Sphere embedded in box
  – Fully convective stars
  – geodynamo
• Other applications
  – Chemistry, combustion
  – Spherical coordinates
Dynamo wave from simulations
**Type of dynamo?**

- Use phase relation
- Closer to $\alpha^2$ dynamo
- Wrong for $\alpha\Omega$ dyn.

Oscillatory $\alpha^2$ dynamo

Mitra et al. (2010)
Calculate full $\alpha_{ij}$ and $\eta_{ij}$ tensors

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \mathbf{E} - \eta \mathbf{J}$$

• Imposed-field method
  – Convection (Brandenburg et al. 1990)
• Correlation method
  – MRI accretion discs (Brandenburg & Sokoloff 2002)
  – Galactic turbulence (Kowal et al. 2005, 2006)
• Test field method
  – Stationary geodynamo (Schrinner et al. 2005, 2007)

$\bar{\mathbf{E}} = \mathbf{u} \times \mathbf{b}$

$\alpha$ effect and turbulent magnetic diffusivity

$$\bar{E}_j = \alpha_{ij} B_j - \eta^*_{ij} \bar{J}_j$$
Calculate full $\alpha_{ij}$ and $\eta_{ij}$ tensors

$$\frac{\partial A}{\partial t} = U \times B - \eta J$$  
Original equation (uncurled)

$$\frac{\partial \overline{A}}{\partial t} = \overline{U} \times \overline{B} + \overline{u} \times \overline{b} - \eta \overline{J}$$  
Mean-field equation

$$\frac{\partial a}{\partial t} = \overline{U} \times \overline{b} + \overline{u} \times \overline{B} + \overline{u} \times \overline{b} - \overline{u} \times \overline{b} - \eta j$$  
fluctuations

Response to arbitrary mean fields

$$\frac{\partial a^{pq}}{\partial t} = \overline{U} \times b^{pq} + \overline{u} \times \overline{B}^{pq} + \overline{u} \times b^{pq} - \overline{u} \times b^{pq} - \eta j^{pq}$$
Test fields

\[ \overline{B}^{11} = \begin{pmatrix} \cos kz \\ 0 \\ 0 \end{pmatrix}, \quad \overline{B}^{21} = \begin{pmatrix} \sin kz \\ 0 \\ 0 \end{pmatrix} \]

\[ \overline{B}^{12} = \begin{pmatrix} 0 \\ \cos kz \\ 0 \end{pmatrix}, \quad \overline{B}^{22} = \begin{pmatrix} 0 \\ \sin kz \\ 0 \end{pmatrix} \]

\[ \overline{E}^{pq} = \alpha_{ij} \overline{B}^{pq}_j + \eta_{ijk} \overline{B}^{pq}_j,k \]

Example:

\[ \overline{E}_1^{11} = \alpha_{11} \cos kz - \eta_{113} k \sin kz \]

\[ \overline{E}_1^{21} = \alpha_{11} \sin kz + \eta_{113} k \cos kz \]

\[ \begin{pmatrix} \alpha_{11} \\ \eta_{113} k \end{pmatrix} = \begin{pmatrix} \cos kz & \sin kz \\ -\sin kz & \cos kz \end{pmatrix} \begin{pmatrix} \overline{E}_1^{11} \\ \overline{E}_1^{21} \end{pmatrix} \]

\[ \begin{pmatrix} \eta_{11}^* & \eta_{12}^* \\ \eta_{21}^* & \eta_{22}^* \end{pmatrix} = \begin{pmatrix} \eta_{123} & -\eta_{113} \\ \eta_{223} & -\eta_{213} \end{pmatrix} \]
Validation: Roberts flow

\[ U = u_{\text{rms}} \begin{pmatrix} -\cos k_x x \sin k_y y \\ + \sin k_x x \cos k_y y \\ \sqrt{2} \cos k_x x \cos k_y y \end{pmatrix} \]

\[ \frac{\partial a^{pq}}{\partial t} = U \times b^{pq} + u \times B^{pq} + u \times b^{pq} - u \times b^{pq} - \eta j^{pq} \]

SOCA

\[ k_x = k_y = k_f / \sqrt{2} \]

SOCA result

\[ \alpha = - \frac{1}{3} R_m u_{\text{rms}} \]

\[ \eta_t = \frac{1}{3} R_m u_{\text{rms}} k_f^{-1} \]

normalize

\[ \alpha_0 = - \frac{1}{3} u_{\text{rms}} \]

\[ \eta_{t0} = \frac{1}{3} u_{\text{rms}} k_f^{-1} \]
Test-field results for $\alpha$ and $\eta_t$

kinematic: independent of $R_m$ (2…200)

