Fundamentals of magnetohydrodynamics

Part III

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General Motivation

- MHD is a fluidistic approach to describe the large scale dynamics of plasmas. The standard approach is also known as one-fluid MHD.

- Today we start from a somewhat more general approach known as two-fluid MHD, which acknowledges the presence of ions and electrons and considers kinetic effects such as Hall, electron pressure and electron inertia.

- For sufficiently diffuse media such as the interstellar medium, the Hall effect eventually becomes non-negligible.

- To study the role of the Hall effect on turbulent dynamos, we present results from three dimensional simulations of the Hall-MHD equations subjected to non-helical forcing. and for different values of the Hall parameter.

- The simulations are performed with a pseudospectral code to achieve exponentially fast convergence.

Fluid equations for multi-species plasmas

- For each species s we have (Goldston & Rutherford 1995):
  - Mass conservation $$\frac{\partial n_s}{\partial t} + \vec{\nabla} \cdot (n_s \vec{U}_s) = 0$$
  - Equation of motion $$m_s n_s \frac{dU_s}{dt} = q_s n_s (\vec{E} + \frac{1}{c}\vec{U}_s \times \vec{B}) - \vec{\nabla} p_s + \vec{\nabla} \bullet \overset{\leftrightarrow}{\sigma}_s + \sum_{s'} \vec{R}_{ss'}$$
  - Momentum exchange rate $$R_{ss'} = -m_s n_s \upsilon_{ss'} (U_s - U_{s'})$$
- These moving charges act as sources for electric and magnetic fields:
  - Charge density (charge neutrality) $$\rho_c = \sum_s q_s n_s \approx 0$$
  - Electric current density $$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = \sum_s q_s n_s \vec{U}_s$$

Two-fluid MHD equations

- For a fully ionized plasma with ions of mass m_i and massless electrons (since $m_e << m_i$):

  - Mass conservation: $$0 = \frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{U}) \quad , \qquad n_e \cong n_i \cong n$$
  - Ions: $$m_i n \frac{d\vec{U}}{dt} = en(\vec{E} + \frac{1}{c}\vec{U} \times \vec{B}) - \nabla p_i + \vec{\nabla} \bullet \sigma + \vec{R}$$
  - Electrons: $$0 = -en(\vec{E} + \frac{1}{c}\vec{U}_e \times \vec{B}) - \vec{\nabla} p_e - \vec{R}$$
  - Friction force: $$\vec{R} = -m_i n \nu_{ie} (\vec{U} - \vec{U}_e)$$
  - Ampere's law: $$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = en(\vec{U} - \vec{U}_e) \qquad \Rightarrow \qquad \vec{R} = -\frac{m\upsilon_{ie}}{e} \vec{J}$$
  - Polytropic laws: $$p_i \propto n^\gamma \quad , \qquad p_e \propto n^\gamma$$
  - Newtonian viscosity: $$\sigma_{ij} = \mu \, (\partial_i U_j + \partial_j U_i)$$

Hall-MHD equations

- The dimensionless version, for a length scale L_0 density n_0 and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{dU}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U} \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_i - \frac{\eta}{\varepsilon n}\vec{J} + \nu \nabla^2 \vec{U} \qquad \nu = \frac{\mu}{m_i n v_A L_0}$$

$$0 = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_e + \frac{\eta}{\varepsilon n}\vec{J}$$

where $\vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon}(\vec{U} - \vec{U}_e)$

- We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma *beta* $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

- Adding these two equations yields:

$$n\frac{dU}{dt} = (\vec{\nabla} \times \vec{B}) \times \vec{B} - \beta \vec{\nabla}(p_i + p_e) + \nu \nabla^2 \vec{U}$$

Hall-MHD equations

- On the other hand, using

$$\vec{E} = -\frac{1}{c}\frac{\partial A}{\partial t} - \vec{\nabla}\phi$$

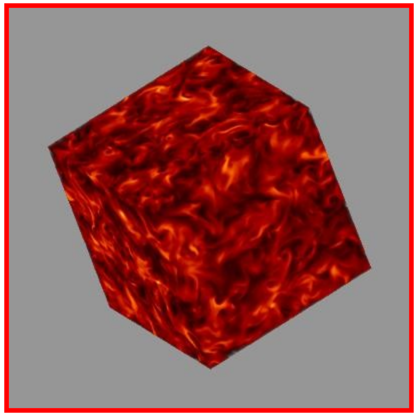
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\rightarrow \quad \frac{\partial A}{\partial t} = (\vec{U} - \frac{\varepsilon}{n}\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{\nabla}\phi + \frac{\varepsilon\beta}{n}\vec{\nabla} p_e - \frac{\eta}{n}\vec{\nabla} \times \vec{B}$$

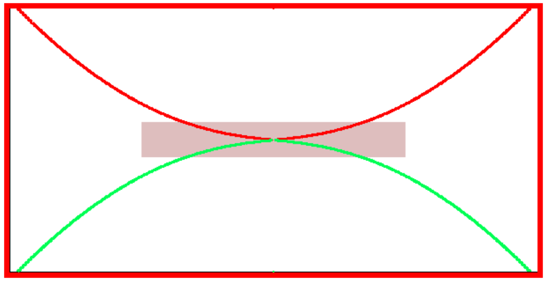
Some applications

➢ We studied a number of astrophysical problems, within the general framework of MHD:

➢ 3D Hall-MHD turbulent dynamos.
(Mininni, Gomez & Mahajan 2003, 2005; Gomez, Dmitruk & Mininni 2010)

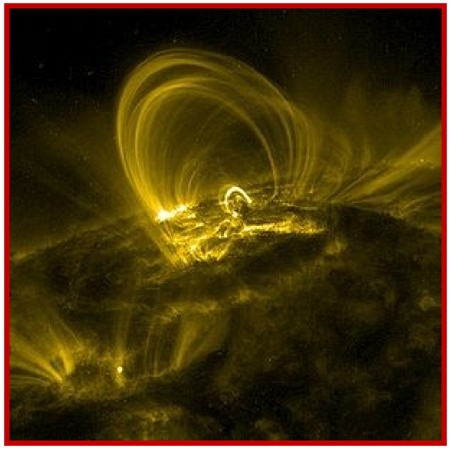


➢ 2.5 D Hall-MHD magnetic reconnection in the Earth magnetosphere
(Morales, Dasso & Gomez 2005, 2006)



➢ 3D HD helical fluid turbulence
(Gomez & Mininni 2004)

➢ RMHD heating of solar coronal loops
(Dmitruk & Gomez 1997, 1999)



➢ RHMHD turbulence in the solar wind
(Martin, Dmitruk & Gomez 2010, 2012)

➢ Hall magneto-rotational instability in accretion disks
(Bejarano, Gomez & Brandenburg 2011)

Hall-MHD equations

➢ The dimensionless version (for a length scale L_0 , density n_0 and typical velocity U_0) of the incompressible Hall MHD equations is

$$\frac{d\vec{U}}{dt} = (\nabla\times\vec{B})\times\vec{B} - \nabla p + \vec{f} + \nu\nabla^2\vec{U}$$

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times[(\vec{U} - \varepsilon\nabla\times\vec{B})\times\vec{B}] + \eta\nabla^2\vec{B}$$

$$U_e = U - \varepsilon\nabla\times B$$

Electron velocity field

➢ We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$ which is simply the dimensionless ion skin depth.

➢ The Prandtl number $Pm = \frac{\nu}{\eta}$ is the ratio of the viscosity to the electric resistivity. Turbulent dynamos for different values of Pm have been studied by Haugen, Brandenburg & Dobler 2004 , and also by Schekochihin et al. 2004, but without Hall effect. We will focus on Pm=1.

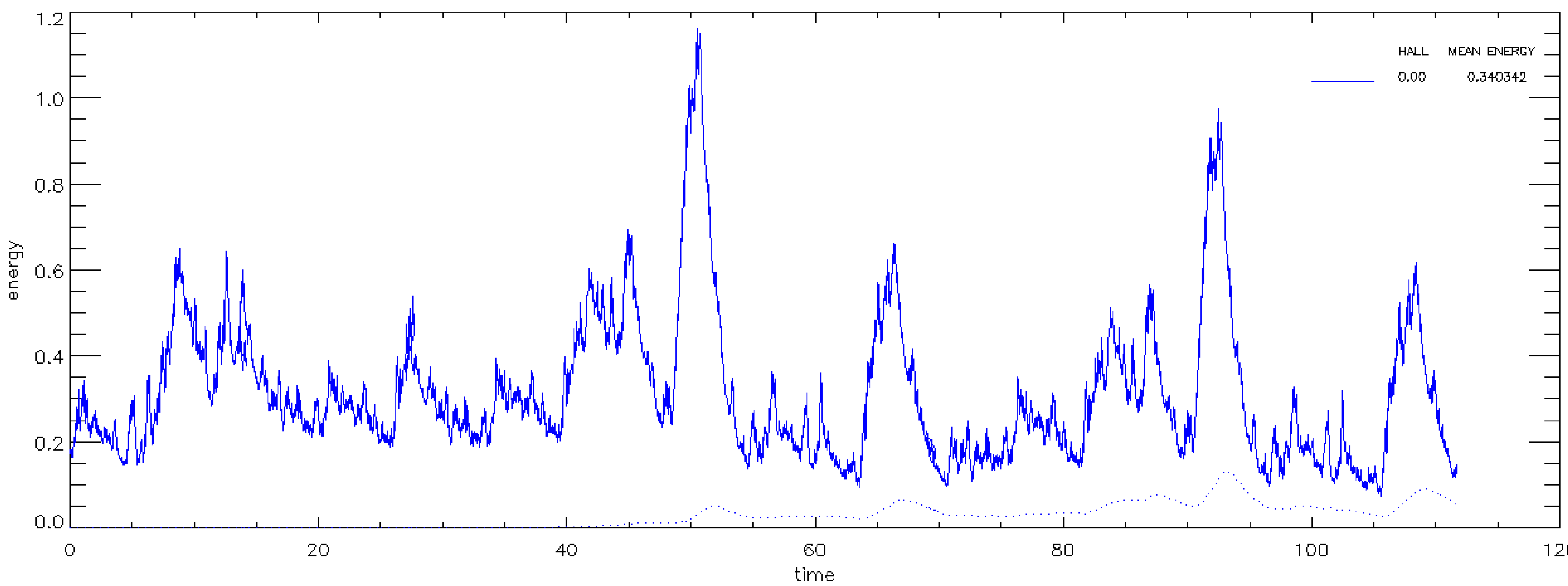
➢ We maintain an external forcing f , which is non-helical, large-scale ($k_f \approx 3$) and delta-correlated in time.

Energy vs. time

- We start off with a purely HD run until it reaches a stationary turbulent regime, where the external forcing is balanced by viscous dissipation. We then plant a magnetic seed at t=0 at large scales and re-start the simulation.

- Among the many outputs, we obtain kinetic and magnetic energy vs time.

- The purely MHD run is shown in **blue**, and the magnetic energy is shown with a dotted trace.



