Fundamentals of magnetohydrodynamics

Part I

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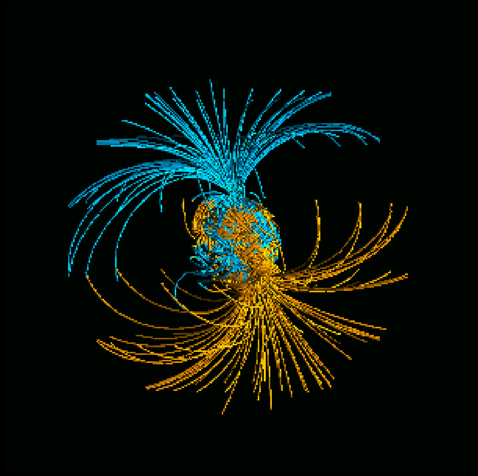
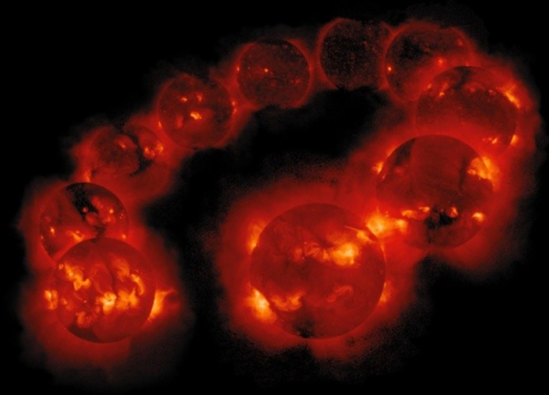
What do we mean by magnetohydrodynamics ?

- It is a fluid-like theoretical description for the dynamics of matter
- Baryonic matter in the Universe is mostly hydrogen.
- At temperatures above 10^4 K it becomes a hydrogen plasma, i.e. a gas made of protons and electrons
- The large scale behavior of this gas can be described through fluidistic equations (Navier-Stokes).
- This fluid is made of electrically charged particles and therefore it suffers electric and magnetic forces.
- Not only that, these charges are sources of self-consistent electric and magnetic fields. Therefore, the fluid equations will couple to Maxwell’s equations.
- At small spatial scales (and fast timescales) non-fluid or kinetic effects become non-negligible.

