# Fundamentals of magnetohydrodynamics

## Part II


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ICTP-SAIFR, Sao Paulo, Brazil, October 14, 2013

# MHD equations

➢ The MHD equations are:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \bullet (\rho \vec{u}) \qquad\qquad p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\rho \frac{\partial \vec{u}}{\partial t} = -\rho(\vec{u} \bullet \vec{\nabla})\vec{u} - \vec{\nabla} p + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{F}_{ext} + \vec{\nabla} \bullet \overleftrightarrow{\sigma}_{visc}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad\qquad \vec{\nabla} \bullet \vec{B} = 0$$

➢ These equations describe a large number of important plasma processes, such as

- instabilities and wave propagación (Alfven and magnetosonic waves)
- dynamo mechanisms to generate magnetic fields
- MHD turbulence
- magnetic reconnection

➢ Note that even though the electric field is not present, it does not mean that it is not relevant

$$\vec{E} + \tfrac{1}{c}\vec{u} \times \vec{B} = \frac{1}{\sigma}\vec{J}, \qquad \eta = \frac{c^2}{4\pi\sigma}$$

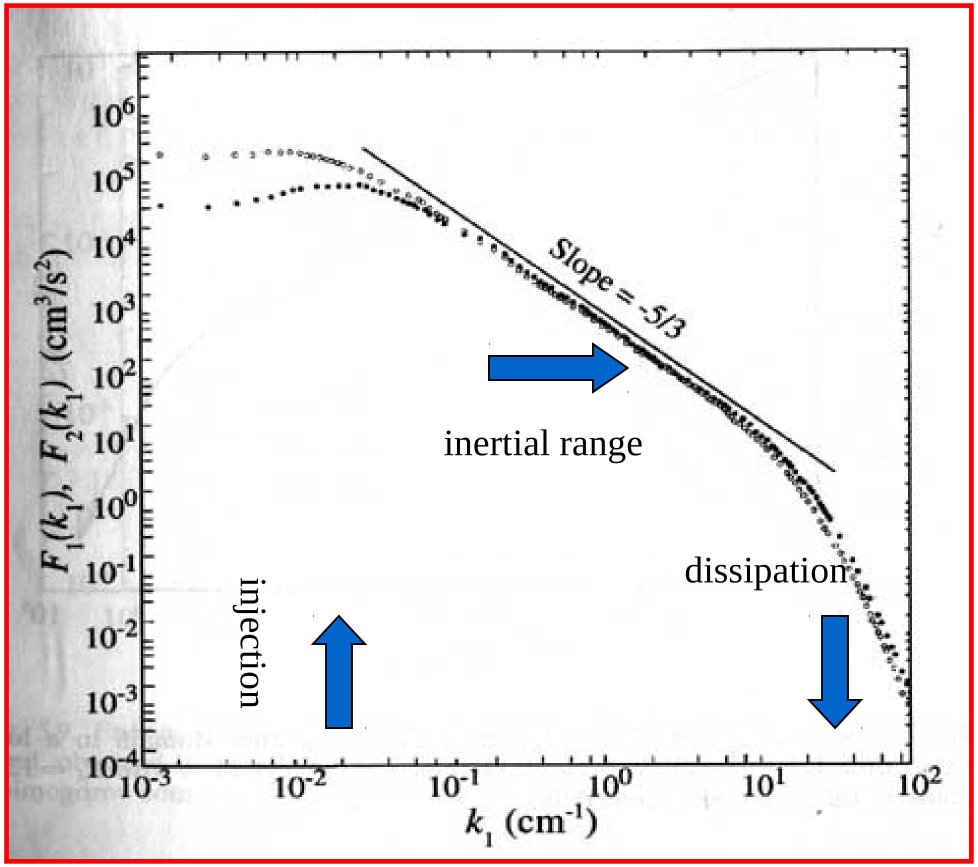
# Fluid turbulence

- Energy cascade
  - energy flux toward high k
  - vortex breakdown

- Scale invariance
  - energy flux in k: $\longrightarrow$ $\varepsilon_k \approx \dfrac{u_k^2}{\tau_k}$
  - energy power spectrum: $\longrightarrow$ $E_k \approx \dfrac{u_k^2}{k}$

$$\tau_k \approx \frac{1}{k u_k} \quad , \quad \varepsilon_k \approx \frac{u_k^2}{\tau_k} = const.$$

- Therefore $\longrightarrow$ $E_k \approx \dfrac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$




Kolmogorov spectrum (K41)

# Simulations

➢ We integrate the MHD equations numerically, using a spectral scheme in all three spatial directions (Gomez, Milano and Dmitruk 2000; also Dmitruk, Gomez & Matthaeus 2003)

➢ We show results from 256x256x256 runs performed in (CAPS), our linux cluster with 80 cores

➢ For the spatial derivatives, we use a pseudo-spectral scheme with 2/3-dealiasing. Spectral codes are well suited for turbulence studies, since they provide exponentially fast convergence.