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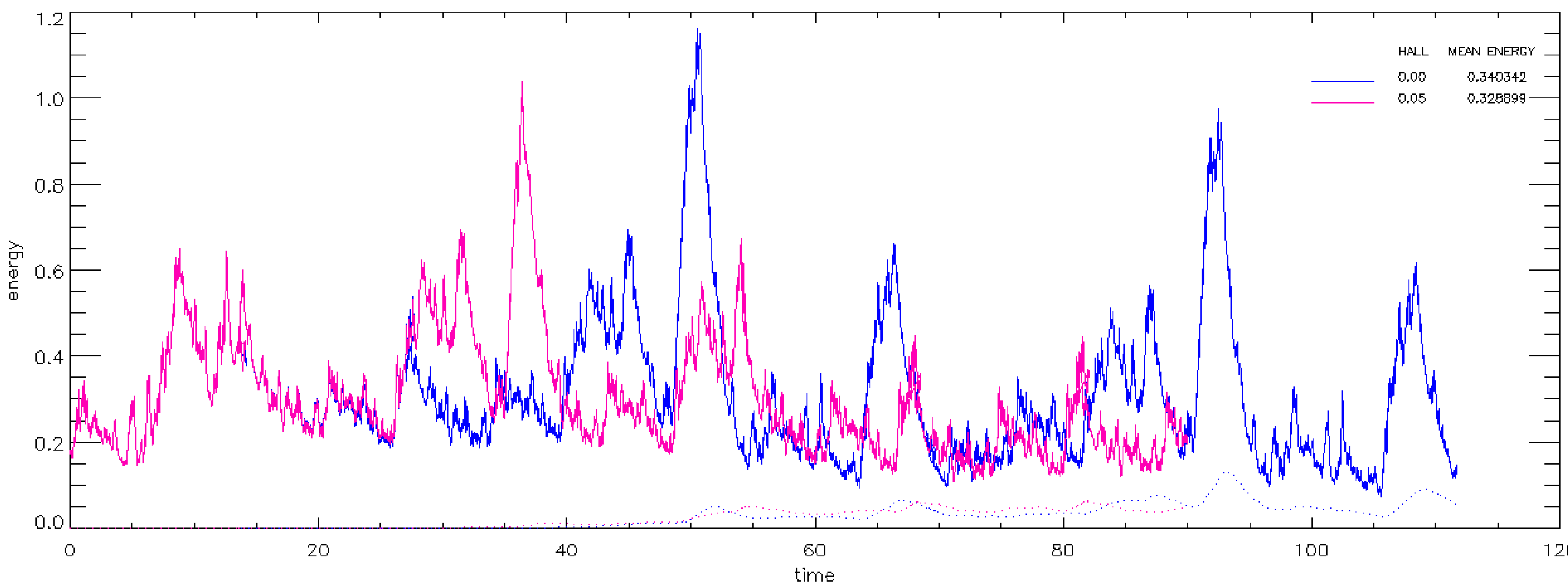

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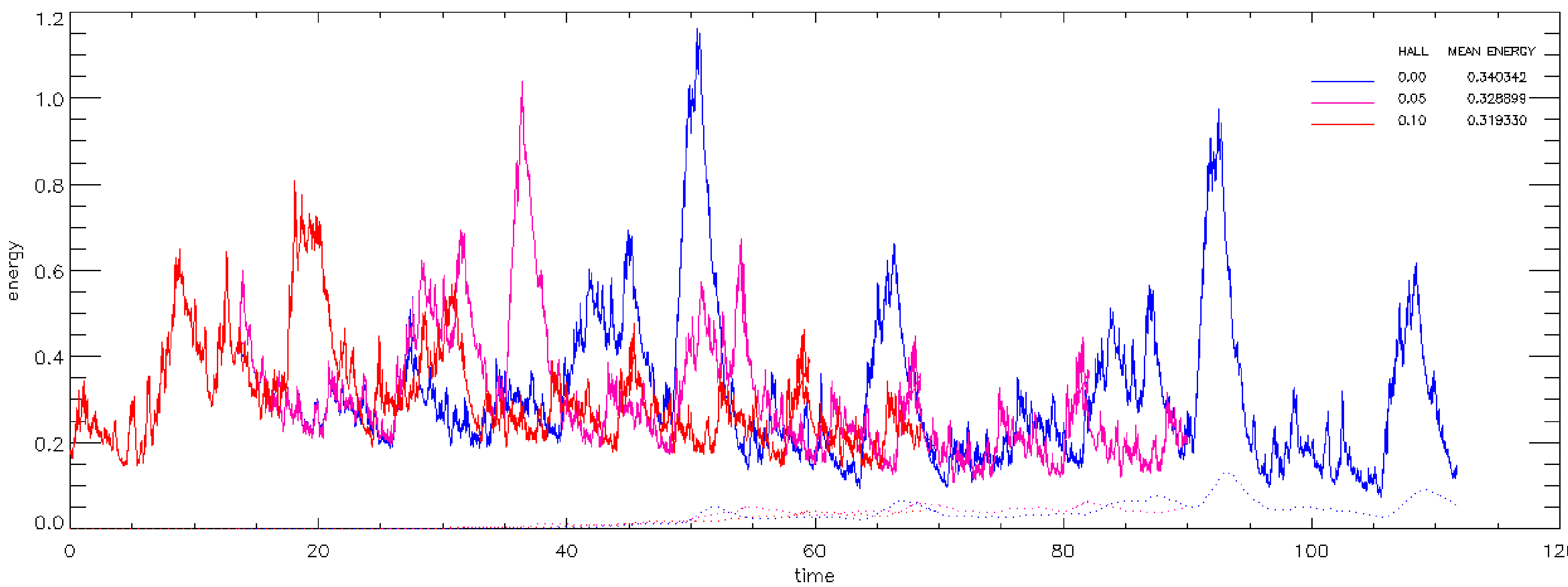
- The run with the largest amount of Hall is shown in **red**.



Dissipation rate vs. time

- We also obtain dissipation rate vs time.
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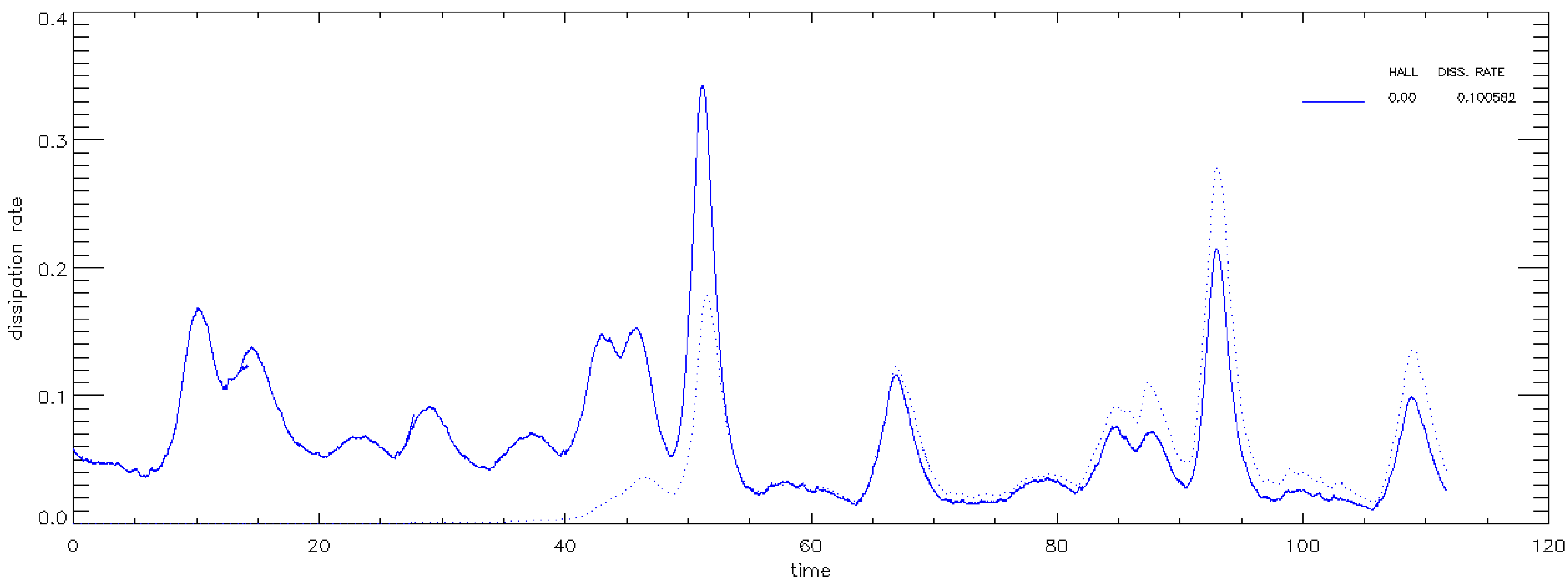

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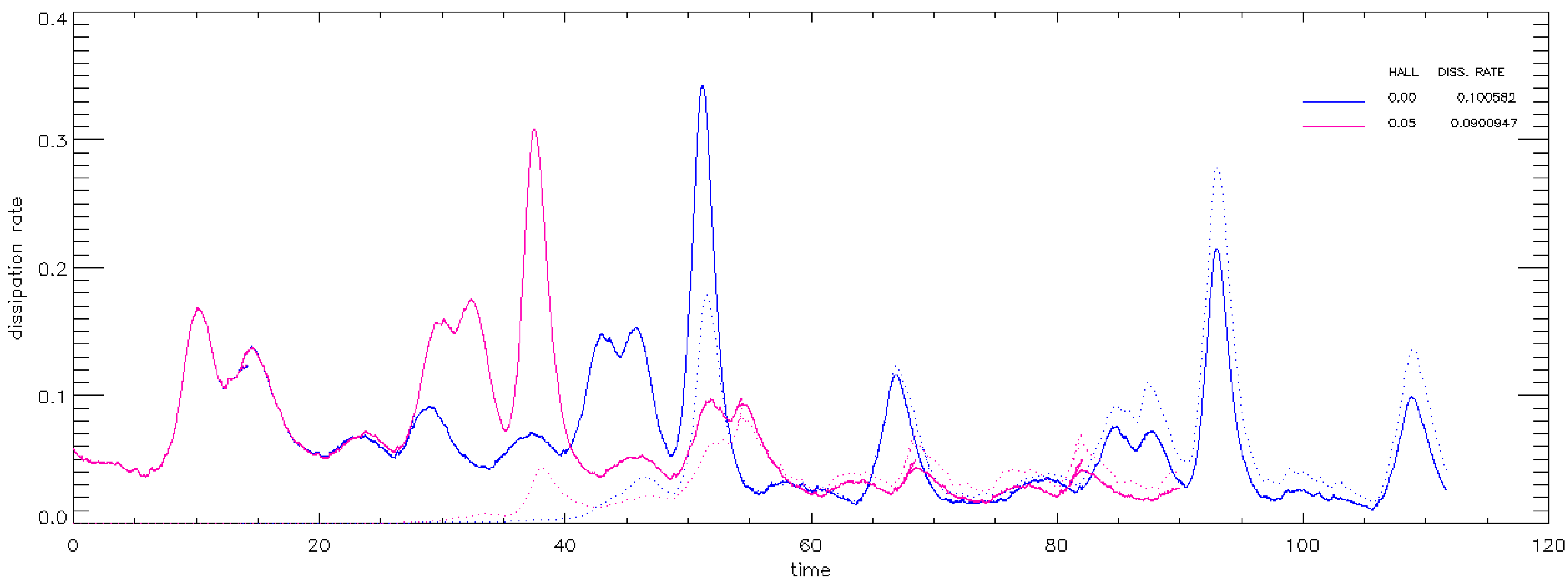

