The very Early universe
Lecture 2.

1. If $\tilde{g}_{ab} = \Omega^2 \mathring{g}_{ab}$ and $\tilde{\phi} = \Omega^{-1}\phi$, then $(\tilde{\Box} - \frac{1}{6}\tilde{R})\tilde{\phi} = \Omega^{-3}(\mathring{\Box} - \frac{1}{6}\mathring{R})\phi$. (This is a statement of conformal invariance of the equation). In FLRW space-times, the physical metric can be written as $g_{ab} = a^2(\eta)\, \mathring{g}_{ab}$ where $\mathring{g}_{ab}$ is flat, (adapted to conformal time η). If $e_k(\eta)$ is a normalized basis for $\mathring{\Box}\phi = 0$, show that $\chi_k(\eta) = a(\eta)\, e_k(\eta)$ satisfies the flat space wave equation in presence of a time dependent potential: $\partial_\eta^2 \chi_k + (k^2 - a^2 \frac{R}{6})\chi_k = 0$
This equation is often used to describe tensor modes in cosmology.

2. Given a basis $e_k(\eta)$ for ~~tensor~~ perturbations $\tau(\vec{x},\eta)$, show that the 2-point function $\langle 0|\, \hat{\tau}(\vec{x}_1,\eta_1)\, \hat{\tau}(\vec{x}_2,\eta_2)\, |0\rangle = \hbar \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\, e_k(\eta_1)\, e_k(\eta_2)$. Further, show that this 2-point function is invariant under spatial translations and rotations. (Here, spatial topology is $\mathbb{R}^3$ rather than $\mathbb{T}^3$.)

3. Suppose we have two (4th adiabatic order) bases $e_k(\eta)$ and $\underline{e}_k(\eta)$, with $\underline{e}_k(\eta) = \alpha_k\, e_k(\eta) + \beta_k\, e_k^*(\eta)$. Using the fact that both satisfy the normalization condition, show that $|\alpha_k|^2 - |\beta_k|^2 = 1$ for all k.
Calculate the total number of "barred" particles
* $\langle 0 |\, \underline{N}\, | 0\rangle$ in the unbarred vacuum in terms of the Bogoluibov coefficients $|\beta_k|^2$
* show that if $\underline{e}_k(\eta) = e^{if(k)}\, e_k(\eta)$, then $|\underline{0}\rangle = |0\rangle$, i.e. the vacua defined by $\underline{e}_k(\eta)$ and $e_k(\eta)$ agree.