$\alpha_0 = -\frac{1}{3} \tau \langle \omega \cdot u \rangle$

$\eta_0 = \frac{1}{3} \tau \langle u^2 \rangle$

$\tau = (u_{\text{rms}} k_f)^{-1}$

$\alpha_0 = -\frac{1}{3} u_{\text{rms}}$

$\eta_0 = \frac{1}{3} u_{\text{rms}} k_f^{-1}$

**$R_m$ dependence for $B \sim B_{eq}$**

(i) $\tilde{\alpha}$ is small $\rightarrow$ consistency

(ii) $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ tend to cancel

(iii) making $\tilde{\alpha}$ small

(iv) $\tilde{\alpha}_2$ small

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<table>
<thead>
<tr>
<th>Run</th>
<th>$Rc_M$</th>
<th>$\tilde{b}^2$</th>
<th>$\tilde{b}^2$</th>
<th>$\tilde{\alpha}$</th>
<th>$\tilde{\eta}_h$</th>
<th>$\tilde{\eta}$</th>
<th>$\tilde{\alpha}$</th>
<th>$\tilde{\eta}_2$</th>
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<td>0.0</td>
<td>0.70 ± 0.03</td>
<td>0.67 ± 0.07</td>
<td>1.57</td>
<td>−0.14 ± 0.01</td>
<td>0.04 ± 0.05</td>
<td>−0.02 ± 0.06</td>
<td>0.09</td>
<td>0.12</td>
<td>1.03</td>
<td>0.01</td>
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<td>B</td>
<td>4</td>
<td>0.9</td>
<td>0.4</td>
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<td>0.00 ± 0.01</td>
<td>0.33 ± 0.01</td>
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<td>0.19 ± 0.01</td>
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<td>0.15 ± 0.00</td>
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<td>0.01 ± 0.01</td>
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<td>0.06</td>
<td>0.16</td>
<td>1.01</td>
<td>0.66</td>
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<td>H</td>
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<td>0.9</td>
<td>0.05 ± 0.01</td>
<td>0.13 ± 0.01</td>
<td>0.005</td>
<td>0.01 ± 0.04</td>
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<td>0.04 ± 0.01</td>
<td>0.05</td>
<td>0.10</td>
<td>1.03</td>
<td>0.64</td>
<td>44</td>
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Small-scale vs large-scale dynamo
**Low $Pr_M$ issue**

- Small-scale dynamo: $R_{m,\text{crit}} = 35-70$ for $Pr_M = 1$ (Novikov, Ruzmaikin, Sokoloff 1983)
- Rogachevskii & Kleeorin (1997): $R_{m,\text{crit}} = 412$
- Boldyrev & Cattaneo (2004): relation to roughness
- Ponty et al.: (2005): levels off at $Pr = 0.2$
Re-appearence at low $Pr_M$

Gap between 0.05 and 0.2 ?

Iskakov et al (2005)
Low $Pr_M$ dynamos with helicity do work

- Energy dissipation via Joule
- Viscous dissipation weak
- Can increase $Re$ substantially!

\[ \frac{\epsilon_M}{\epsilon_T} = 0.37 \ Pr_M^{1/2} \]
Discs & magneto-rotational instability

Vertical field $B_0$

\[
\frac{\dot{u}}{u} = 2\Omega u_y = B_0z b' \quad \quad \Omega(r) \propto r^{-q}
\]

\[
\frac{\dot{u}}{u} + (2 - q) \Omega u_x = B_0z b'
\]

\[
b_x = B_0z u_x
\]

\[
b_y = B_0z u_y - q\Omega b_x
\]

Dispersion relation

\[
\omega^4 - \omega^2 \left[ 2\omega_A^2 + 2(2 - q)\Omega^2 \right] + \omega_A^2 \left( \omega_A^2 - 2q\Omega^2 \right) = 0
\]

Alfven frequency: $\omega_A = v_A k$
Alfvén and slow magnetosonic waves

Degeneracy lifted by $q$ or $\Omega \neq 0$

$$\Omega(r) \propto r^{-q}$$

slow magnetosonic
Comparison with stratified runs

Dynamo waves
Versus
Buoyant escape

Brandenburg et al. (1995)
**Phase relation in dynamos**

Mean-field equations:

\[
\frac{\partial \overline{B}_x}{\partial t} = -\alpha \frac{\partial \overline{B}_y}{\partial z} + \eta \frac{\partial^2 \overline{B}_x}{\partial z^2}
\]

\[
\frac{\partial \overline{B}_y}{\partial t} = -\frac{3}{2} \Omega \overline{B}_x + \eta \frac{\partial^2 \overline{B}_y}{\partial z^2}
\]