➢ Time integration is performed with a second order Runge-Kutta scheme.The time step is chosen to satisfy the CFL condition.

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Cluster for Astrophysical Plasma Simulations

# Simulations: spatial integration

- We focus on Fourier-Galerkin methods. Let us ilustrate on Burgers equation

$$\partial_t u + u\partial_x u = \nu\, \partial_{xx} u$$

for u(x,t) on the interval $0 \le x < 2\pi$ assuming periodic boundary conditions and the initial condition $u(x,0) = u_0(x)$

- We expand in a truncated Fourier expansion $\Longrightarrow$ $u^N(x,t) = \sum_{k=-N/2}^{N/2} u_k(t)\, e^{ikx}$

- Demanding zero projection of the solution u(x,t) on the truncated Fourier space

$$\partial_t u_k = -(u\partial_x u)_k - \nu\, k^2\, u_k \quad , \quad (u\partial_x u)_k = \sum_{l+m=k} i m u_l u_m$$

- This truncated expansion $u^N(x,t)$ converges exponentially fast to the exact solution as $N \to \infty$

However, it is computationally very demanding, it involves $O(N^2)$ operations.

# Simulations: spatial integration

- The FFT algorithm yields the discrete set $\{\hat{u}_k\}$ from the set $\{u(x_j)\}$ after $O(N \log N)$ floating point operations.

$$\left\{u(x_j)\ ,\ x_j = \frac{2\pi}{N} j\ ,\ j = 0,\ldots,N-1\right\} \ \overrightarrow{FFT}\ \ \{\hat{u}_k\ ,\ k = -N/2+1,\ldots,N/2\}$$

- The strategy of computing spatial derivatives in Fourier space and nonlinear terms in physical space, is known as pseudo-spectral, i.e.

$$\partial_t u_k = -(u\partial_x u)_k - \nu k^2 u_k \quad , \quad (u\partial_x u)_k = FFT(FFT^{-1}(u_k)\ FFT^{-1}(iku_k))$$

- The relation between discrete Fourier coefficients $\{\hat{u}_k\}$ and the continuous ones is

$$\hat{u}_k = u_k + \sum_{m \neq 0} u_{k+Nm}$$

- This sum causes a spurious effect known as aliasing when computing nonlinear terms. Aliasing effects can be suppressed by applying the “two-thirds rule”, i.e.

$$\hat{u}_k = 0 \quad , \quad \forall \quad |k| \geq N/3$$

# Simulations: temporal integration

- We advance the solution through discrete time steps $\Longrightarrow$ $t_i = i\Delta t$
- In compact notation, if $\dfrac{dU}{dt} = F(U,t)$

where F is a nonlinear and spatial differential operator, we use a second order Runge-Kutta scheme.

- We first advance half a step $\Longrightarrow$ $U^{i+\frac{1}{2}} = U^i + \dfrac{\Delta t}{2} F(U^i, t_i)$

and use $U^{i+\frac{1}{2}}$ to jump the whole step $\Longrightarrow$ $U^{i+1} = U^i + \Delta t\, F(U^{i+\frac{1}{2}}, t_{i+\frac{1}{2}})$

- This is second order accurate (i.e. $O((\Delta t)^2)$ ). The size of the step is limited by