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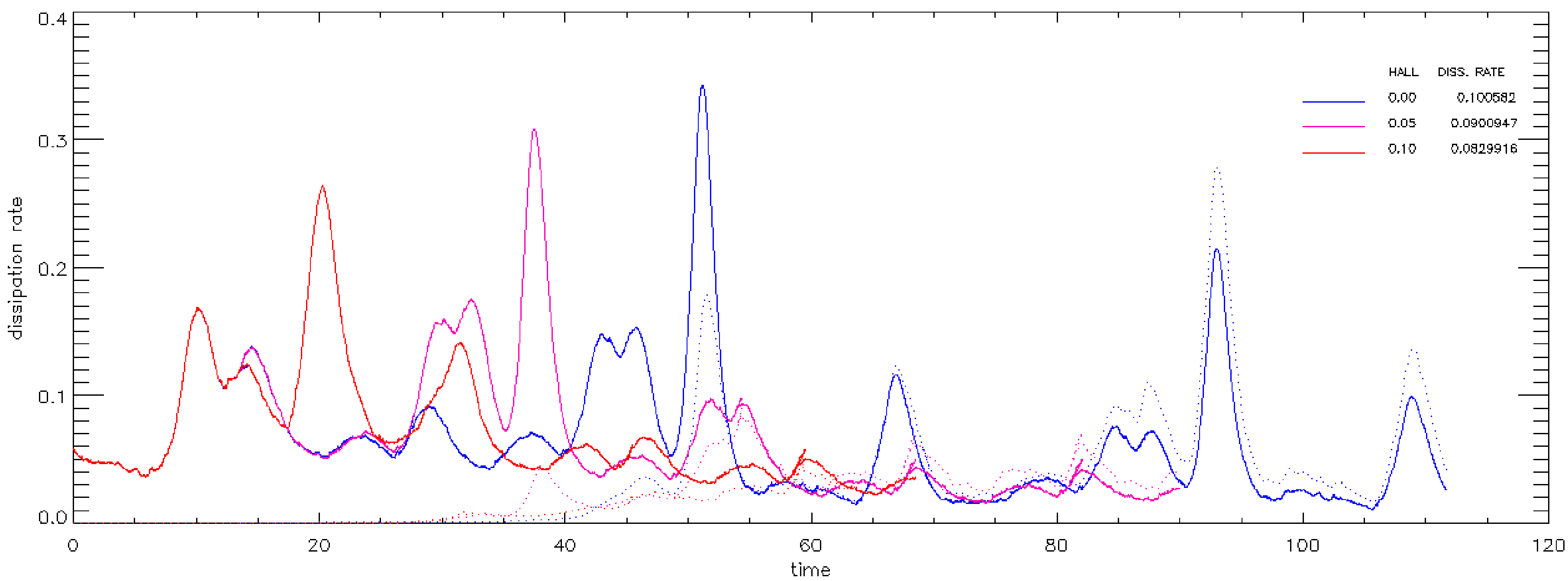
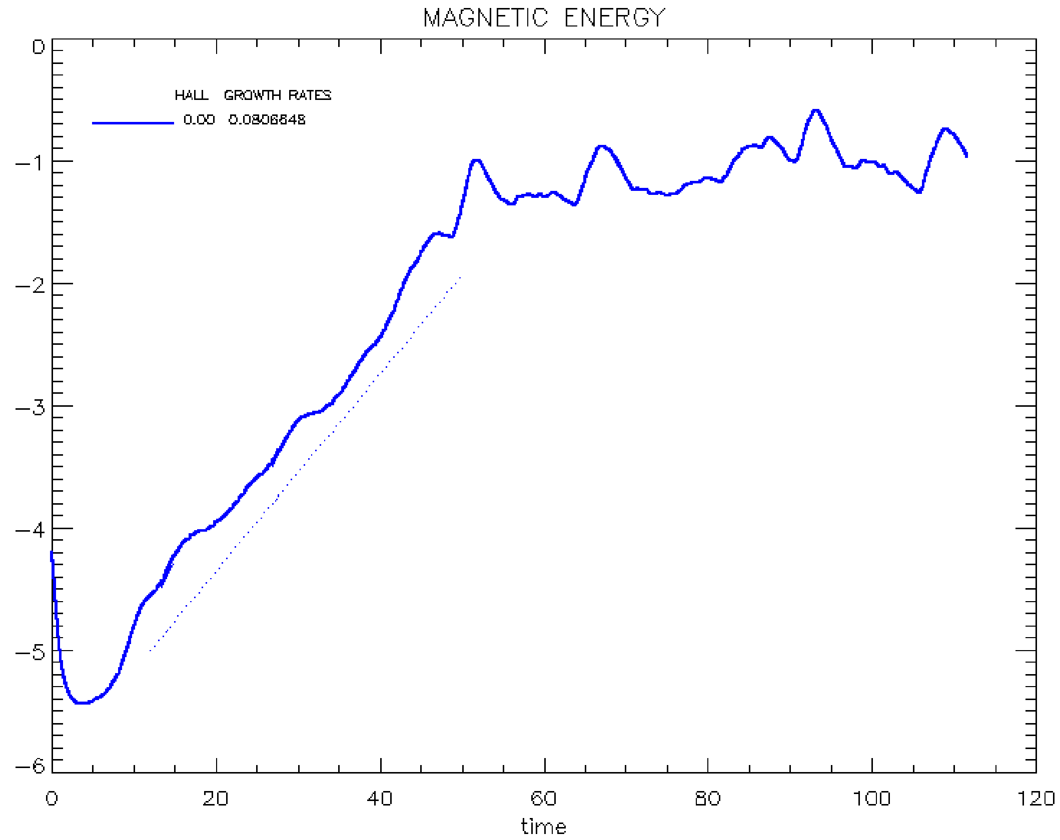
- The purely MHD run is shown in **blue**, and magnetic dissipation is shown with a dotted trace.
- The run with a moderate amount of Hall is shown in **purple**.
- The run with the largest amount of Hall is shown in **red**. The dissipation rate slightly decreases as the Hall parameter increases. The effect is a bit stronger for the magnetic dissipation rate.



Dynamo efficiency

➢ We display magnetic energy vs. time in lin-log plot to estimate its growth rate during the kinematic dynamo stage. The purely MHD run is shown in **blue**.



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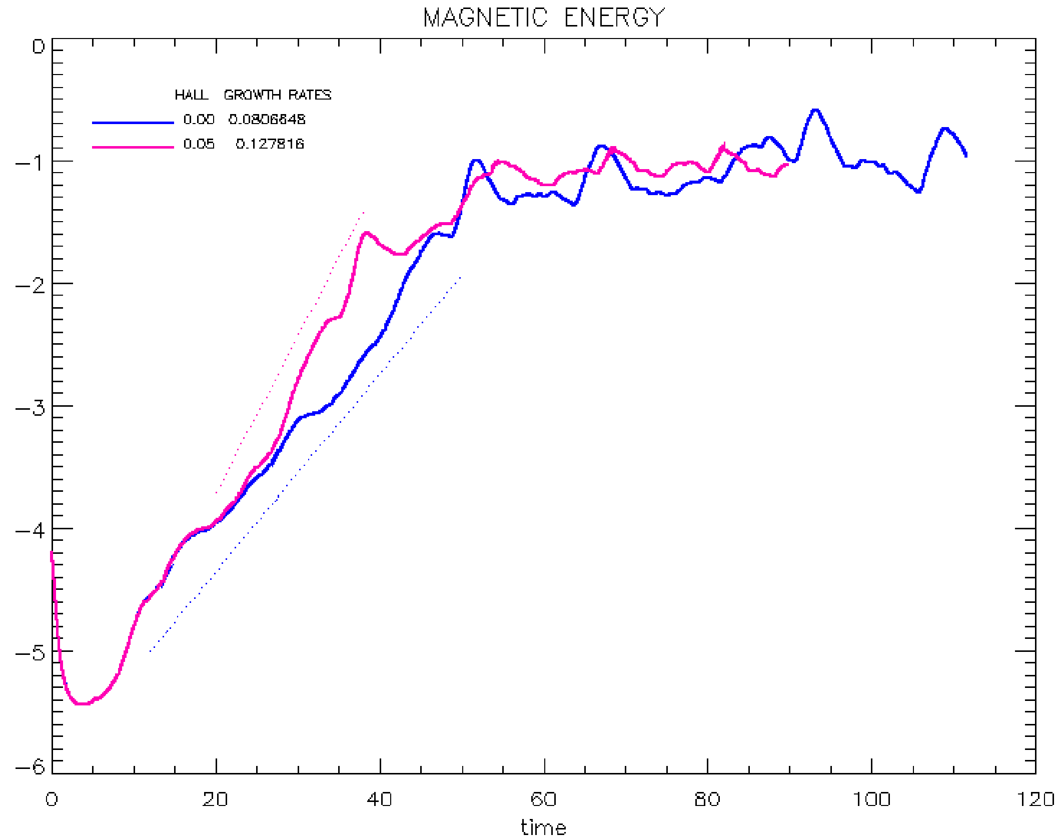
➢ An enlargement of the "linear" regime shows that the role of the Hall term is to start increasing the growth rate at a given time.

➢ This departure in the growth rate is consistent with the nonlinear nature of the Hall term with the magnetic field.



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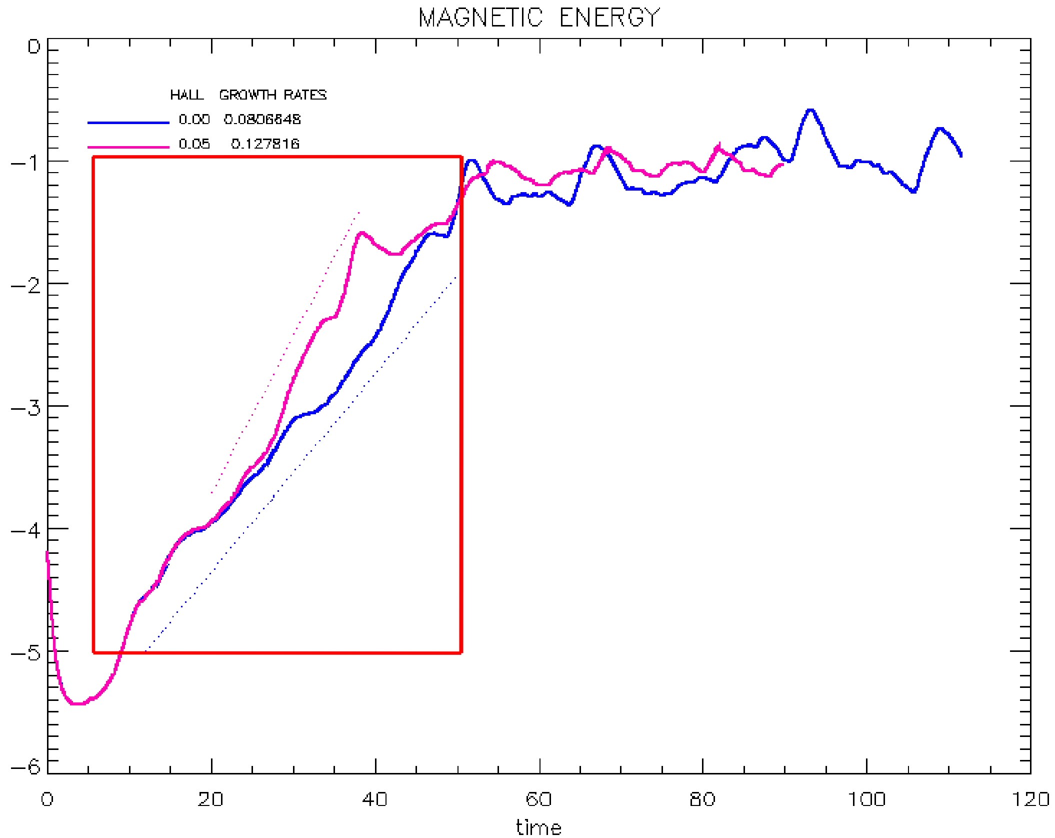

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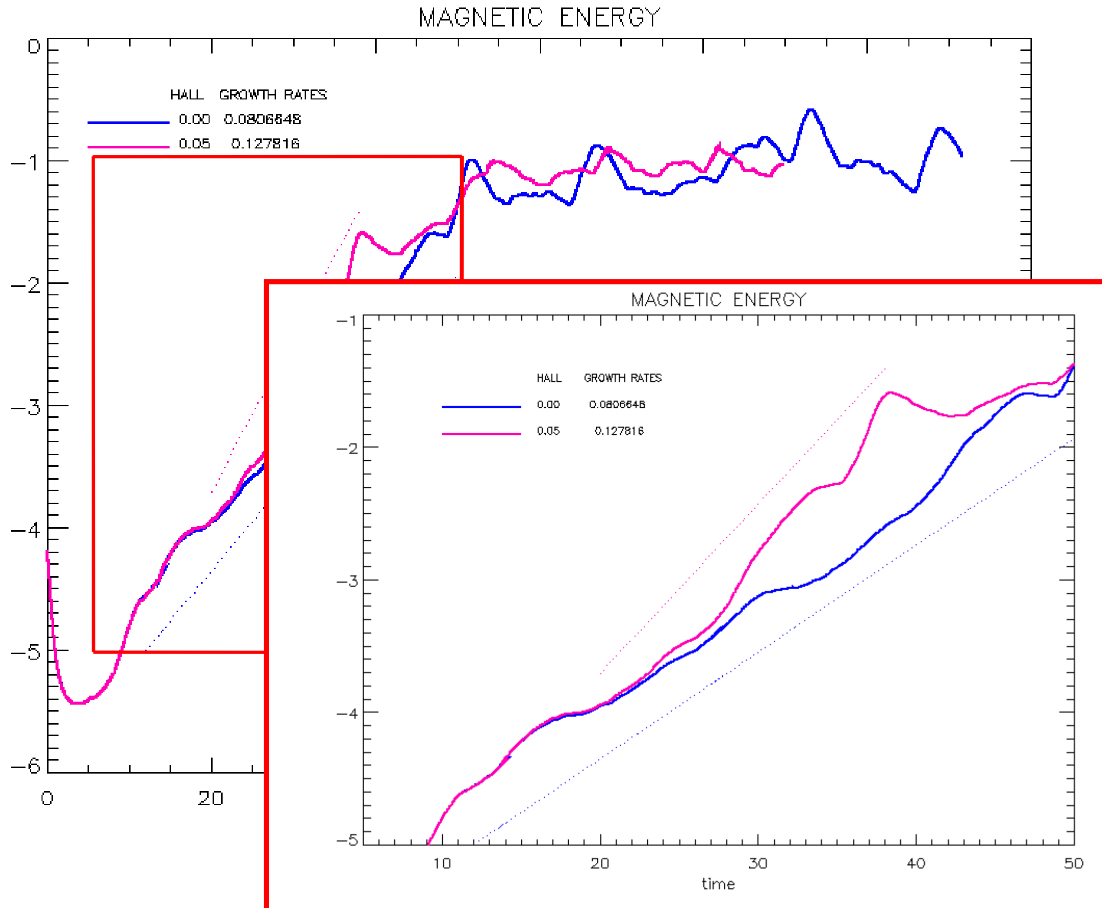