Magnetic fields in Astrophysics

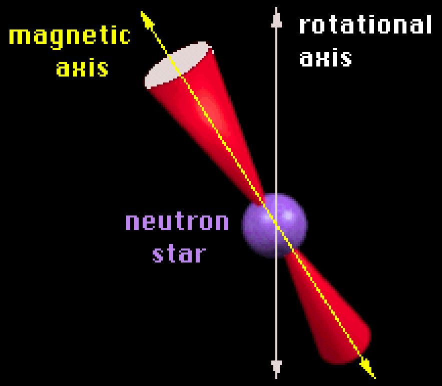


Earth and planets



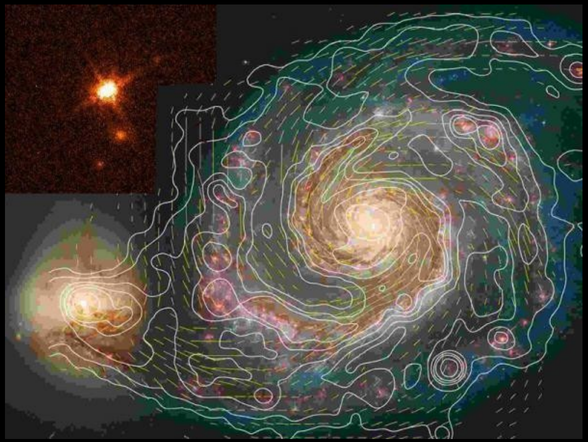
Sun and stars

Interstellar medium



Pulsars

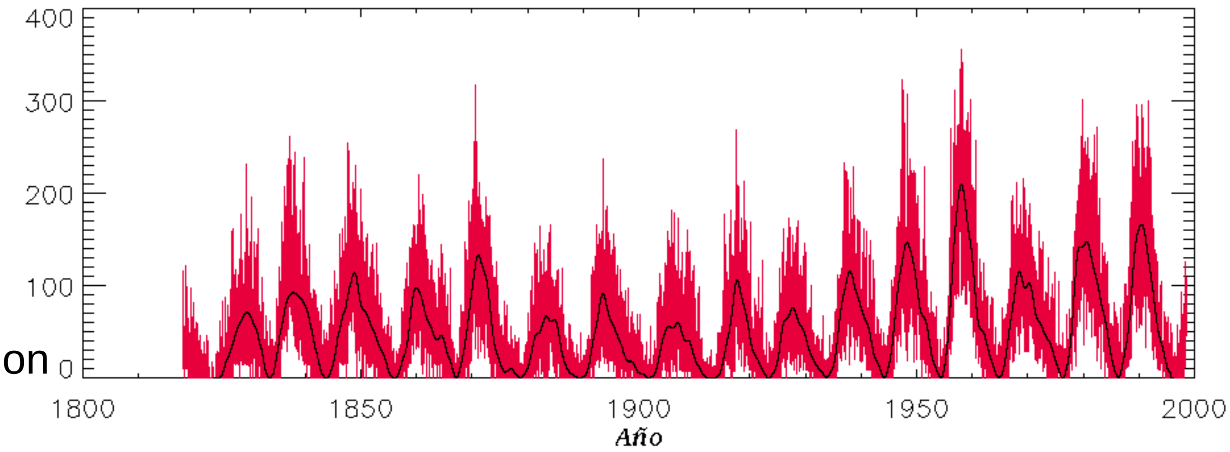
Accretion disks



Galaxies

Magnetic field of the Sun

- Number of sunspots vs. time

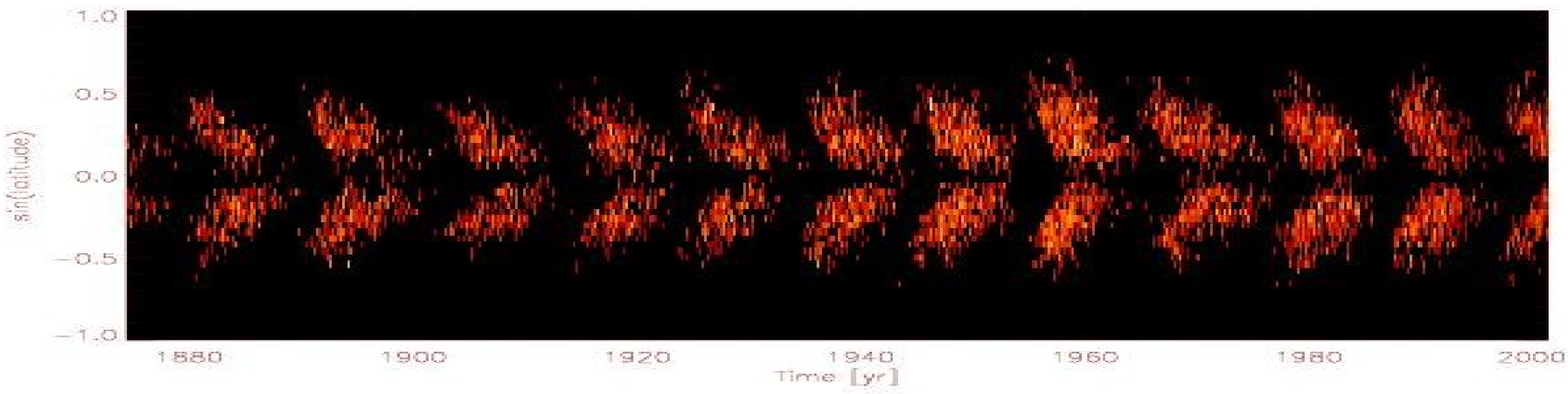


- It clearly shows an 11 yr period with irregularities in its maxima, its periods and rise-fall times.

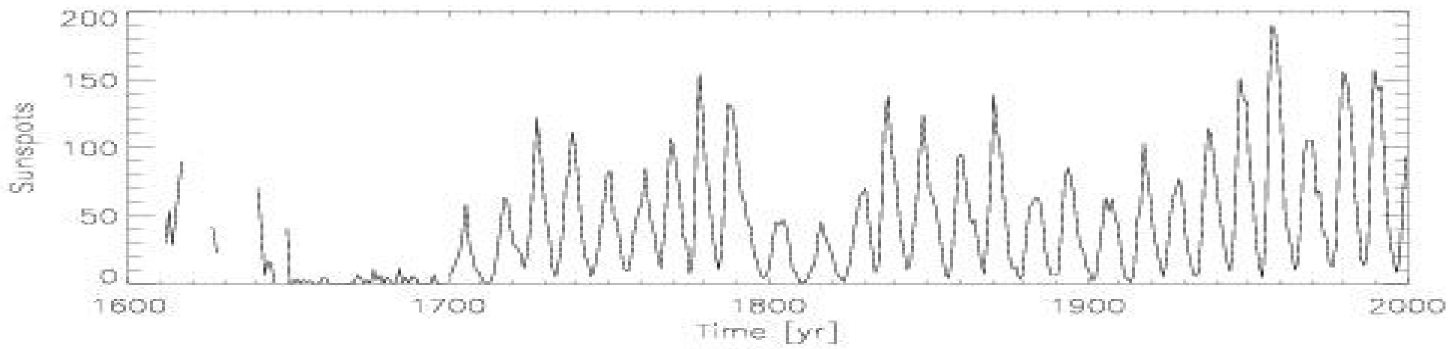
- Area covered by spots as a function of latitude and time.

- At the beginning of each cycle, sunspots are born at latitudes of $\pm 30^{\circ}$ and migrate to the Equator.

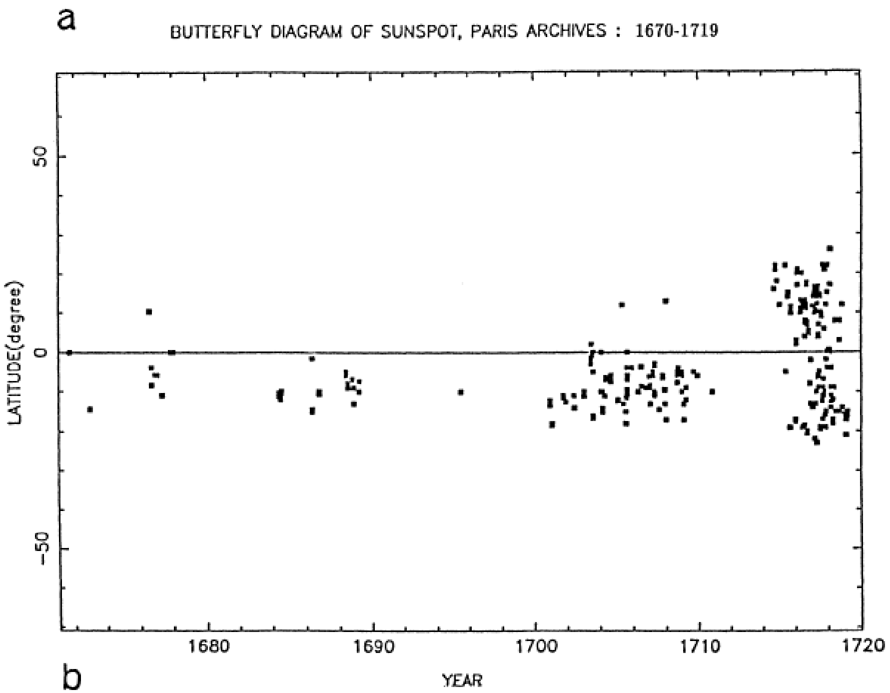
- Magnetic polarities are reversed from one cycle to the next and are different at different hemispheres (Hale´s law)



Maunder minimum



- Wolf Number vs. time
- Maunder minimum lasts from 1650 to 1700.
- There is evidence of more Maunder-like events (Beer 2000).
- N-S asymmetries were enhanced during the Maunder minimum (Ribes & Nesme-Ribes 1993).