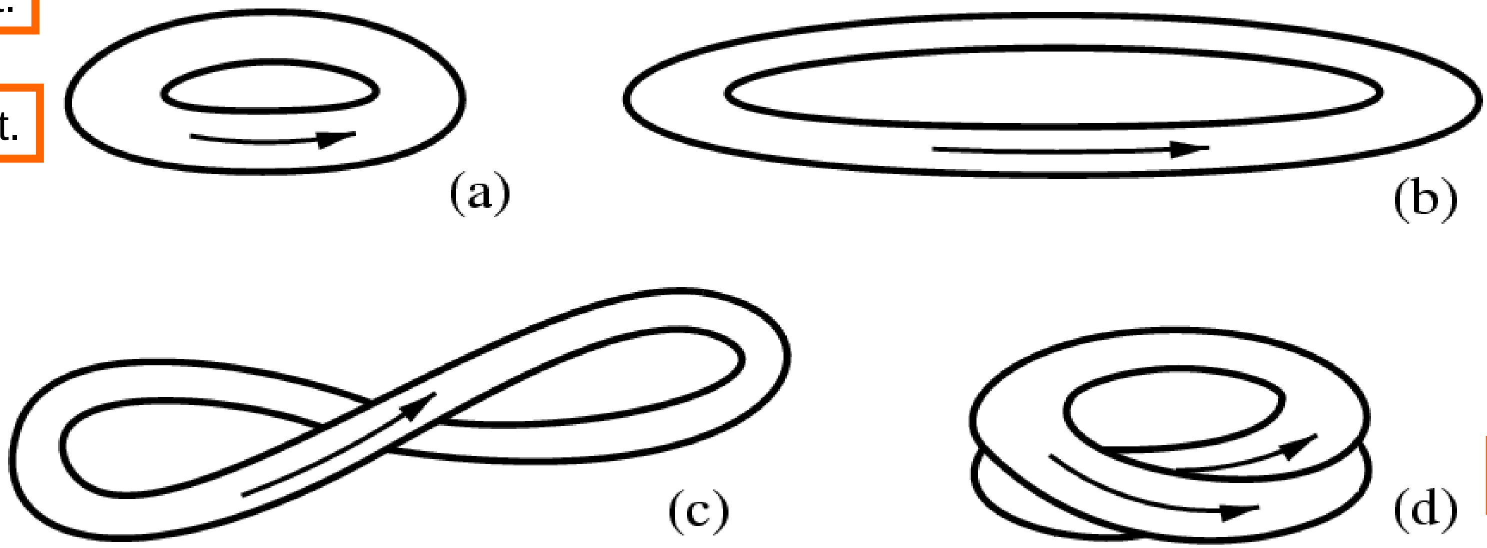
the CFL condition, i.e $\Delta t \leq \Delta x / u_0$ for $\partial_t u = u_0 \partial_x u$

# Stretch-twist-fold dynamo

- Stretch-twist-fold (Vainshtein-Zeldovich 1972)

A.L=const.

A.B=const.




2.L

2.B

L

2.B

- (a) Magnetic flux tube
- (b) Stretching (frozen to inhomogeneous flows
- (c) Twisting (helicoidal flows)
- (d) Folding (with reconnection)

... with twice as much magnetic flux !!

# Mean field theory

- It provides a quantitative expresion for the coefficient alpha. The first assumption is that there is a scale separation between the large scale magnetic field being  generated and the small scale convective motions, i.e

$$B \rightarrow B + b \quad , \quad u \rightarrow U + u \quad , \quad < b > = 0 = < u >$$

where <...> is an average over small scales. To compute the evolution of the mean field, we average the induct ion equation

$$\frac{\partial B}{\partial t} = \vec{\nabla} \times (\vec{U} \times \vec{B}) + \vec{\nabla} \times < \vec{u} \times \vec{b} > \quad , \qquad \vec{\nabla} \bullet \vec{B} = 0$$

- The extra term can be interpreted as an electromotive force exerted by small scale motions

$$\varepsilon_{EMF} = < u \times b >$$

- We still need to obtain an expresion for the electromotive force, and that requires some assumptions  (Steenbeck, Krause & Radler 1966).

# Mean field theory

$$\frac{\partial \vec{b}}{\partial t} = \underbrace{\cancel{\nabla\times(\vec{U}\times\vec{b})}}_{[1]} + \nabla\times(\vec{u}\times\vec{B}) + \underbrace{\cancel{\nabla\times(\vec{u}\times\vec{b} - <\vec{u}\times\vec{b}>)}}_{[2]} \quad , \quad \nabla\bullet\vec{b} = 0$$

[1] Can be removed with a Galilean transformation.
[2] It´s a departure from average of a second order quantity (FOSA).

- Let us further assume that this system evolves in a typical correlation time of these small scale convective motions.
- Therefore

$$\varepsilon_{EMF} = \tau < u\times\nabla\times(u\times B) > = \alpha\bullet B - \beta\bullet\nabla\times B$$

where we neglected the gradient of the large scale magnetic field.