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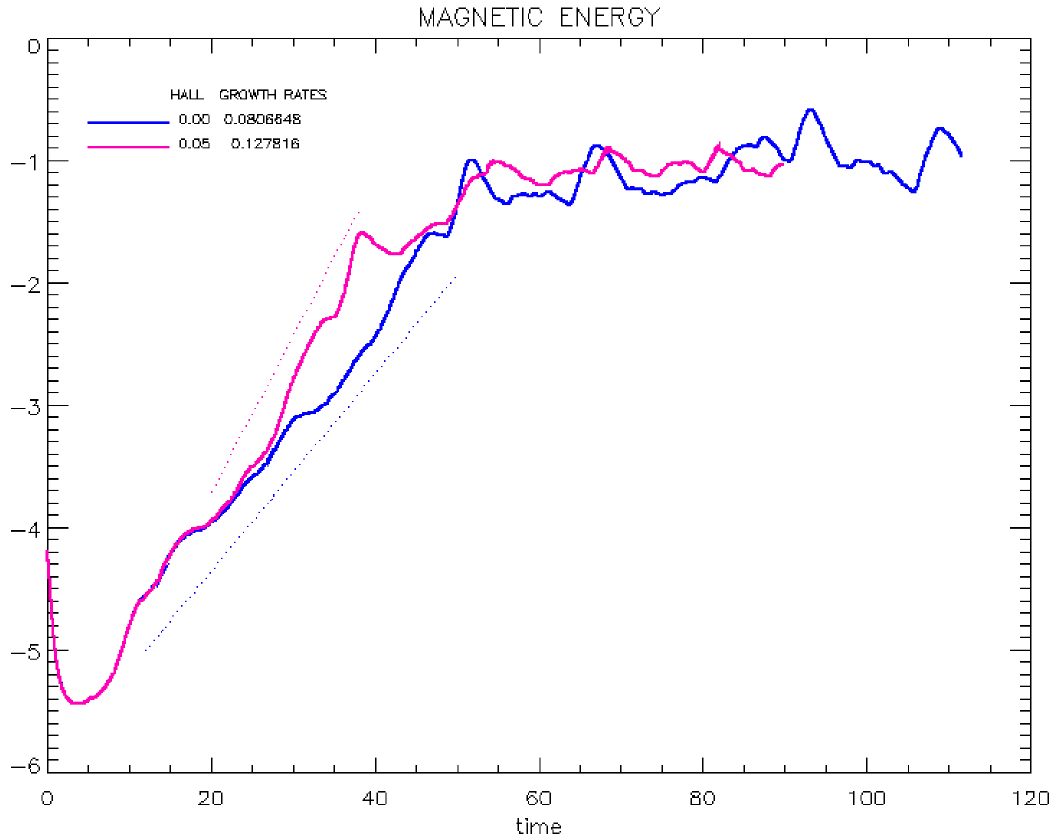

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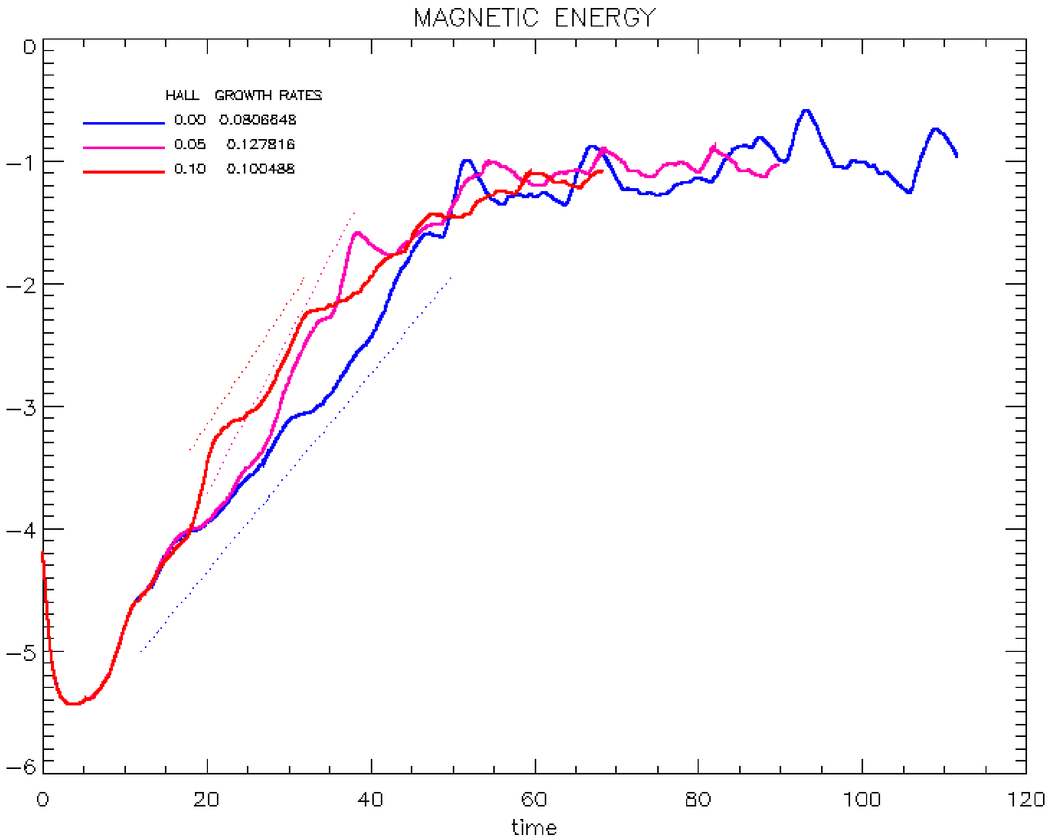
➢ The case with large Hall is shown in **red**. In this case it is not obvious that we can fit a linear growth rate.

➢ At the saturation stage, magnetic energy keeps growing at a much slower pace. At moderate Hall, the mean magnetic energy is larger. For larger Hall, the mean magnetic energy reduces.



Hall dynamo

- Looking at the induction equation $\frac{\partial B}{\partial t} = \nabla \times [(\vec{U} - \varepsilon \vec{J}) \times \vec{B}] + \eta \nabla^2 \vec{B}$

we find that the growth rate is related to the gradient of the electron velocity

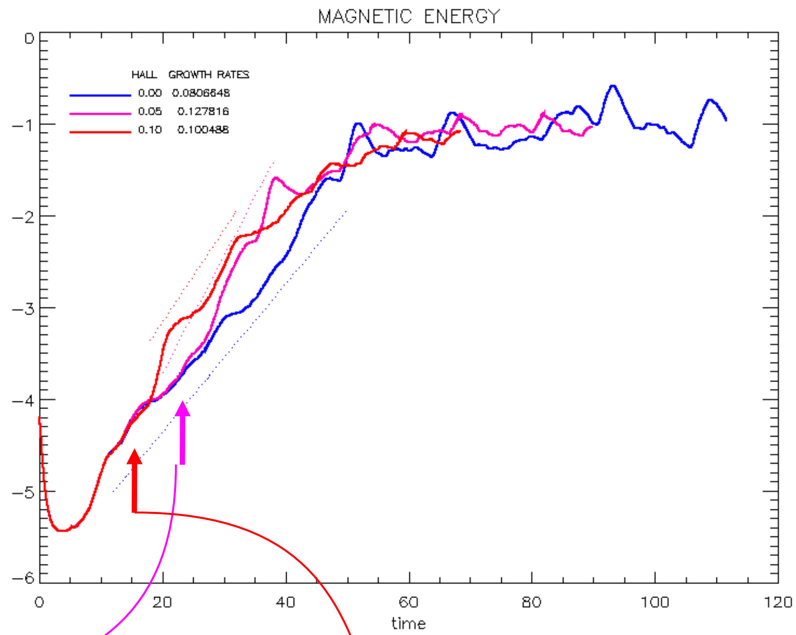
$$U_e = U - \varepsilon J$$

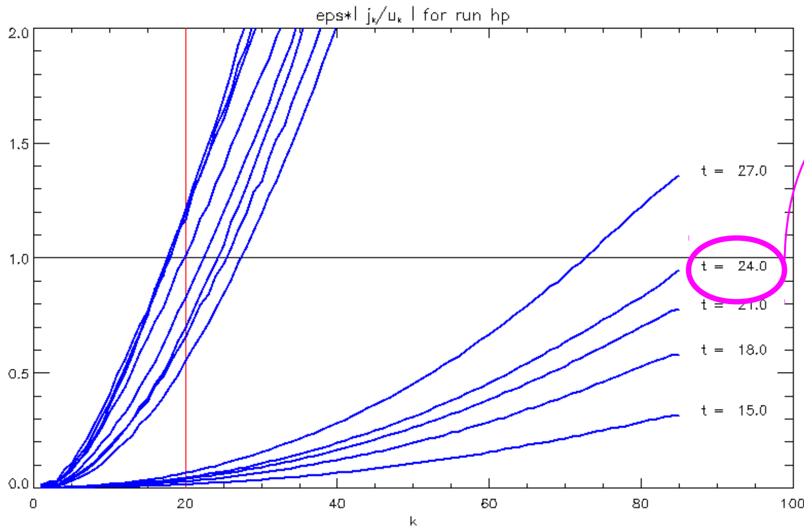
- We plot the ratio $\varepsilon | J_k | / | U_k |$ at different times.

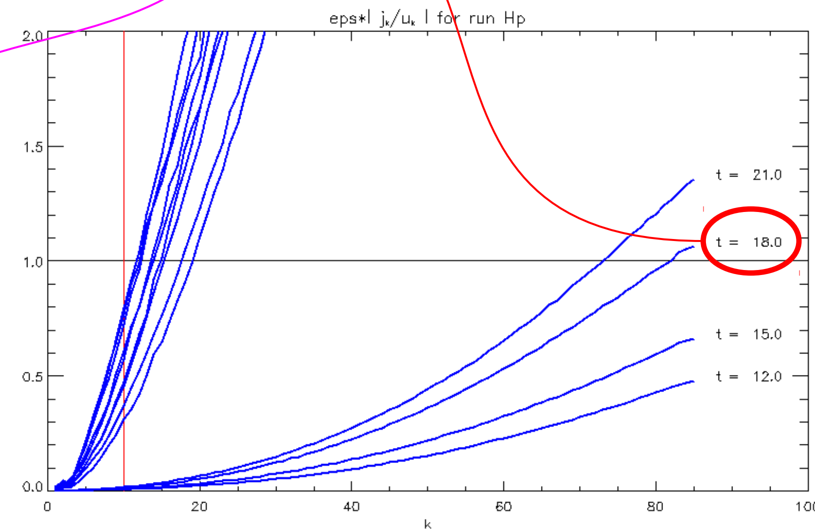
- We see that $\varepsilon | J_k |$ eventually overtakes $| U_k |$ at small scales.