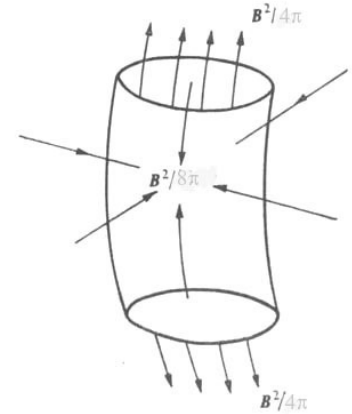


MHD equations

- The equations for the fluid are:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \bullet (\rho \vec{u}) \qquad p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\rho \frac{\partial \vec{u}}{\partial t} = -\rho(\vec{u} \bullet \vec{\nabla})\vec{u} - \vec{\nabla} p + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{F}_{ext} + \vec{\nabla} \bullet \vec{\sigma}_{visc}$$



$$\frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}(\vec{B} \cdot \vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{8\pi}\right)$$

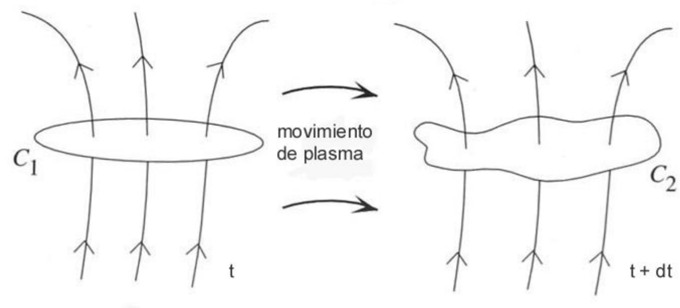
Magnetic pressure and magnetic tension

- The magnetic field is generated by the plasma, and satisfies the so-called induction equation.

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad \vec{\nabla} \bullet \vec{B} = 0$$

- It is obtained as a result of Ohm's law (see below) and Faraday's equation.

$$\vec{E} + \tfrac{1}{c}\vec{u} \times \vec{B} = \frac{1}{\sigma}\vec{J}$$



Frozen-in condition

MHD equations

- The MHD equations are:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \bullet (\rho \vec{u}) \qquad p = p_0 (\frac{\rho}{\rho_0})^{\gamma}$$

$$\rho \frac{\partial \vec{u}}{\partial t} = -\rho(\vec{u} \bullet \vec{\nabla})\vec{u} - \vec{\nabla} p + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{F}_{ext} + \vec{\nabla} \bullet \sigma_{visc}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad \vec{\nabla} \bullet \vec{B} = 0$$

- These equations describe a large number of important plasma processes, such as

- instabilities and wave propagación (Alfven and magnetosonic waves)
- dynamo mechanisms to generate magnetic fields
- MHD turbulence
- magnetic reconnection

- Note that even though the electric field is not present, it does not mean that it is not relevant

$$\vec{E} + \tfrac{1}{c}\vec{u} \times \vec{B} = \frac{1}{\sigma}\vec{J}, \qquad \eta = \frac{c^2}{4\pi\sigma}$$

Kinematic dynamos

- If we assume the magnetic field **B** to be very small, the MHD equations decouple. We can first solve the equations of motion. For instance, in the incompressible limit

$$\frac{\partial u}{\partial t} = -(\vec{u} \bullet \vec{\nabla})\vec{u} - \frac{1}{\rho}\vec{\nabla} p + \nu \nabla^2 \vec{u} \quad , \qquad \vec{\nabla} \bullet \vec{u} = 0$$

- Now that we know $u(x,t)$, we can solve the induction equation to obtain $B(x,t)$

$$\frac{\partial B}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad \vec{\nabla} \bullet \vec{B} = 0$$

- This particular and convenient approximation is known as the kinematic dynamo. Note that the induction equation is linear in $B(x,t)$ for any given $u(x,t)$. For stationary flows, there will be a dynamo solution whenever

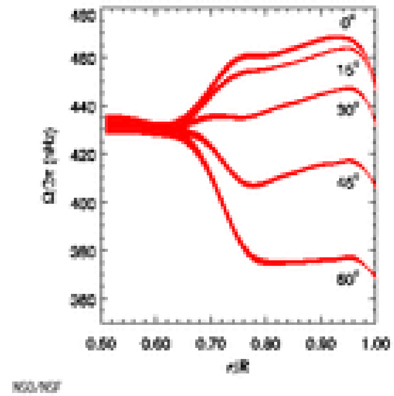
$$B(x,t) = B_0(x) e^{\gamma t} \quad , \qquad \gamma > 0$$

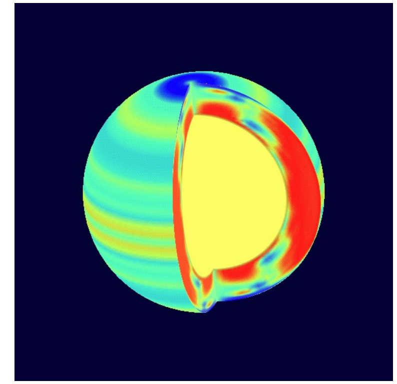
What kind of permanent flows are ubiquitous in astrophysical objects ?

Rotation and Convection

Rotation (macro)

- Radial differential rotation
- Latitudinal differential rotation
- Meridional flow

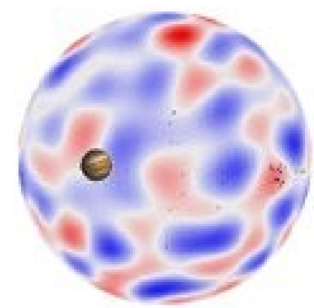




Omega effect

Convection (micro)

- Helicoidal convective turbulence
- Giant cells (driven by Coriolis)
- Regular and stochastic components



Alpha effect

Differential rotation (Omega effect)

- The Sun rotates with a well documented differential rotation profile, obtained from helioseismology.