- For an isotropic state of these small scale flows, these tensors become

$$\alpha_{ij} = -\frac{\tau}{3} < \vec{u}\bullet\vec{\nabla}\times\vec{u} > \delta_{ij} \qquad \beta_{ij} = \frac{\tau}{2} < \vec{u}\bullet\vec{u} > \delta_{ij}$$

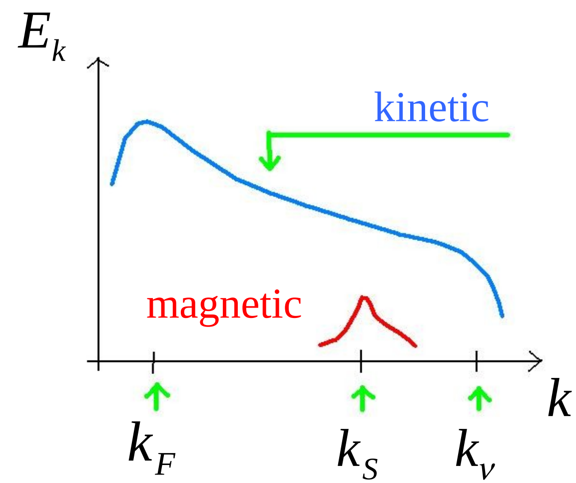
- The kinetic helicity of convective flows is important for dynamo activity.

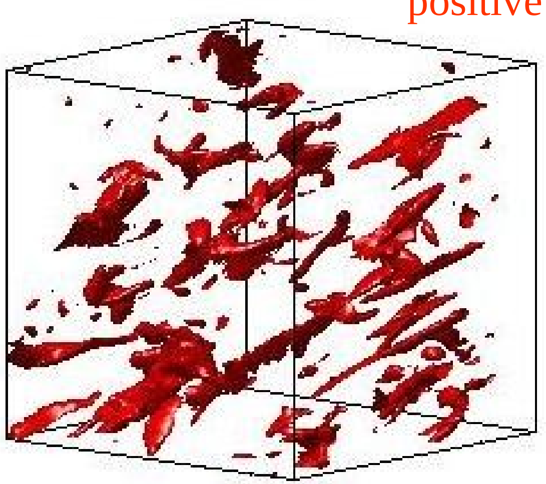
# MHD 3D dynamos

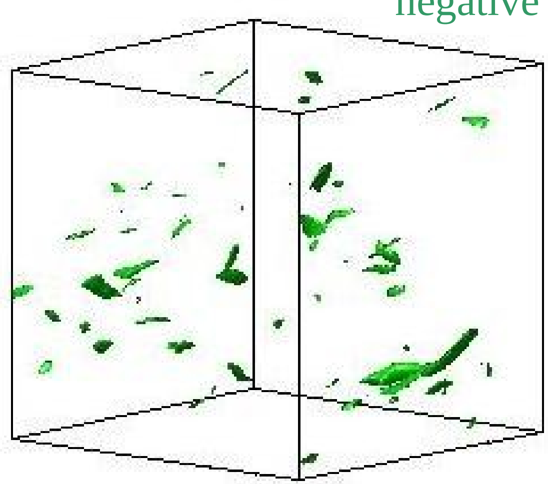
- From mean field theory (Krause & Radler 1980), we know that the turbulent generation of magnetic fields (the **alpha effect**) is proportional to the **kinetic helicity** of the flow. $H = \frac{1}{2}\langle \vec{u} \bullet \nabla \times \vec{u} \rangle$

- To study this mechanism through direct simulations, we externally drive the flow with a helical force at large scales (an ABC pattern), until a stationary turbulent state is reached (Mininni, Gómez & Mahajan, 2003, ApJ, 587, 472; Mininni, Gómez & Mahajan, 2005, ApJ, 619, 1019)

- At that point, a magnetic seed is implanted at small scales and the 3D MHD equations are evolved (Meneguzzi, Frisch & Pouquet 1981).