- At large scales $(k < k_\varepsilon = 1/\varepsilon)$, electrons and ions move together since

$$U_e \approx U$$

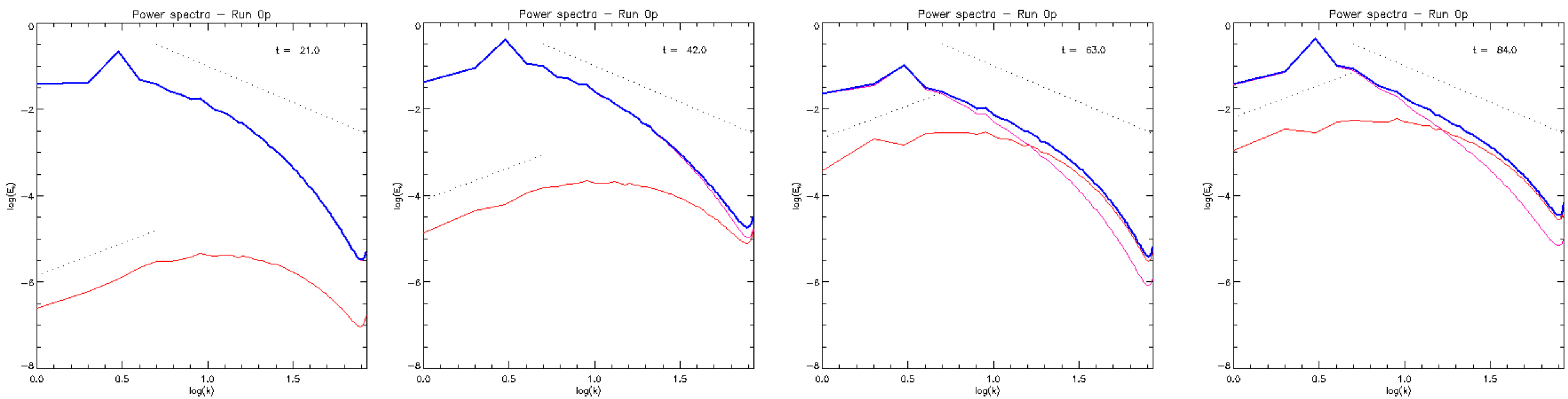






Energy power spectra

- We compute energy power spectra. Total energy is shown in **blue** for the purely MHD run.
- Magnetic energy spectra are shown in **red** at four different times, while kinetic energy is **purple**.
- The Kolmogorov slope $E(k) \approx k^{-5/3}$ is overlaid for reference.
- Kazantsev's slope $E(k) \approx k^{3/2}$, corresponding to a dynamo driven by a non-helical, large-scale and delta-correlated velocity field (Kazantsev 1968; also Brandenburg & Subramanian 2004) is also shown. It is also obtained by Kleeorin & Rogachevskii 1994 including Hall.
- Magnetic energy remains much smaller than kinetic energy, except at very small scales, when a state of super-equipartition is reached.



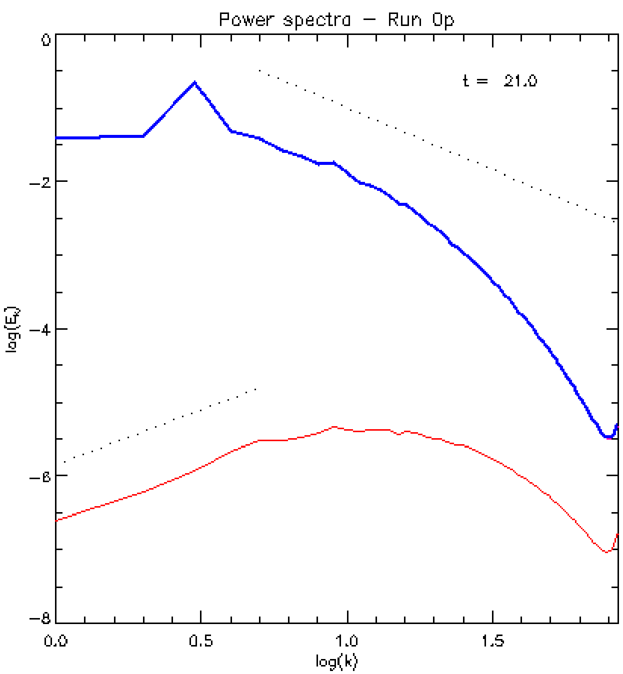
Energy power spectra

- We can compare energy spectra for three runs with different Hall strength for t=21.0

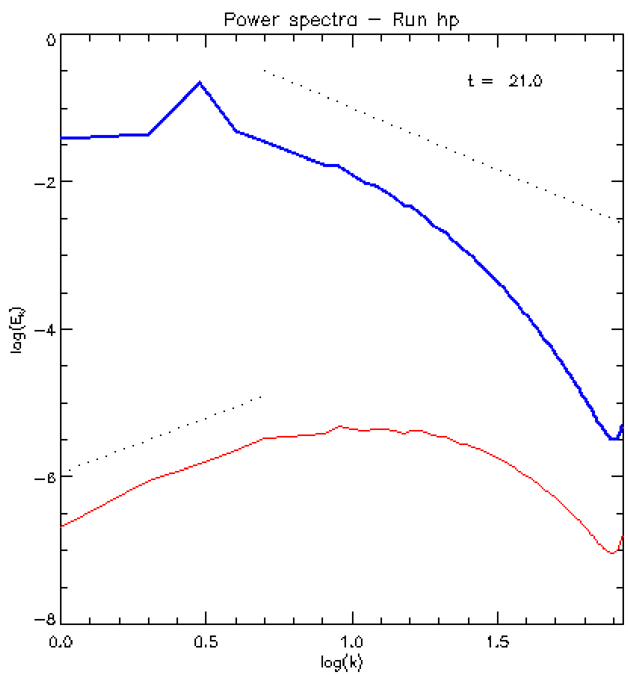
- To the left we have the purely MHD run (i.e. eps=0.00), the case with moderate Hall (eps=0.05) is at the center, and the case with intense Hall effect (eps=0.10) is the one to the right.

- Comparsions like this need to performed carefully, because of the intermittent behavior of turbulence.

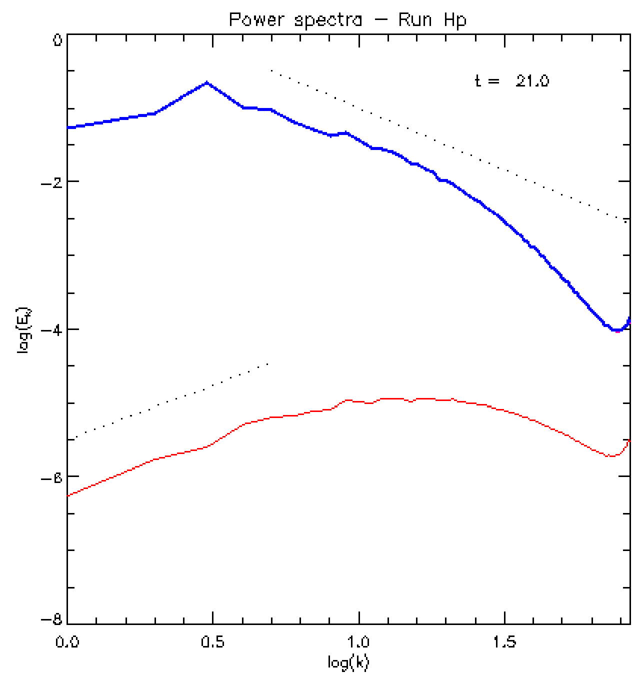
$\varepsilon = 0.00$



$\varepsilon = 0.05$



$\varepsilon = 0.10$



Energy power spectra

- We can compare energy spectra for three runs with different Hall strength for t=42.0

- To the left we have the purely MHD run (i.e. eps=0.00), the case with moderate Hall (eps=0.05) is at the center, and the case with intense Hall effect (eps=0.10) is the one to the right.

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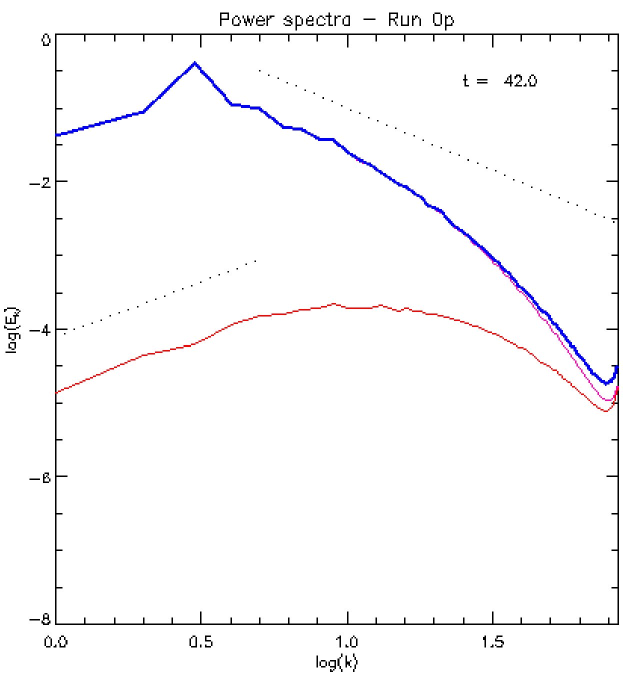
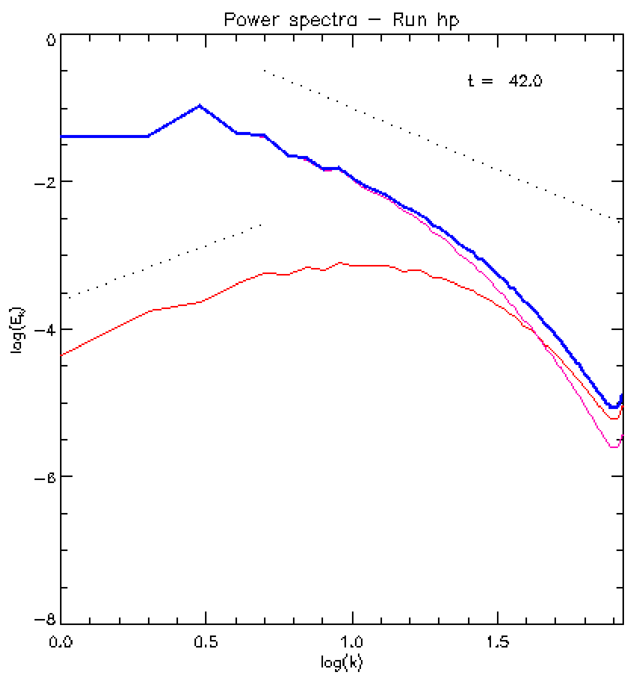
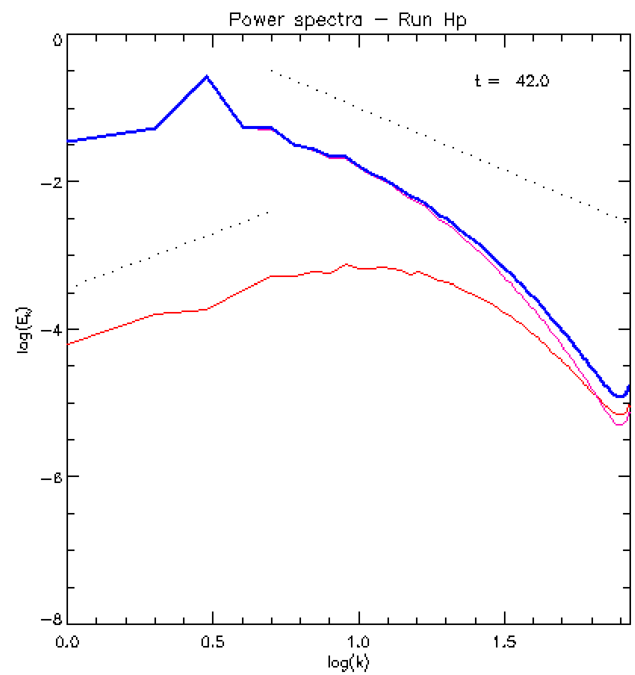
$\varepsilon = 0.00$ | $\varepsilon = 0.05$ | $\varepsilon = 0.10$