Solution:

\[
\overline{B}_x = \sin k(z - ct)
\]

\[
\overline{B}_x = \sqrt{2} \frac{c}{\alpha} \sin[k(z - ct) + \frac{3}{4} \pi]
\]

\[
c = \frac{\omega}{k} = -\alpha \sqrt{3\Omega / 4\alpha k} = \mp \eta_T k
\]
Unstratified: also LS fields?
Low $Pr_M$ issue in unstratified MRI

Käpylä & Korpi (2010): vertical field condition
**LS from Fluctuations of $\alpha_{ij}$ and $\eta_{ij}$**

Incoherent $\alpha$ effect

\[ \partial_t B_r = -\partial_z (\alpha B_{\theta}) + D_t \partial_z^2 B_r , \]
\[ \partial_t B_{\theta} = -\frac{3}{2} \Omega B_r + D_t \partial_z^2 B_{\theta} , \]
**Magnetic fields in galaxies**

**Fig. 1.** Total intensity contours with magnetic field orientations indicated by vectors according to observations of NGC 4631 at 6.2 cm by Golla & Hummel (1992). The contours start at 100 mJ/beam and the interval is a factor of 2. The noise in the total intensity map is about 28 μJ/beam. Vectors are plotted only if the polarised intensity exceeds ~70 μJ/beam. The half power beam width (HPBW) is 40′. Assuming a distance of 7.5 Mpc the scale is 1′ = 2.2 kpc
Cioffi & Jones (1980)

\[ \frac{dI}{dz} = \varepsilon \]

\[ \frac{dP}{dz} = i\varepsilon P + P\varepsilon \]

\( P = \text{complex polarization} \)

\[ p = p_0 e^{2i\chi} \]

complex polarized emissivity

\[ \chi = \pi / 2 + \arctan \frac{B_y}{B_x} \]

intrinsic polarization

\[ f = 2n_{th} K B_\| \lambda^2 \]

\( K = 0.81 \text{ rad m}^{-2} \text{ cm}^3 \mu \text{G}^{-1} \text{ pc}^{-1} \)

\( P = \text{complex polarization} \)

\( R = \text{intrinsic Faraday rotation measure} \)
The general formulae governing synchrotron emission can be found in e.g. Pacholczyk (1970, Chap. 3 and references therein). For an optically thin source and neglecting circular polarisation the radiation can be described by means of the Stokes parameters \( I, Q, U \) governed by the equations

\[
\frac{dI}{d\ell} = \varepsilon, \quad (7)
\]

\[
\frac{dQ}{d\ell} = -f \cdot U - p \cdot \varepsilon \cdot \cos 2 \chi, \quad (8)
\]

\[
\frac{dU}{d\ell} = f \cdot Q - p \cdot \varepsilon \cdot \sin 2 \chi, \quad (9)
\]
Generation and interpretation of galactic magnetic fields

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Received April 5; accepted July 31, 1990

Fig. 12. RM for the S0 and A0 modes for two values of $H_e$, $i = 60^\circ$. Otherwise the same as Fig. 11

to emphasise that the variations of RM may be quite complex even in very simple models and that the interpretation of RM observations requires some care.
Fig. 6. Synthetical polarisation maps for the S0 mode computed for different values of $H_{rot}$, $i = 60^\circ$. The upper row is a grey scale representation of the total intensity $I$. The short lines in this map show the field orientation as inferred from the integrated Stokes parameters. The length of these lines is proportional to the polarised intensity. The lower row gives a grey scale representation of the degree of polarisation relative to its value for a homogeneous field. The strip on the left-hand side gives the calibration of the grey scale in per cent.

observes the polarisation angle $\chi_F$ at two different wavelengths and determines then the rotation measure $\Delta \chi_F/\Delta \lambda^2$. With this definition of the rotation measure we obtain

$$RM = (Q_i U_0 - U_i Q_0)/(Q_0^2 + U_0^2),$$

(22)

which can be computed from (18)-(21).
Integrating factor

Integrating factor

\[ \frac{dP}{dz} = i\Phi P + p\varepsilon \]

\[ e^{-iF} \frac{d}{dz} Pe^{iF} = p\varepsilon \]

\[ \frac{dF}{dz} = -f \]