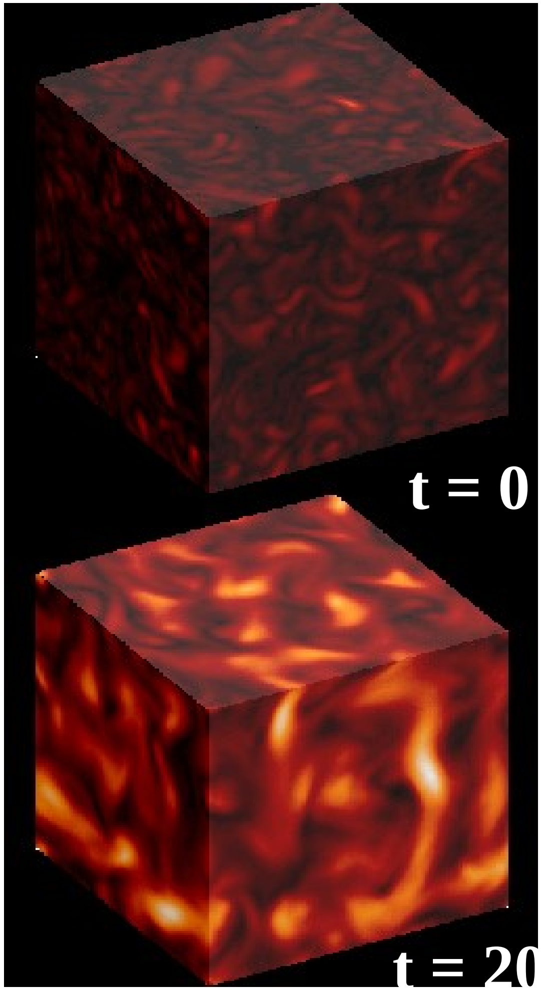


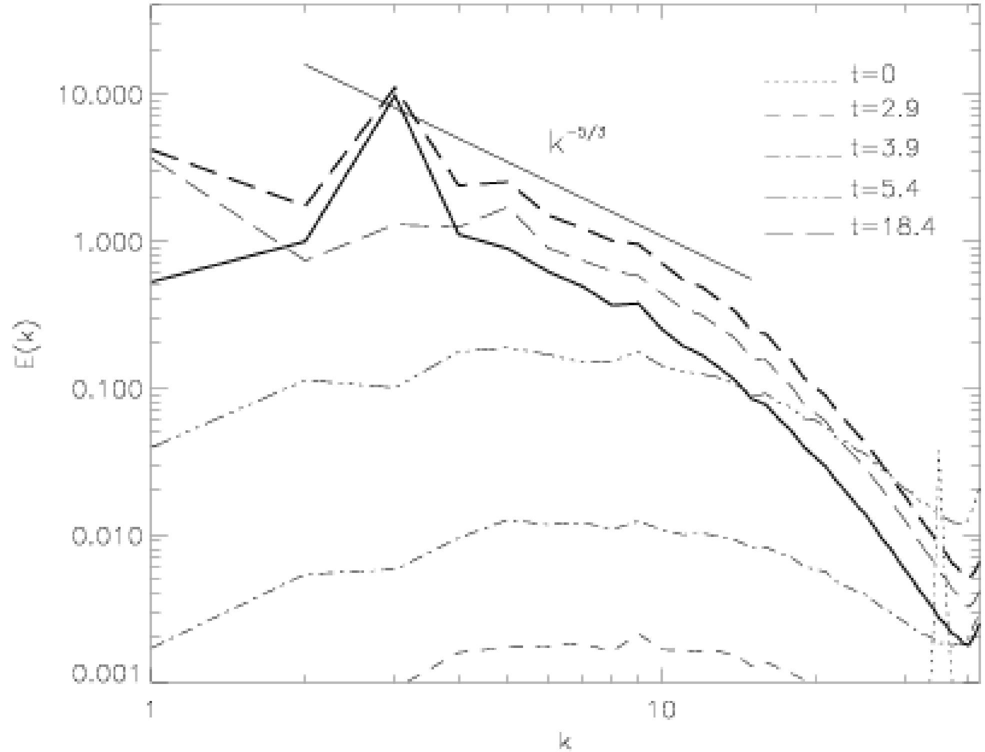




- The boxes show the intermittent spatial distribution of positive and negative kinetic helicity H, clearly displaying a net unbalance.