- Radial: Almost solid body rotation in the interior at a rate

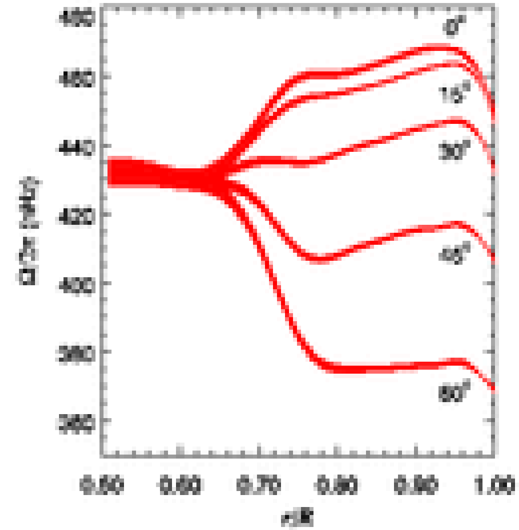
$$\omega_{core} \cong 2\,10^{-6}\, s^{-1}$$

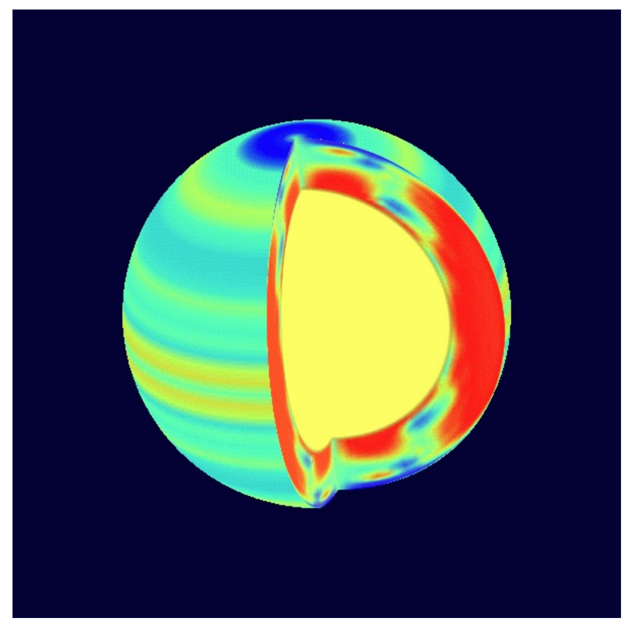
- There is an abrupt jump at the base of the convective zone (tachocline).

- Latitudinal: Differential rotation at the surface is faster at the Equator and is given by (Beck 1999)

$$\omega_{surf}(\theta) \cong a + b\cos^2(\theta) + c\cos^4(\theta)$$

where θ is the colatitude.





Meridional flow

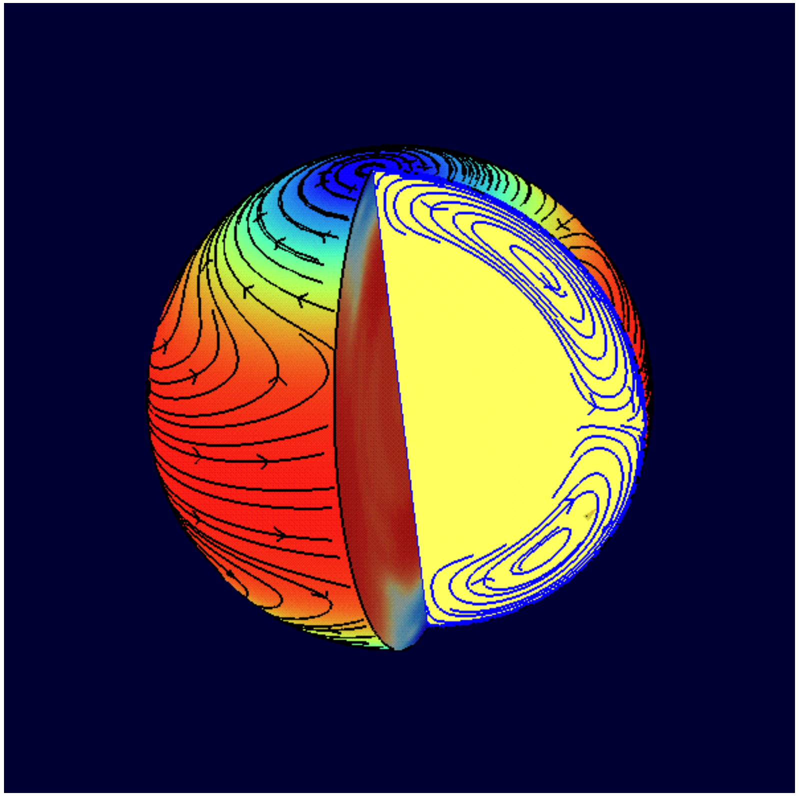
- We assume an incompressible meridional flow with a surface speed of approx. $20\,m\,s^{-1}$

- The mass flux is given by the following stationary stream function

$$\psi(\theta, r) = -\psi_0 \sin^{5/2}(\theta)\cos(\theta)\, r^{1/2}\,(r - r_{min})\,(r_{max} - r)$$

where $r_{min} = 0.6\,R_\Theta$ and $r_{min} = R_\Theta$

- We adopt a density profile $\rho(r) = \rho_0\, r^{-1/2}$



and perform a radial average of the poloidal velocity components.

Convection (Alpha effect)

- MDI (SoHO) observations remarkably show the various components involved in convective and rotational motions (Beck et al. 1998).

- Rising convective flows might become helical as a result of the Coriolis force. This process is relevant whenever the <u>Rossby number</u> is less than unity.

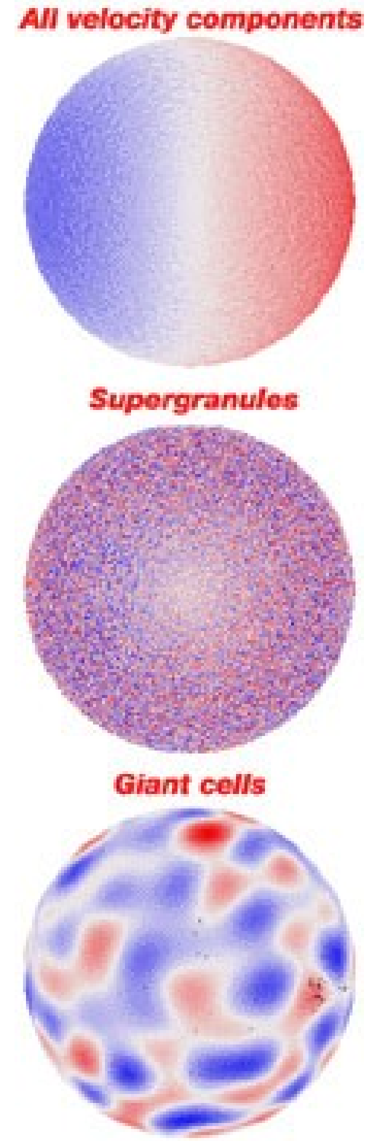
- The Rossby number is

$$Ro = \frac{|(\vec{u} \cdot \nabla)\vec{u}|}{|2\Omega \times u|} \approx \frac{T_{Sun}}{T_{vortex}}$$

- $Ro \approx \frac{T_{Sun}}{T_{vortex}} \leq 1$ only for the <u>giant cells</u>.