Energy power spectra

- We can compare energy spectra for three runs with different Hall strength for t=63.0
- To the left we have the purely MHD run (i.e. eps=0.00), the case with moderate Hall (eps=0.05) is at the center, and the case with intense Hall effect (eps=0.10) is the one to the right.
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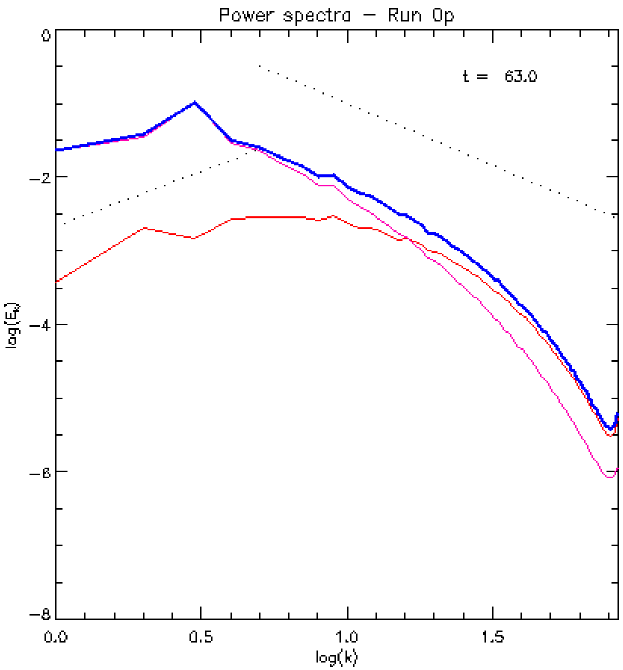
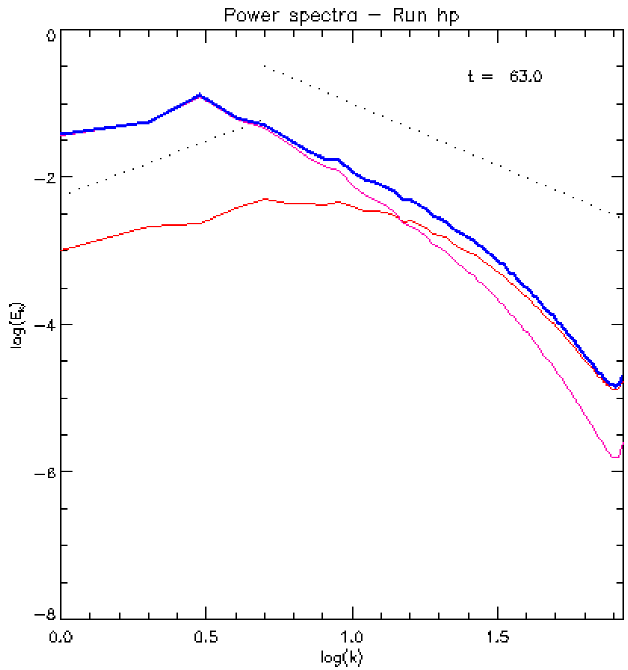
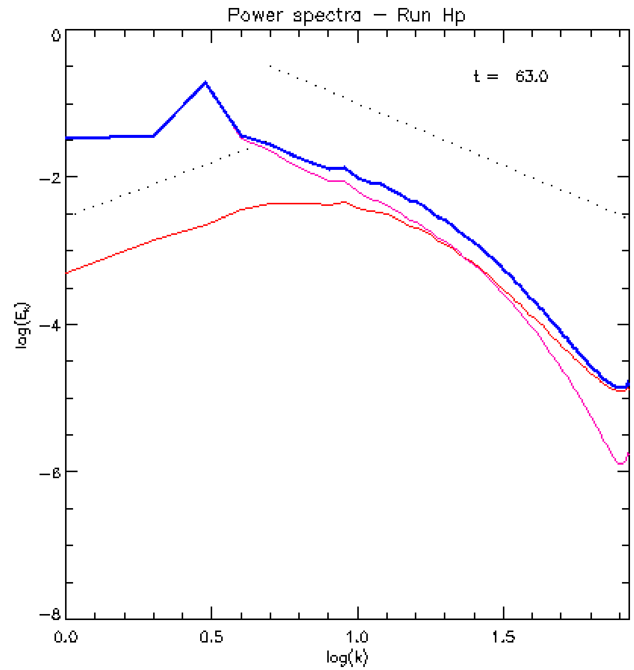
$\varepsilon = 0.00$



$\varepsilon = 0.05$



$\varepsilon = 0.10$



Energy power spectra

- We can compare energy spectra for three runs with different Hall strength for t=84.0

- To the left we have the purely MHD run (i.e. eps=0.00), the case with moderate Hall (eps=0.05) is at the center, and the case with intense Hall effect (eps=0.10) is the one to the right.

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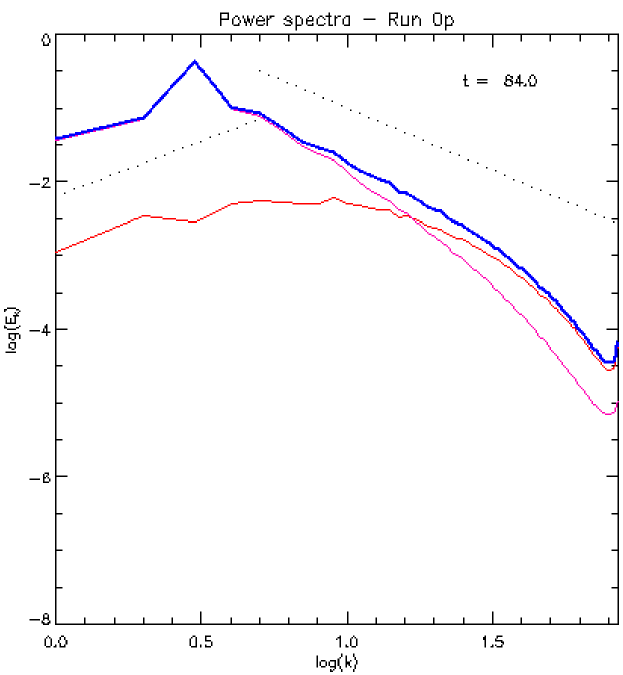
$\varepsilon = 0.00$



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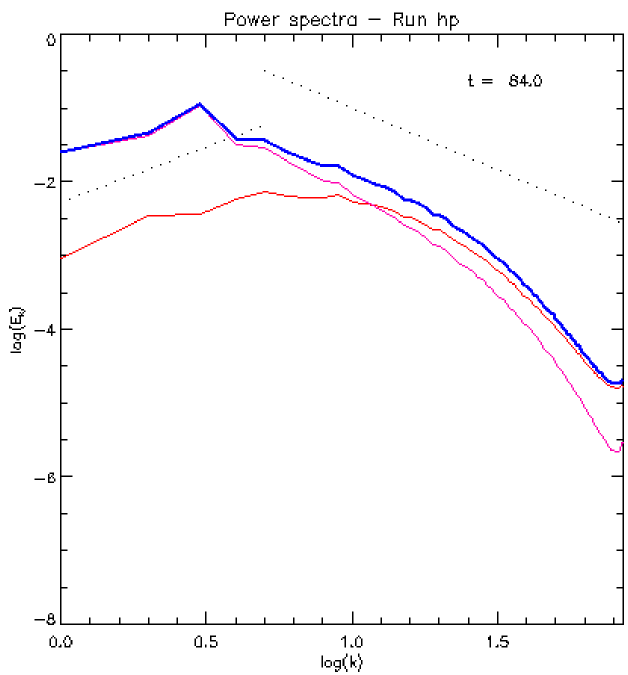
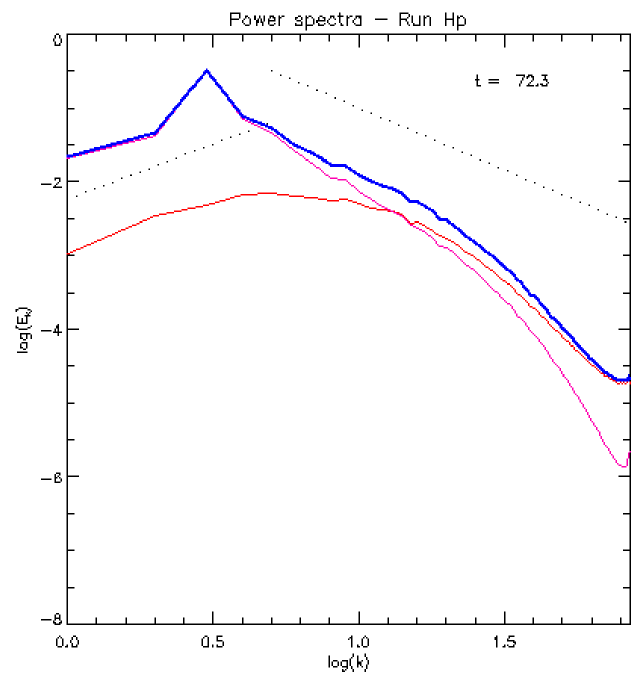


$\varepsilon = 0.10$



Current density distribution

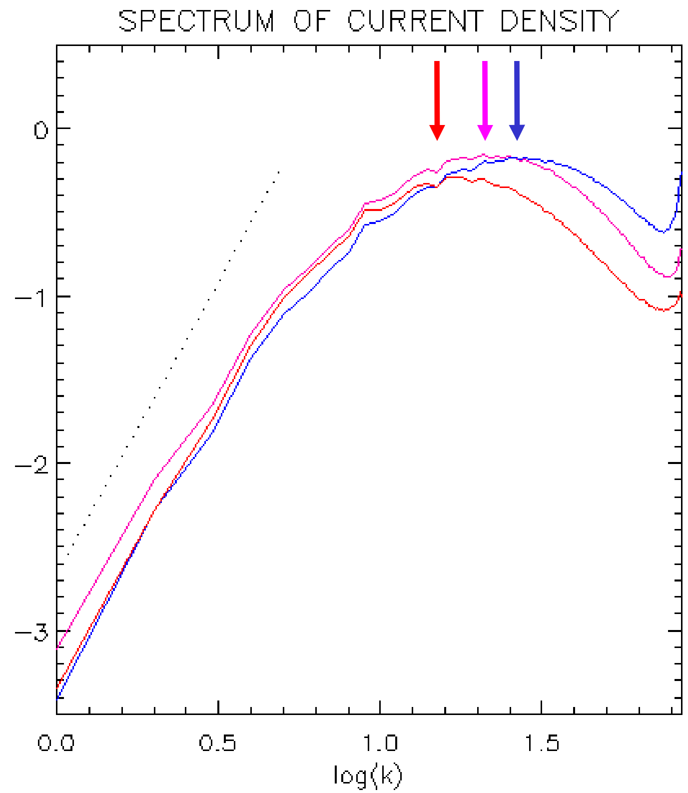
➢ We compute and compare power spectra for current density, to see how it distributes along spatial scales.

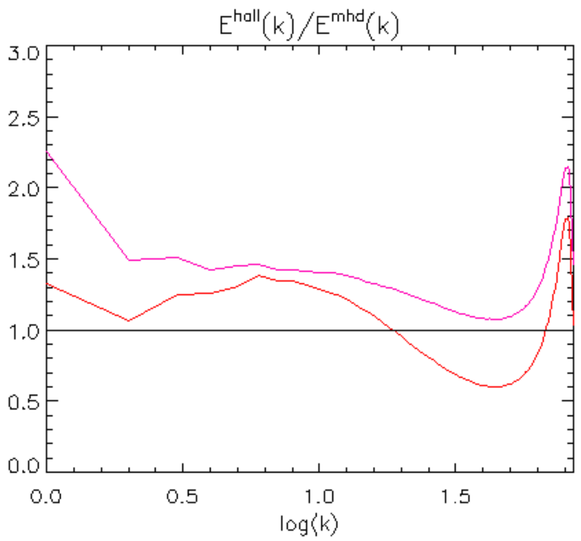
➢ The MHD run is shown in **blue**, the case with moderate Hall is **purple** and intense Hall is **red**.

➢ An average lengthscale for current density distribution can be defined as

$$k_J^2 = \frac{\int dk\, k^2 E_B(k)}{\int dk\, E_B(k)}$$

and is shown with the coloured arrows.





➢ Dissipative structures are therefore relatively “thicker” in the presence of the Hall effect.

➢ The ratio of magnetic spectra also confirms the relatively larger amount of magnetic energy when Hall is present.