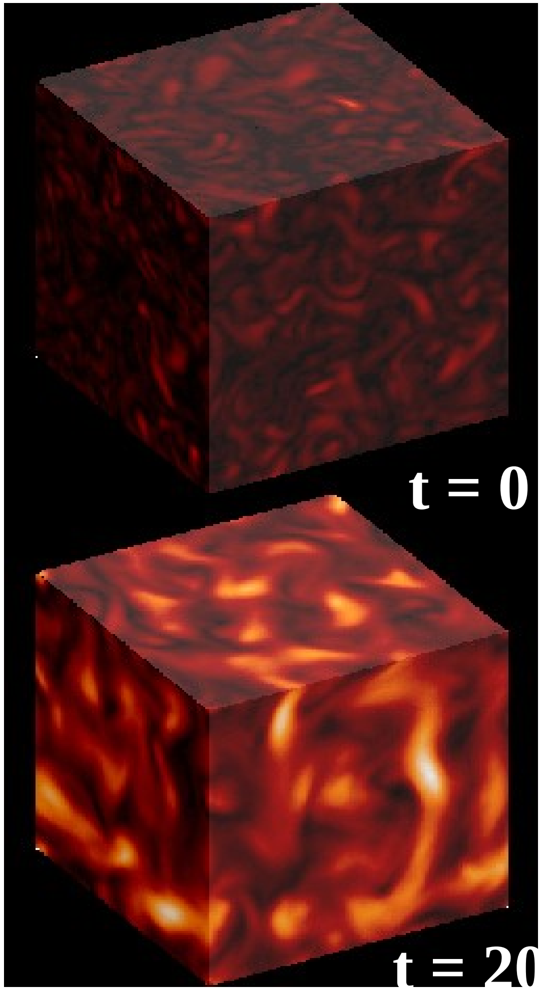
# Energy power spectra

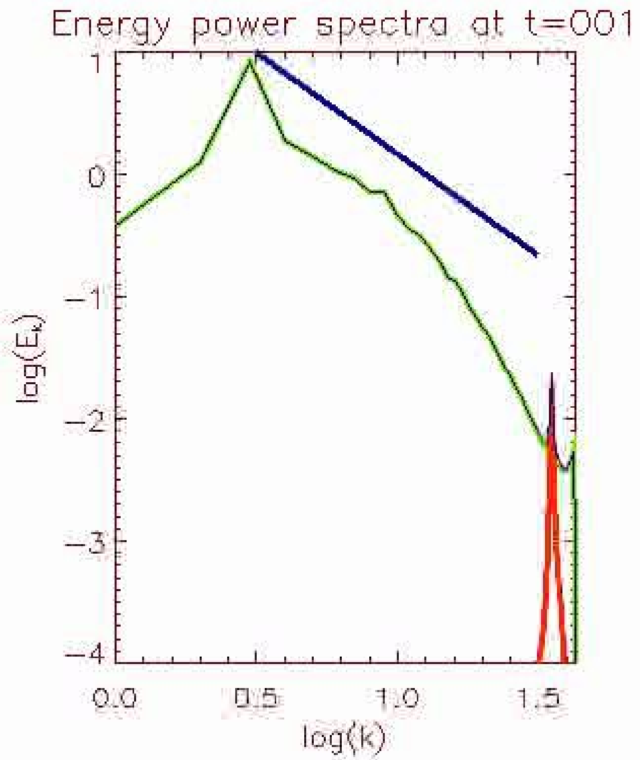


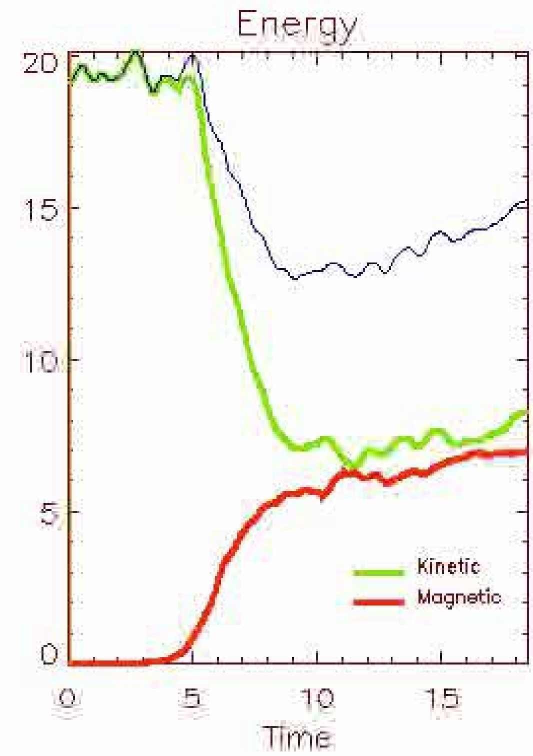




- The power spectrum of magnetic energy grows in time until it reaches equipartition at each scale (Brandenburg et al. 2003).
- The Kolmogorov slope is also displayed for reference.
- The full line is the kinetic energy power spectrum and the dotted line is the total energy.