- Note that with only 20-40 of these vortices we cover the whole solar surface.

- Because of this poor statistics, we assume the <u>alpha</u> coefficient to have a <u>regular</u> and a <u>stochastic</u> part (Choudhuri 1992, Ossendrijver 1996).



1D simulations

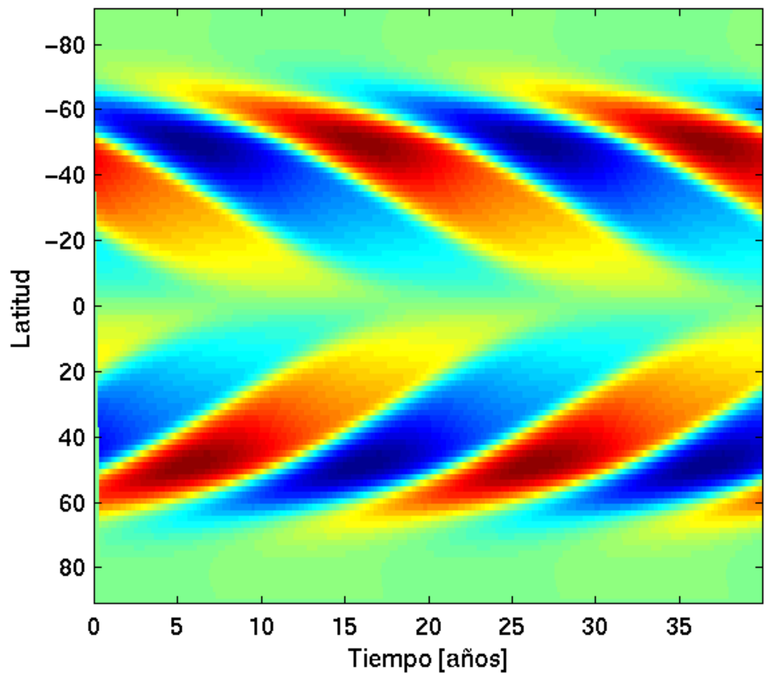
- We integrate the induction equation numerically, assuming axi-symmetry.
- We use empirical profiles of differential rotation and meridional flow. (Mininni & Gómez 2002, ApJ 573, 454).

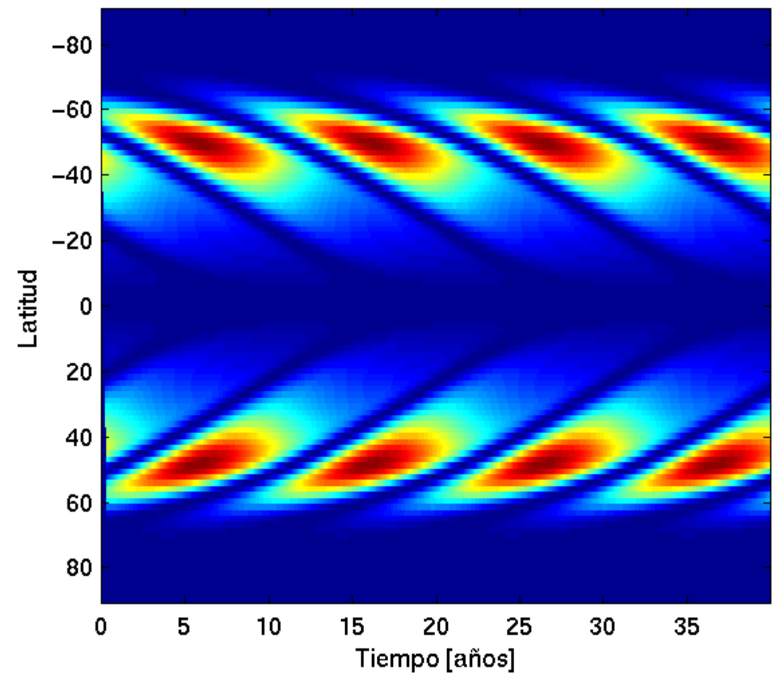
$$\frac{\partial B_\phi}{\partial t} = -(U_r + \varepsilon \frac{\partial U_\theta}{\partial \theta}) B_\phi - \varepsilon U_\theta \frac{\partial B_\phi}{\partial \theta} + \overbrace{(\Delta\omega \cos\theta - \sin\theta \frac{\partial \omega}{\partial \theta}) A + \Delta\omega \sin\theta \frac{\partial A}{\partial \theta}}^{\text{Differential rotation}} + \frac{1}{\Re} \nabla_\theta^2 B_\phi$$

$$\frac{\partial A}{\partial t} = \underbrace{-(U_r + \varepsilon U_\theta \cot\theta) A - \varepsilon U_\theta \frac{\partial A}{\partial \theta}}_{\text{Meridional flow}} + \underbrace{\alpha(B_\phi)\, B_\phi}_{\text{Small-scale convection}} + \underbrace{\frac{1}{\Re} \nabla_\theta^2 A}_{\text{Dissipation}}$$

$$where \quad \Re = \frac{U_0 \delta R}{\eta} \quad , \quad \varepsilon = \frac{\delta R}{R} \quad , \quad \Delta\omega = \omega_{surf}(\theta) - \omega_{core} \quad , \quad \alpha = \frac{\alpha_0 + \delta\alpha}{1 + B_\phi^2 / B_0^2} \sin(\theta)\cos(\theta)$$