Energy transfer rates in k-space

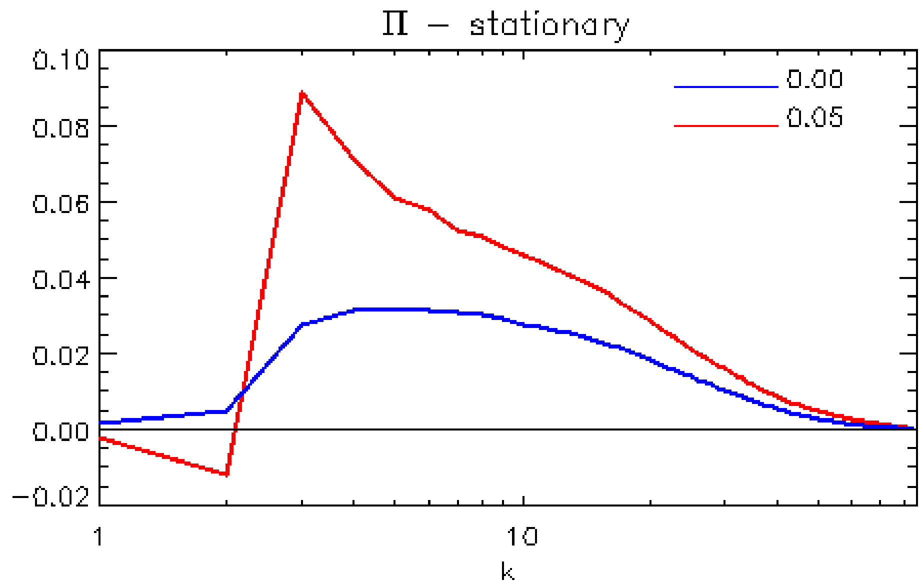
➢ We quantitatively evaluate the shell-to-shell energy transfer rates as derived by Verma 2004 and later extended by Mininni et al. 2006 for Hall MHD. Detailed energy balance equations can be written as

$$\frac{\partial E_U(K)}{\partial t} = \int d^3r\{\sum_Q [\underbrace{-\vec{U}_K \cdot (\vec{U}\cdot\nabla)\vec{U}_Q}_{T_{UU}(K,Q)} + \underbrace{\vec{U}_K \cdot (\vec{B}\cdot\nabla)\vec{B}_Q}_{T_{UB}(K,Q)}] + \nu \vec{U}\cdot\nabla^2\vec{U}_K + \vec{f}\cdot\vec{U}_K\}$$

$$\frac{\partial E_B(K)}{\partial t} = \int d^3r\{\sum_Q [\underbrace{-\vec{B}_K \cdot (\vec{U}\cdot\nabla)\vec{B}_Q}_{T_{BB}^{mhd}(K,Q)} + \underbrace{\vec{B}_K \cdot (\vec{B}\cdot\nabla)\vec{U}_Q}_{T_{BU}(K,Q)} + \underbrace{\varepsilon \vec{J}_K \cdot (\vec{B}\times\vec{J}_Q)}_{T_{BB}^{hall}(K,Q)}] + \eta \vec{B}\cdot\nabla^2\vec{B}_K\}$$

where F_K is a filter in a Fourier shell defined as

$$\vec{F}_K(\vec{r}) = \sum_{k=K}^{K+1} \vec{F}(\vec{k}) e^{i\vec{k}\cdot\vec{r}}$$



➢ The corresponding energy fluxes are defined as

$$\Pi_{FG}(k) = \frac{1}{2}\sum_{K=0}^{k}\sum_Q (T_{FG}(K,Q) + T_{GF}(K,Q))$$

➢ We show the total energy flux in k-space for the purely MHD run (**blue**) and also for moderate Hall (**red**).

Energy transfer rates in k-space

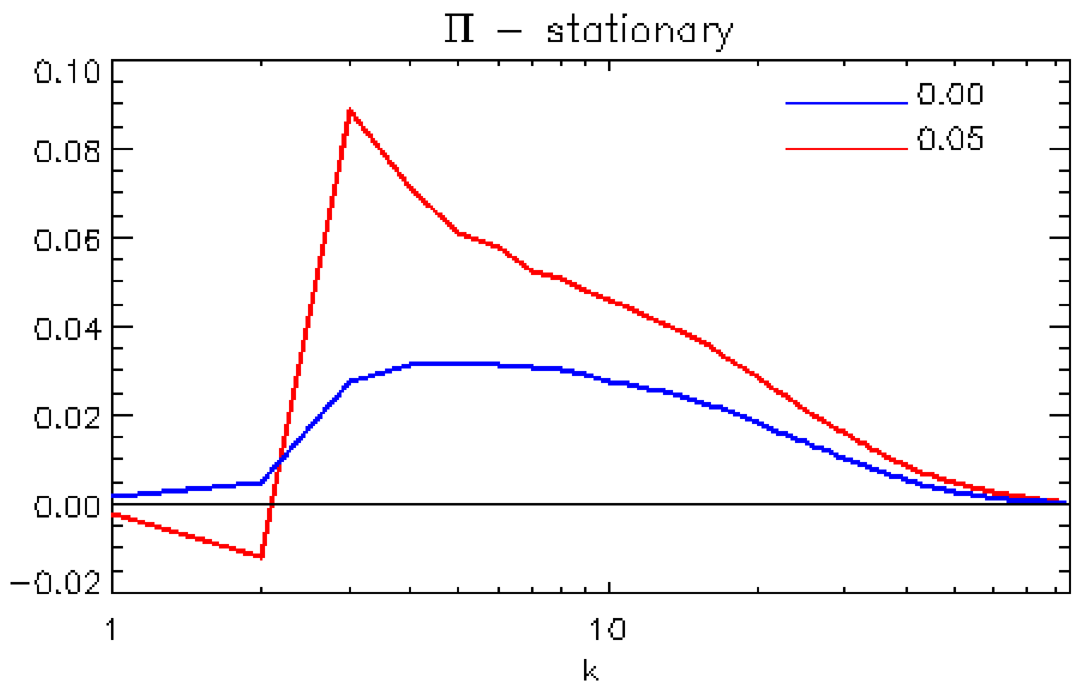
➢ We display the energy fluxes for the same runs, but split into its various terms.

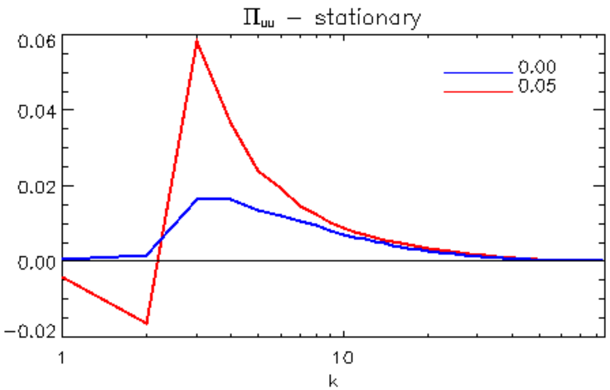
➢ None of these terms is negligible in any of the simulations. More important, the Hall effect modifies all the terms and not just the Hall flux.

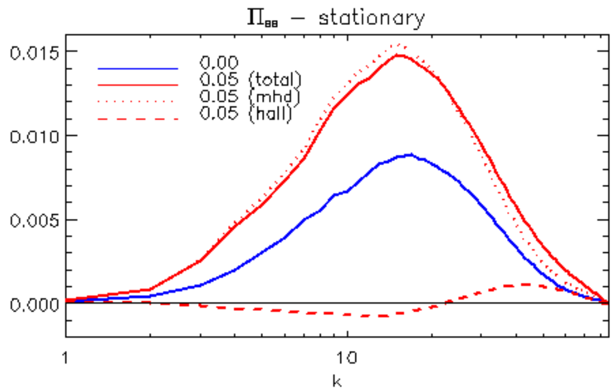
➢ As reported in Mininni et al. 2006, the energy flux due to Hall reverses sign exactly at

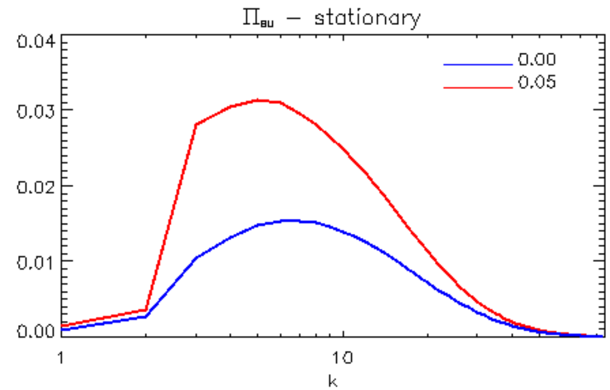
$$k_\varepsilon = 1/\varepsilon$$

➢ Note that the UU flux in the Hall run is responsible for some backscattering at scales larger than externally forced ones.









Conclusions

- We performed several runs of the Hall-MHD equations, considering different values of the Hall parameter to study the efficiency of turbulent dynamo action.

- The Hall effect causes magnetic energy to grow faster in the kinematic stage and also to saturate at a higher level. This is the case up to an optimal value of the Hall parameter, the efficiency is reduced for values larger than this (as shown in Mininni et al. 2005).

- The dissipation rate of magnetic energy is lower when Hall is present, and the dissipative structures are relatively “thicker”.

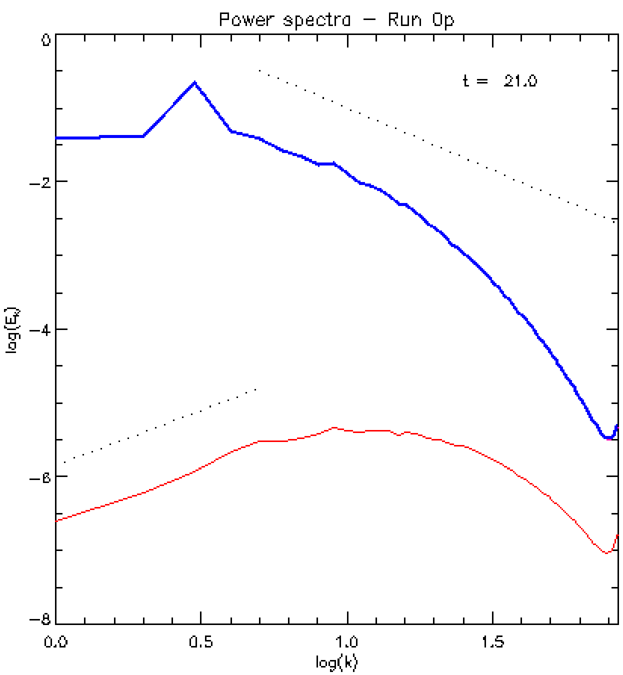
- All the terms participating in the energy flux in k-space change considerably in the presence of Hall. The term explicitly related to Hall, contributes to inhibit the direct cascade, which is consistent with a higher level of magnetic energy and smaller dissipation rate.

- With Hall, the UU energy flux becomes negative at large scales, which can be interpreted as large scale flows driven by small-scale magnetic fields.

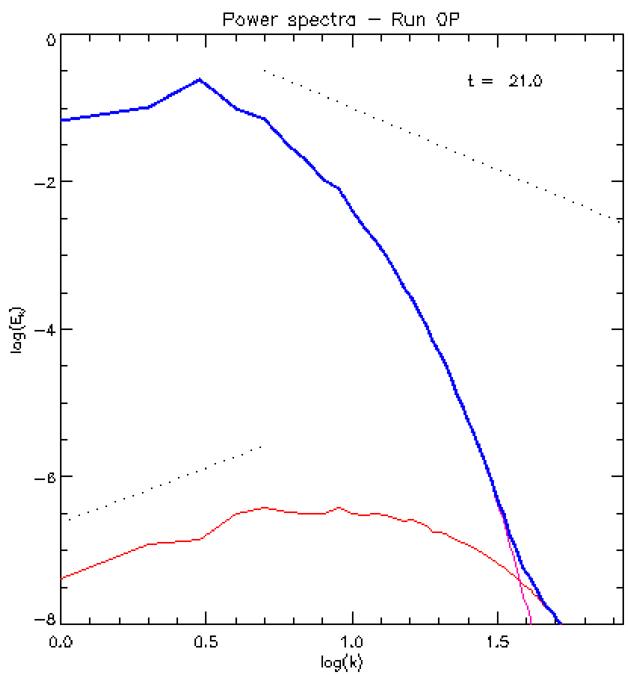
Large Pm: energy power spectra

- We can compare energy spectra for the following three runs at t=21.0
- To the left we have the purely MHD run (i.e. eps=0.00) and Pm=1. The case at the center is also MHD, but with Pm=10. The run with moderate Hall (eps=0.05) and Pm=10 is the one to the right.
- Many of the features that we have shown for Pm=1 are also present here, except for the large separation of the dissipative scales.