# Energy power spectra



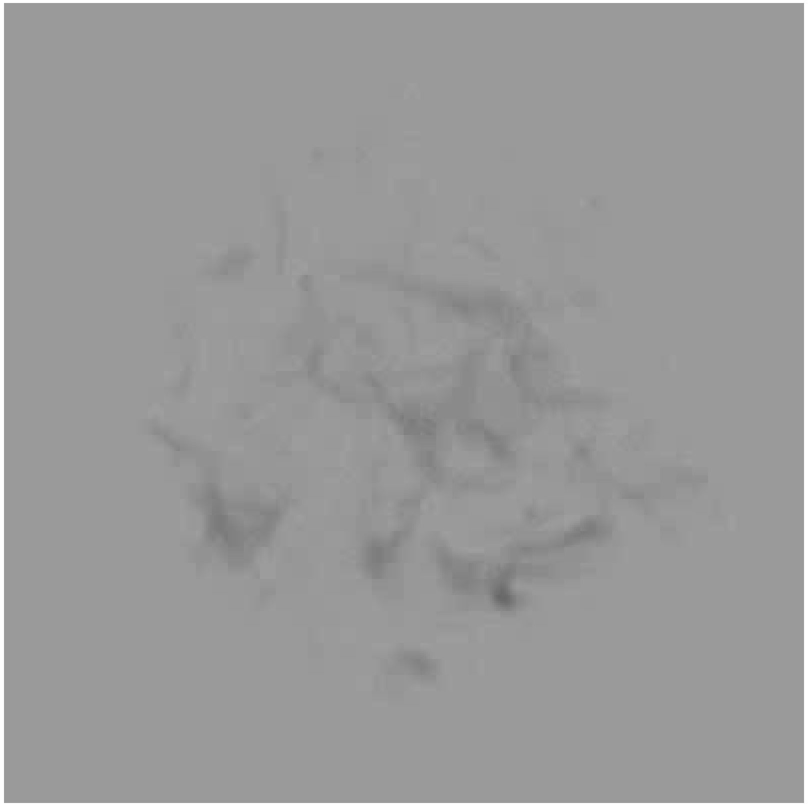

➢ The power spectrum of magnetic energy grows in time until it reaches equipartition at each scale (Brandenburg et al. 2003).

➢ The Kolmogorov slope is also displayed for reference.

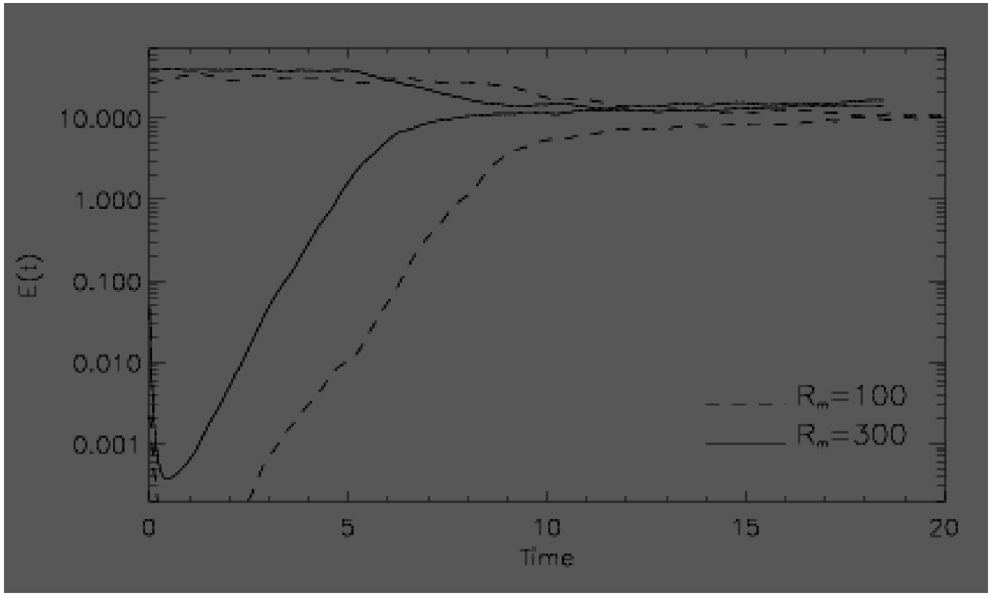
➢ The full line is the kinetic energy power spectrum and the dotted line is the total energy.