Non-stochastic butterfly diagrams



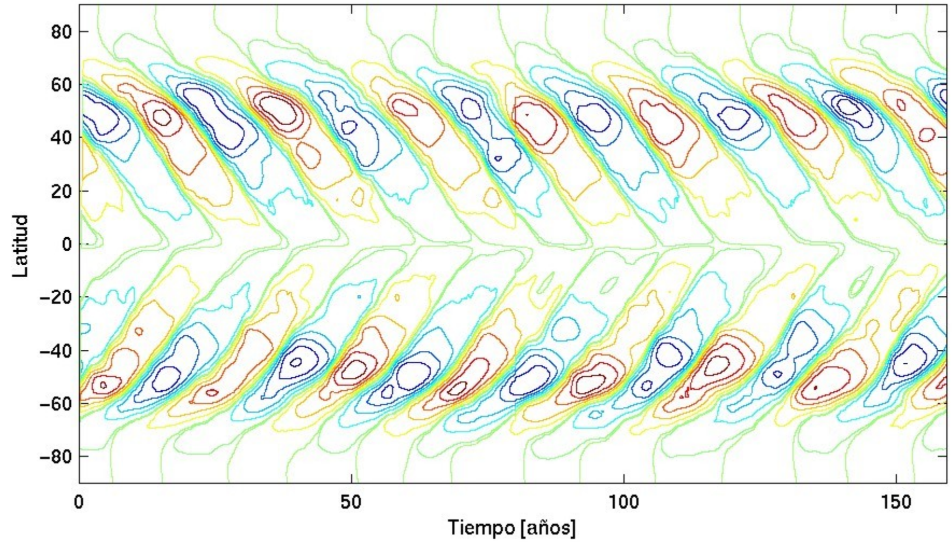


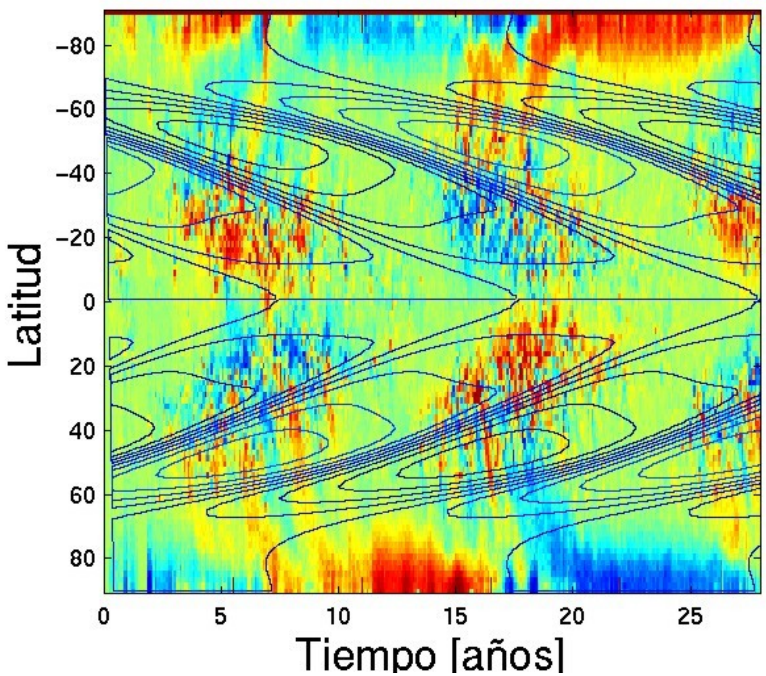
- Toroidal field vs. latitude and time.
- Hale´s law can cleary be observed.
- Magnetic energy vs. latitude and time.
- It is a proxy of Wolf´s number.

Role of stochasticity

- We model $\delta\alpha$ as a gaussian stochastic process, with spatial and temporal correlations corresponding to typical giant cells.

$$\tau_{corr} \cong 30\, days \quad , \quad \lambda_{corr} \cong 2.10^{5}\; km$$



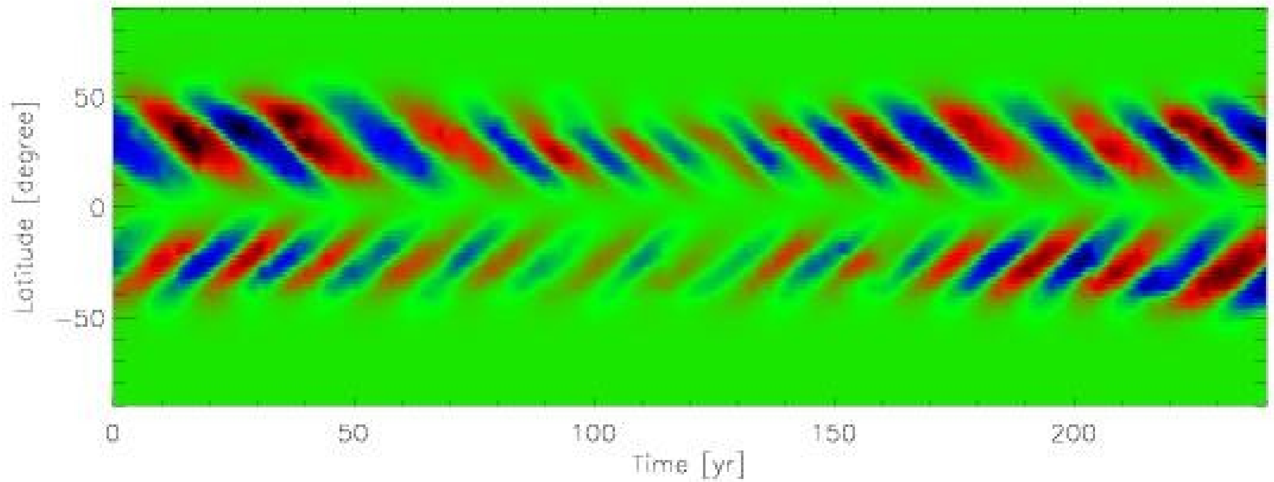


- Toroidal magnetic field obtained from solar magnetograms, displaying the change of polarity in the polar regions.

- Our results correctly reproduce the general behavior, although our butterflies arise at higher latitudes