Equation to be integrated

\[ \frac{dP}{dz} = P_0 \varepsilon e^{2i(\chi + \phi \lambda^2)} \]

\[ 2\phi \lambda^2 = F \]

\[ \Phi(z) = \text{Faraday depth} \]

\[ P = Q + iU \]
Faraday space

\[ P = p_0 \int_{-\infty}^{\infty} \varepsilon e^{2i(\chi + \phi \lambda^2)} \, dz \]

\[ P = p_0 \int_{-\infty}^{\infty} \frac{dz}{d\phi} \varepsilon e^{2i(\chi + \phi \lambda^2)} \, d\phi \]

\[ P \left( \lambda^2 \right) = \int_{-\infty}^{\infty} p_0 \frac{\varepsilon e^{2i\chi}}{Kn_{th} B_z} e^{2i\phi \lambda^2} \, d\phi \]
Stationary phase

Helical perpendicular fields with pos and neg helicity

\[ B = \begin{pmatrix} B_1 \sin kz \\ B_1 \cos kz \\ B_0 \end{pmatrix} \quad B = \begin{pmatrix} B_1 \cos kz \\ B_1 \sin kz \\ B_0 \end{pmatrix} \]

Integration

\[ Q = p_0 \int_{-\infty}^{\infty} \varepsilon \cos 2(\chi + \phi \lambda^2) \, dz \]
\[ U = p_0 \int_{-\infty}^{\infty} \varepsilon \sin 2(\chi + \phi \lambda^2) \, dz \]
\[ \phi = Kn_{th} B_0 z \]
Stationary: \[ Q = p_0 \int_{-\infty}^{\infty} \varepsilon \cos 2(\chi + \phi \lambda^2) \, dz \]
Positive helicity as input

\[ \hat{Q}(\phi) = \int_{\lambda_{\text{min}}^2}^{\lambda_{\text{max}}^2} Q(\lambda^2) e^{2i\phi \lambda^2} d\lambda^2 \]

\[ \hat{U}(\phi) = \int_{\lambda_{\text{min}}^2}^{\lambda_{\text{max}}^2} U(\lambda^2) e^{2i\phi \lambda^2} d\lambda^2 \]

\[ H(\phi) = \text{Im}(\hat{Q}\hat{U}^*) \]

Why? By analogy; just try
Negative helicity as input

Helicity proxy also reversed in sign!
Dynamo effect in SNR

\[ \rho \frac{\partial U}{\partial t} = -\nabla P + \frac{1}{c} J \times B + e(n_i - n_e)E + F_v \]

\[ \nabla \times B = \frac{4\pi}{c} \left( J + J_{cr} \right) \quad n_i + n_{cr} = n_e \]

To be solved with induction equation and continuity equation, isothermal EOS
Introduces pseudoscalar: helicity

\[ J_{cr} \cdot B_0 \rightarrow \alpha \text{ effect} \]

\[ \alpha \text{ effect important for large-scale field in the Sun} \]

\[ \frac{\partial \overline{B}}{\partial t} = \nabla \times \left( \overline{U} \times \overline{B} + \overline{u} \times \overline{b} - \overline{J} / \sigma \right) \]

\[ \overline{E} \equiv \overline{u} \times \overline{b} = \alpha \overline{B} + \ldots \]

\[ \overline{E}_i = \alpha_{ij} \overline{B}_j - \eta_{ij} \overline{J}_j + \ldots \]
Bell instability

\[ \gamma_B^2 = \left( \frac{4\pi J_{cr}}{c_B} k_z - k^2 \right) v_A^2 \]

\[ J = \frac{4\pi}{c} J_{cr} / kB_0 \]

Bell (2004): \( J = 2 \)
Zirakashvili et al (2008): \( J = 16 \)

Continued growth in both cases! \( \Rightarrow \alpha \) effect important?
**New simulations**

- $512^3$ resolution, non-ideal ($\text{Re} = \text{Lu} < 300$)
- Larger $J$ parameter (80 and 800)
- Most unstable $k/k_1 = 40$ and 400 (unresolved)
- Measure alpha and turbulent diff. tensor
- Related to earlier work by Bykov et al. (2011)
Bell instability $\rightarrow$ turbulence ($\ell=80$)
3 stages

- Bell instability, small scale, $k/k_1 = 40$
- Accelerated large-scale growth
- Slow growth after initial saturation
Dynamo number, turb diff

Critical value 1, turb diff
>> microscopic value
New theory for magnetic spots

- Minimalistic model
- 2 ingredients:
  - Stratification & turbulence
- Extensions
  - Coupled to dynamo
  - Compete with rotation
  - Radiation/ionization
Conclusions

• Spherical dynamos
  – Equatorward migration
  – TFM
• Low PrM issue
• MRI dynamos
  – Phase relation
• Galactic dynamos
  – RM synthesis
• Bell-dynamo instability