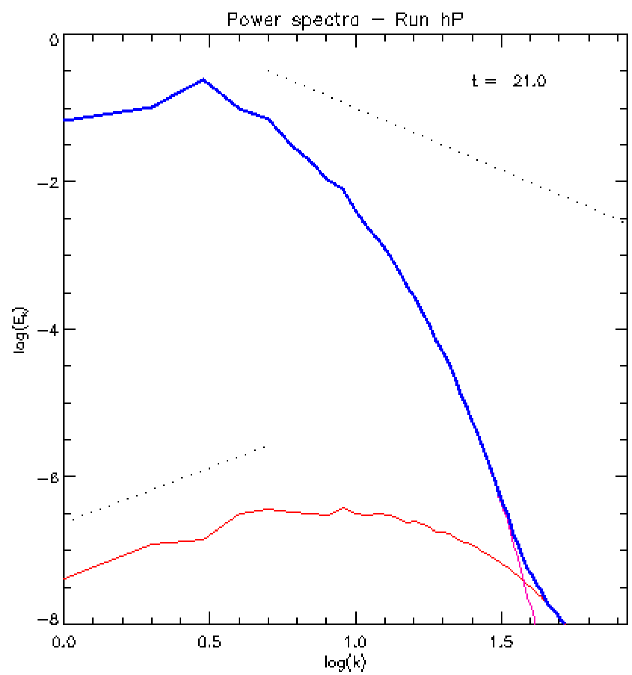
$\varepsilon = 0.00 \quad Pm = 1$



$\varepsilon = 0.00 \quad Pm = 10$



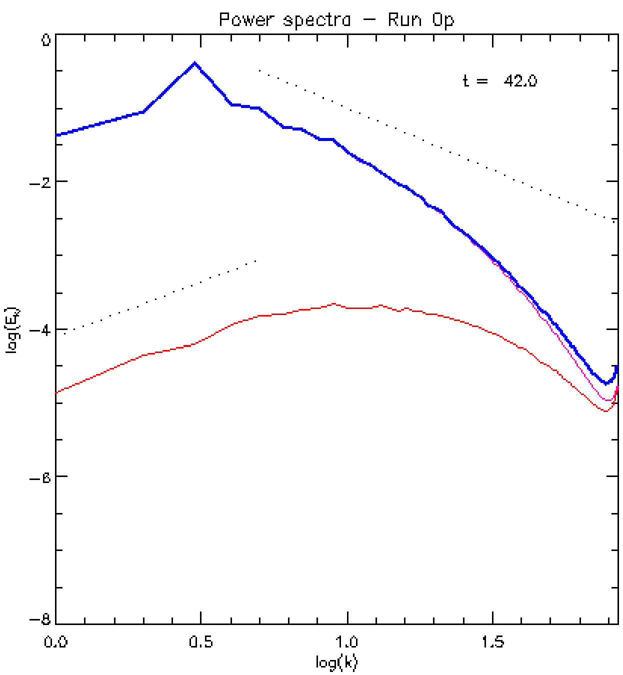
$\varepsilon = 0.05 \quad Pm = 10$



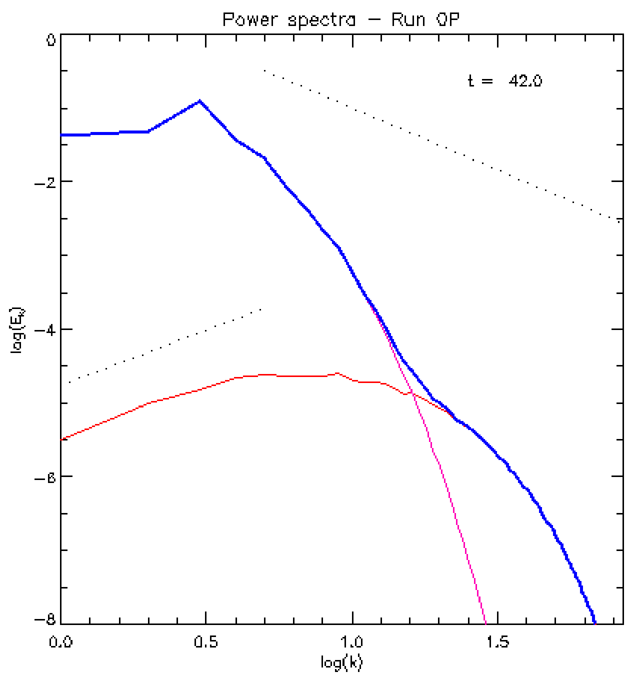
Large Pm: energy power spectra

- We can compare energy spectra for the following three runs at t=42.0
- To the left we have the purely MHD run (i.e. eps=0.00) and Pm=1. The case at the center is also MHD, but with Pm=10. The run with moderate Hall (eps=0.05) and Pm=10 is the one to the right.
- Many of the features that we have shown for Pm=1 are also present here, except for the large separation of the dissipative scales.

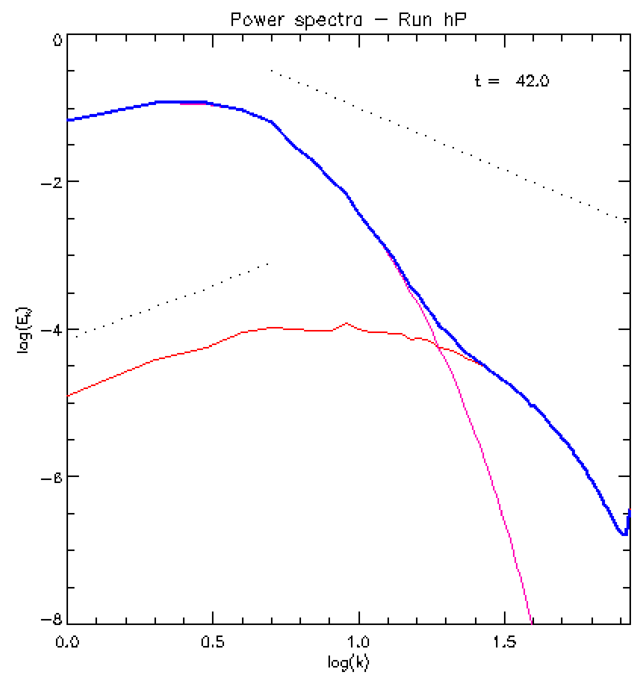
$\varepsilon = 0.00 \quad Pm = 1$



$\varepsilon = 0.00 \quad Pm = 10$



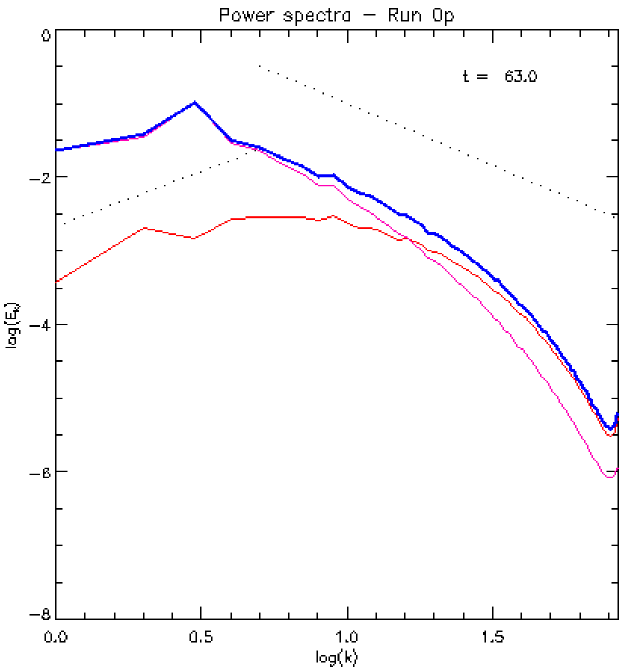
$\varepsilon = 0.05 \quad Pm = 10$

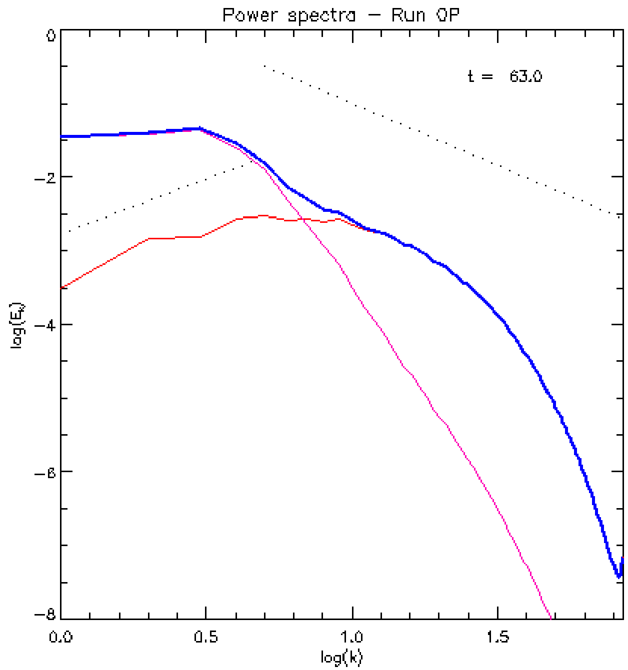


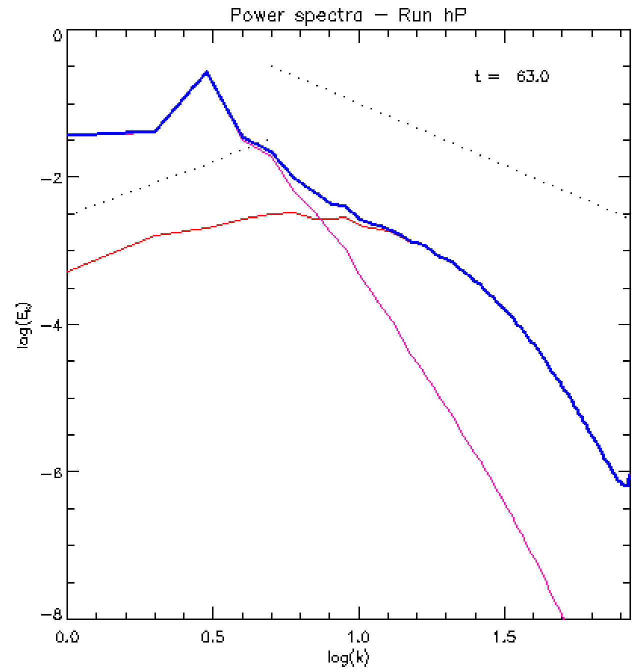
Large Pm: energy power spectra

- We can compare energy spectra for the following three runs at t=63.0
- To the left we have the purely MHD run (i.e. eps=0.00) and Pm=1. The case at the center is also MHD, but with Pm=10. The run with moderate Hall (eps=0.05) and Pm=10 is the one to the right.
- Many of the features that we have shown for Pm=1 are also present here, except for the large separation of the dissipative scales.

$\varepsilon = 0.00 \quad Pm = 1$ $\qquad$ $\varepsilon = 0.00 \quad Pm = 10$ $\qquad$ $\varepsilon = 0.05 \quad Pm = 10$







Large Pm: energy power spectra

- We can compare energy spectra for the following three runs at t=84.0
- To the left we have the purely MHD run (i.e. eps=0.00) and Pm=1. The case at the center is also MHD, but with Pm=10. The run with moderate Hall (eps=0.05) and Pm=10 is the one to the right.
- Many of the features that we have shown for Pm=1 are also present here, except for the large separation of the dissipative scales.

$\varepsilon = 0.00 \quad Pm = 1$ $\qquad \varepsilon = 0.00 \quad Pm = 10$ $\qquad \varepsilon = 0.05 \quad Pm = 10$