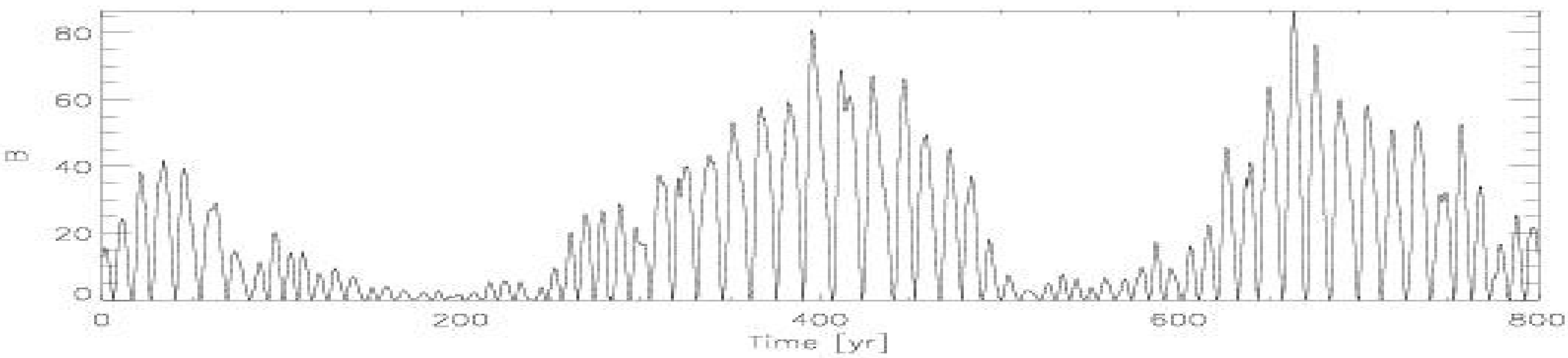
Maunder-like events

- Toroidal magnetic field for a long time integration (Gómez & Mininni 2006).



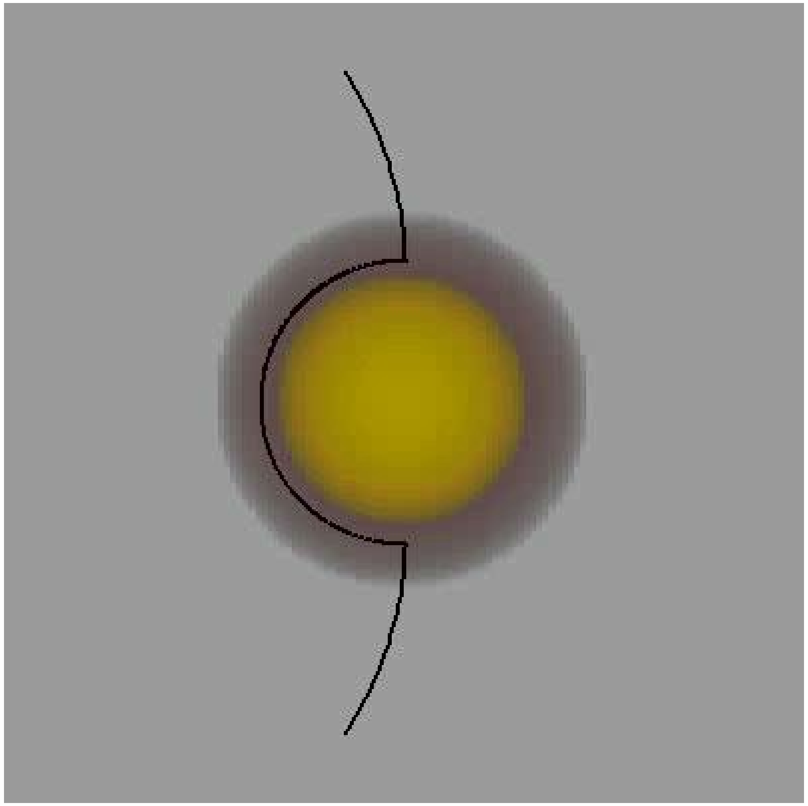
- A minimum of activity is observed at the center. After a few cycles, normal activity is restablished.

- Magnetic energy at mid-latitudes vs. time. Two Maunder-like events are observed.



Babcock-Leighton picture

- The movie shows the solar magnetic cycle and the emergence of coronal loops as a result of differential rotation and magnetic buoyancy.



Induced magnetospheres

- Most planets in the solar system have their own magnetic field. The impact of the solar wind on the planetary fields generate the so-called magnetospheres.

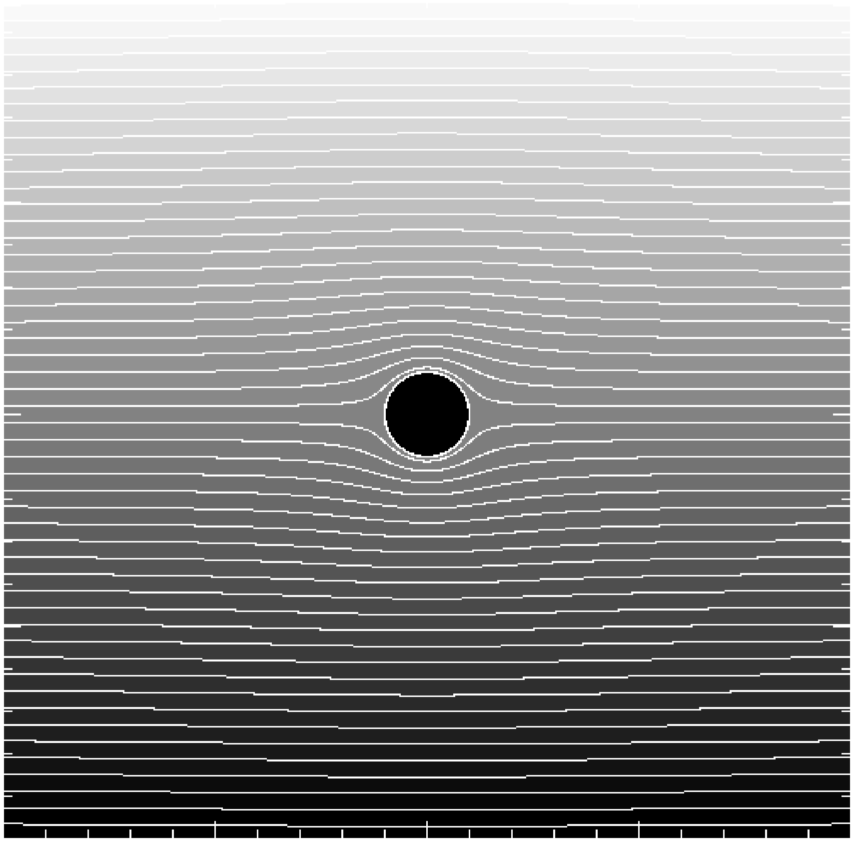
- Mars and Venus do not have magnetic fields, but do have atmospheres.

- The frozen-in condition of the magnetic field carried by the SW with atmospheric ions create the so-called induced magnetospheres.