# Turbulent dynamos

➢ The image on the right shows the spatio-temporal distribution of magnetic energy.



➢ The image below shows an initial exponential growth stage (kinematic dynamo) for the total magnetic energy. At later times it saturates when it reaches approximate equipartition with the total kinetic energy of the turbulent flow.




➢ As predicted by MFT (Steenbeck et al. 1966), kinematic helicity (H) at the microscale produces magnetic field at macroscopic scales (large-scale dynamos).

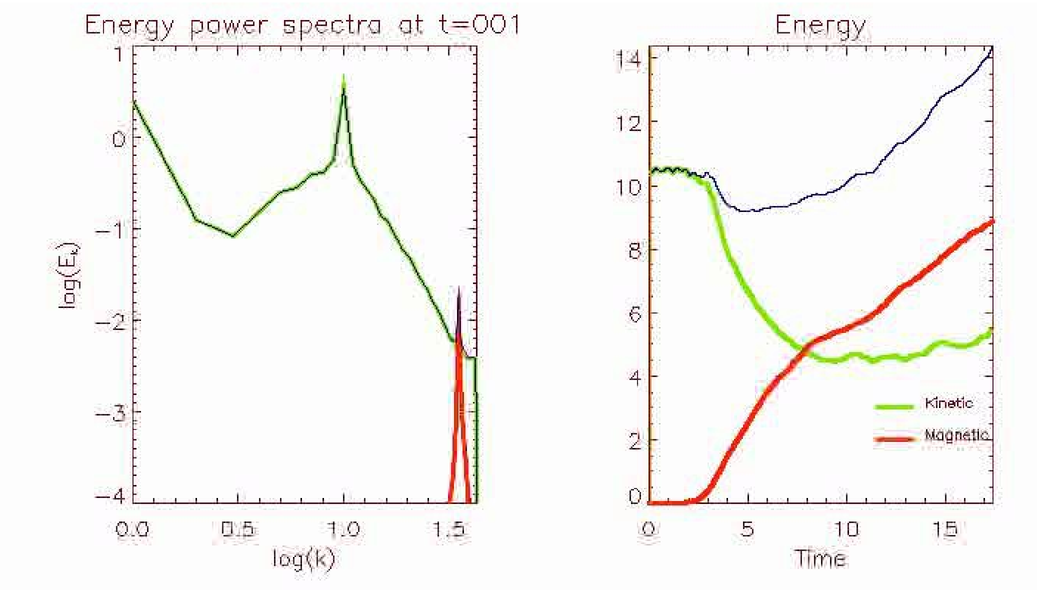
# Force-free equilibria

- When the forcing is applied at intermediate scales, an accumulation of magnetic energy is observed at the largest scales.

- This behavior is caused by the inverse cascade of magnetic helicity.

- The magnetic field at large scales is approximately force-free, i.e.

$$\nabla \times B \mathbin{//} B$$




- Small scales, however, are consistent with a strongly turbulent MHD regime.

- This configuration can be representative of active regions of the solar corona, which are approximately force-free at large scales and at the same time are being heated by a strong MHD turbulence at smaller scales (Gómez & F.Fontán 1988)

# Conclusions

- Using Mean Field Theory, we have seen that large-scale dynamos are generated by turbulent helical flows, i.e. with a high concentration of kinematic helicity.

- We briefly mentioned the main features of the spectral codes used in our numerical simulations. In particular, spectral schemes present exponential convergence as the number of grid points increases.

- We showed MHD 3D simulations of a turbulent and helical flow. An initial magnetic seed is seen to exponentially grow and to saturate when it reaches approximate equipartition with the kinetic energy of the flow.

- When the flow is externally driven at intermediate scales, an accumulation of magnetic energy is observed at the largest scales in the box. Furthermore, the magnetic field at these large scales is well described by a force-free regime.

- This state is relevant to coronal active regions, which are observed to be close to force-free equilibria. Yet, at smaller scales they might show a strongly turbulent regime responsible for the heating of the plasma confined in these regions.