MHD turbulence in the solar corona and solar wind

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Waves, turbulence, low-frequency fluctuations

Consider the 3D MHD equations,

\[ u = u(x, y, z, t) = \text{plasma velocity} \]

\[ B = B(x, y, z, t) = \text{magnetic field} \]

\[ \rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + J \times B + \rho \nu \nabla^2 u + f_u(t) \]

\[ \frac{\partial B}{\partial t} = \nabla \times (u \times B) + \eta \nabla^2 B + f_b(t) \]

\[ J = \nabla \times B = \text{current density}, \quad \nabla \cdot B = 0, \quad \nabla \cdot u = 0 \]

And we will assume also a background magnetic field \( B_0 \), so \( B = B_0 + b \).
We numerically (DNS) solve the MHD equations using a pseudospectral code. The fields are evolved in k-space, and we add forcing terms $f^u_k(t)$ and $f^b_k(t)$ to achieve a steady state. The forcing terms are narrow in k-space ($1 \leq k \leq 2$) and include a memory part and a random part. A component of the forcing is of the form:

$$\alpha_{i+1} = m\alpha_i + \sqrt{1 - m^2}r_{i+1}$$

with $0 \leq m \leq 1$ the memory parameter and $r_i$ a (uniform) random number.
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It can be seen that

$$<\alpha_n\alpha_{n+l}> \rightarrow m^l, \quad \text{when } n \rightarrow \infty$$

By constructing a (discrete) time series with $t_n = n\Delta t$,

$$<\alpha(t)\alpha(t + \tau_f)> \rightarrow e^{-t/\tau_f}$$

with $\tau_f = l\Delta t, \ m = 1 - \Delta t/\tau_f$.

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We use units of length $L_0 = 1/2\pi$ size of the box, that is $L_0 = 1$, and $u_0 = \langle b^2 \rangle^{1/2} (t = 0) = \langle u^2 \rangle^{1/2} (t = 0) = 1$, and $t_0 = L_0/u_0 = 1$. We chose $\tau_f = 1$. 
We consider probes inside the box, where we can measure magnetic and velocity fluctuations, \( b(x, y, z, t) \), \( u(x, y, z, t) \) as a function of time.

Frequency spectra

![Graphs of frequency spectra for different values of B.](image-url)
Recall Alfvén waves, satisfy,

\[ w = k \cdot B_0 = k_{\parallel} B_0 \]

Here \( B_0 = 8 \) and we see the peaks at multiples of \( B_0 \). Also plotted with light line the spectrum in the case of linear ideal MHD (i.e., only waves).
We define the Signal to Noise Ratio,

$$\text{SNR} = \log_{10} \left[ \frac{P(w_0)}{P_0(w_0)} \right].$$

and the Wave Power Ratio,

$$\text{WPR} = \frac{\int_{w_1}^{w_2} [P(w) - P_0(w)] dw}{\int_{w>0} P(w) dw}$$

SNR = 0, 0.3, 0.6, 1.5, 3.1 for $B_0=0, 1, 2, 8, 16$

WPR = 0, 0.1, 0.13, 0.03, 0.02 for $B_0=0, 1, 2, 8, 16$

Although waves can be clearly distinguished (large SNR ratio), most of the power (WPR ratio) is on eddies (turbulent fluctuations).
We can also look at the frequency spectrum of individual modes in k-space,
And the real and imaginary parts of a mode $b_k$, which for the case of a wave should be a circle
Low-frequency fluctuations, $1/f$ noise

Look at low frequency fluctuations (Dmitruk & Matthaeus, Phys. Rev. E 76, 036305, 2007)

see solar wind observations in Matthaeus & Goldstein, Phys. Rev. Lett. 57, 495, 1986
Frequency spectra for different modes $b(k)$
Consider hydrodynamics (3D HD) → no $1/f$ !!
Other systems with low frequency fluctuations: MHD2D, HD2D
Frequency spectra of individual modes

![Graphs showing frequency spectra for MHD2D and HD2D modes](image)
Ideal MHD ($\eta = \nu = 0$) with non-zero magnetic helicity, $H_m < \mathbf{a} \cdot \mathbf{b}$, where $\nabla \times \mathbf{a} = \mathbf{b}$ is the potential vector. No DC field (i.e. $B_0 = 0$).

Time behavior of the lowest k mode

\[ \text{Re}\{b_{z[k=(1,0,0)]}\}, N=16, H_m = -0.008 \]

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Frequency spectra

N=16, $H_m = -0.008$

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N=16, $H_m = -0.395$
Correlation function $< b(t_0)b(t_0 + t) >$

Lack of single correlation time $\rightarrow \frac{1}{f}$
Phase space behavior of modes in complex plane, $\mathbf{k} = (1, 0, 0)$

![Graphs showing phase space behavior for different time intervals](image-url)
\textbf{\textit{k}} = (2, 0, 0)
We also consider MHD and HD inside a sphere, with or without rotation $\Omega$,

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho} \nabla P + \mathbf{j} \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

with null boundary conditions, $\mathbf{B} \cdot \hat{n} = 0 \mathbf{u} \cdot \hat{n}$ at the radius $R = 1$ of the sphere.

We use a Galerkin spectral code, with Chandrasekar-Kendall functions in the sphere (Dmitruk, Mininni, Pouquet, Servidio, Matthaeus, Phys. Rev. E 2011)
Ideal MHD in the sphere, with rotation

$q_{\text{max}} = 5$, $\Omega = 0$

$q_{\text{max}} = 5$, $\Omega = 16$
Frequency spectra

$q_{\text{max}}=5, \Omega=0$

$q_{\text{max}}=5, \Omega=16$
Ideal HD in the sphere, with rotation

\[ \text{Re}\{u_z[k=(1,0,0)]\} \quad N=32, \quad \Omega=0 \]

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Frequency spectra

$P(f)$

$N=32, \Omega=0$

$N=32, \Omega_y=16$
Recall that in our defined units, \( t_0 = L_0/u_0 = 1 \), and the eddy turnover time (non-linear time) is

\[
\tau_k = l_k/u_k = (k u_k)^{-1}
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So, for \( k = 1 \), we get \( \tau_k = t_0 = 1 \). Following Kolmogorov scaling, \( u_k \sim k^{-1/3} \) we can get

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**Which is the origin then of these long time fluctuations that we see in many systems?**

Non-local interactions between large scale modes and a thermal bath of small scale fluctuations give long time fluctuations, i.e., low-frequency fluctuations.

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\frac{\partial b(k)}{\partial t} = -ik \sum_{k=p+q} u(q)b(p)
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A nice example: geomagnetic field reversals
We solve ideal MHD equations inside a rotating sphere, and consider the dynamics of the magnetic dipole moment

\[ \mu = \frac{1}{2} \int \mathbf{r} \times \mathbf{j} \, dV \]
We solve ideal MHD equations inside a rotating sphere, and consider the dynamics of the magnetic dipole moment

\[ \mu = \frac{1}{2} \int r \times j \, dV \]

We found reversals!! These are long-time fluctuations...
Magnetic dipole $1/f$ frequency spectrum
**Waiting time** between reversals is compatible with geological observations (Cande-Kent 95).