- In a stationary regime, the magnetic field should satisfy

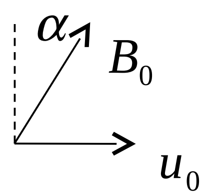
$$\nabla \times (u \times B) = 0 \quad , \qquad \nabla \cdot B = 0$$

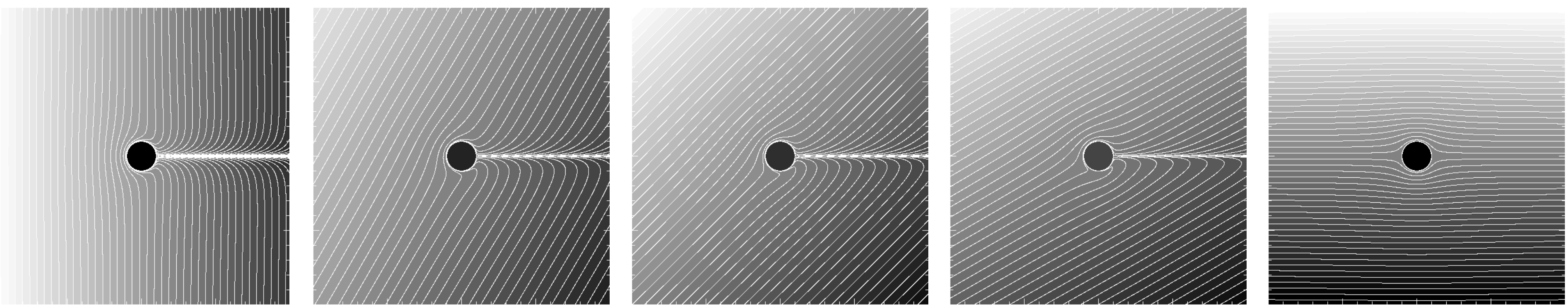
- The velocity field is the stationary and irrotational flow past a sphere. The Figure shows the corresponding streamlines for this flow.



Induced magnetospheres

- Integration of $\nabla \times (u \times B) = 0$, $\nabla \cdot B = 0$ seems straightforward, but it is not.
- It can be integrated analitically using the method of characteristics along the flow streamlines.
- Below I show the magnetic fieldlines for the 2D version. The 3D version is in the poster by Romanelli et al.
- Let the magnetic field at infinity be tilted at an angle α with respect to the vertical





$\alpha = 0°$ $\alpha = 30°$ $\alpha = 45°$ $\alpha = 60°$ $\alpha = 90°$

Conclusions

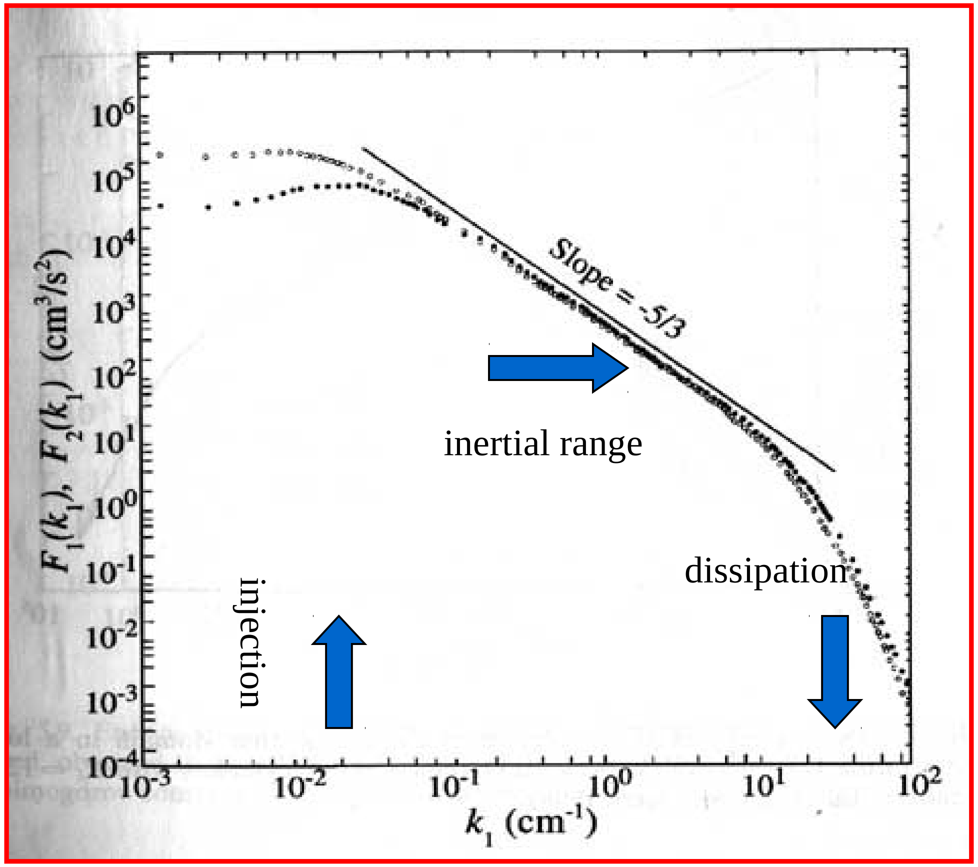
- Today we presented the MHD equations as a valid description of the large-scale behavior of astrophysical plasmas.
- As a first application, we presented the Alpha-Omega dynamos to describe the basic features of the solar dynamo.
- Using empirical profiles of differential rotation and meridional flows, we manage to reproduce various observed aspects of the solar cycle, such as its period, rise-fall asymmetry and sunspot migration toward the Equator.
- Moreover, considering a stochastic part for the Alpha effect, we not only reproduce the irregularities observed in the cycle, but also the potential occurrence of Maunder-like events where magnetic activity on the Sun switches off for several decades.
- Finally, we solved the stationary induction equation for a stationary flow past a sphere, as a simple model to describe induced magnetospheres.

Fluid turbulence

- Energy cascade
  - energy flux toward high k
  - vortex breakdown

- Scale invariance
  - energy flux in k: → $\varepsilon_k \approx \frac{u_k^2}{\tau_k}$
  - energy power spectrum: → $E_k \approx \frac{u_k^2}{k}$

$$\tau_k \approx \frac{1}{ku_k} \;,\; \varepsilon_k \approx \frac{u_k^2}{\tau_k} = const.$$



- Therefore → $E_k \approx \frac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$

Kolmogorov spectrum (K41)