Fundamentals of magnetohydrodynamics

Part IV

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The dimensionless version, for a length scale $L_0$, density $n_0$ and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

\[
\frac{dU}{dt} = \frac{1}{\varepsilon} (E + U \times B) - \frac{\beta}{n} \nabla p_i - \frac{\eta}{\varepsilon n} J + \nu \nabla^2 U
\]

\[
0 = -\frac{1}{\varepsilon} (E + U_e \times B) - \frac{\beta}{n} \nabla p_e + \frac{\eta}{\varepsilon n} J
\]

where

\[
J = \nabla \times B = \frac{n}{\varepsilon} (U - U_e)
\]

We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma beta $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 v_{ie}}{\omega_{pi}^2 L_0 v_A}$

Adding these two equations yields:

\[
n \frac{dU}{dt} = (\nabla \times B) \times B - \beta \nabla (p_i + p_e) + \nu \nabla^2 U
\]

On the other hand, using

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi
\]

\[
B = \nabla \times A
\]

\[
\frac{\partial A}{\partial t} = (U - \frac{\varepsilon}{n} \nabla \times B) \times B - \nabla \phi + \frac{\varepsilon \beta}{n} \nabla p_e - \frac{\eta}{n} \nabla \times B
\]
Hall-MHD reconnection in 2.5D

Hall reconnection has extensively been studied for the Earth's magnetopause and also the magnetotail. The Hall effect is expected to increase the reconnection rate.

The simplest geometrical setup is 2.5D, for which the velocity and magnetic field can be written in terms of four scalar fields (Gómez 2006, Space Sci. Rev. 122, 231; Gómez et al. 2006, Adv. Sp. Res. 37, 1287)

\[
\begin{align*}
B_\perp(x, y, t) &= \nabla \times [\hat{z} a(x, y, t)] + \hat{z} b(x, y, t) \\
U_\perp(x, y, t) &= \nabla \times [\hat{z} \phi(x, y, t)] + \hat{z} u(x, y, t) \\
U_e &= U - \epsilon J = \nabla \times [\hat{z} (\phi - \epsilon b)] + \hat{z} (u - \epsilon j)
\end{align*}
\]

The 2.5D Hall-MHD equations are

\[
\begin{align*}
\partial_t a &= [\phi - \epsilon b, a] + \eta \nabla^2 a \\
\partial_t \omega &= [\phi, \omega] + [j, a] + \nu \nabla^2 \omega \\
\partial_t b &= [\phi, b] + [u - \epsilon j, a] + \eta \nabla^2 b \\
\partial_t u &= [b, a] + [\phi, u] + \nu \nabla^2 u 
\end{align*}
\]

where \( \omega = -\nabla^2 \phi \), \( j = -\nabla^2 a \)

In the absence of Hall, the parallel components \((u,b)\) have no influence on the perp. dynamics.

When Hall is present, the parallel components will be turned on and couple to the perp. components.
When the Hall effect is neglected (pure MHD) 2D reconnection is possible.

Magnetic fieldlines (left) and flow streamlines (right) are shown at three successive Alfvén times. Blue contours are positive and the red ones are negative.
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Magnetic fieldlines (left) and flow streamlines (right) are shown at three successive Alfvén times. Blue contours are positive and the red ones are negative.
When the Hall effect is considered, the out-of-plane fields are generated. They were initially set to zero.

We show contour plots of the four scalar fields at three successive Alfven times for $\varepsilon = 0.07$.

The out-of-plane magnetic field develops a quadrupolar pattern, while the velocity field develops a net flow at the reconnection region.

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Blue contours are positive and the red ones are negative.
The out-of-plane **current density** is shown for the cases $\varepsilon = 0.00$ and 0.07.

The current sheets becomes narrower and smaller as the Hall parameter is increased.

The **reconnected flux** also increases with the Hall parameter, confirming previous results from collisionless and also Hall-MHD simulations.

The plot shows reconnected flux vs. time for different values of the Hall parameter.
For each species \( s \) we have (Goldston & Rutherford 1995):

1. **Mass conservation**
   \[
   \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s U_s) = 0
   \]

2. **Equation of motion**
   \[
   m_s n_s \frac{dU_s}{dt} = q_s n_s \left( E + \frac{1}{c} U_s \times B \right) - \nabla p_s + \nabla \cdot \sigma_s + \sum_{s'} R_{ss'}
   \]

3. **Momentum exchange rate**
   \[
   R_{ss'} = -m_s n_s \nu_{ss'} (U_s - U_{s'})
   \]

These moving charges act as sources for electric and magnetic fields:

1. **Charge density**
   \[
   \rho_c = \sum_s q_s n_s \approx 0
   \]

2. **Electric current density**
   \[
   J = \frac{c}{4\pi} \nabla \times B = \sum_s q_s n_s U_s
   \]
Let us now retain electron inertia (i.e. $0 < m_e \ll m_i$):

- **Mass conservation**:
  \[ 0 = \frac{\partial n}{\partial t} + \nabla \cdot (nU) \quad , \quad n_e \equiv n_i \equiv n \]

- **Ions**:
  \[ m_i n \frac{dU}{dt} = -en(E + \frac{1}{c} U \times B) - \nabla p_i + R \]

- **Electrons**:
  \[ m_e n \frac{dU_e}{dt} = -en(E + \frac{1}{c} U_e \times B) - \nabla p_e - R \]

- **Friction force**:
  \[ R = -m_i n \nu_{ie} (U - U_e) \]

- **Ampere's law**:
  \[ J = \frac{c}{4\pi} \nabla \times B = en(U - U_e) \quad \Rightarrow \quad R = -\frac{m\nu_{ie}}{e} J \]

- **Polytropic laws**:
  \[ p_i \propto n^\gamma \quad , \quad p_e \propto n^\gamma \]
Retaining electron inertia: EIHMHD equations

The dimensionless version, for a length scale $L_0$, density $n_0$ and Alfvén speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{dU_i}{dt} = \frac{1}{\beta} (E + U_i \times B) - \frac{\beta}{n} \nabla p_i - \frac{\eta}{\epsilon n} J$$

$$\frac{m_e}{m_i} \frac{dU_e}{dt} = -\frac{1}{\beta} (E + U_e \times B) - \frac{\beta}{n} \nabla p_e + \frac{\eta}{\epsilon n} J$$

where $J = \nabla \times B = \frac{n}{\epsilon} (U_i - U_e)$

- We defined the Hall parameter $\epsilon = \frac{c}{\omega_{pi} L_0}$

- as well as the plasma beta $\beta = \frac{p_0}{m_i n_0 v_A^2}$

- and the electric resistivity $\eta = \frac{c^2 v_{ie}}{\omega_{pi}^2 L_0 v_A}$

Adding these two equations yields:

$$\frac{dU}{dt} = (\nabla \times B) \times (B + \epsilon^2 \nabla \times J) - \nabla p$$

where $U = \frac{m_i U_i + m_e U_e}{m_i + m_e}$

and $p = p_i + p_e$
Retaining electron inertia: EIHMHD equations

In the equation for electrons (assuming incompressibility)

\[
\begin{align*}
\frac{m_e}{m_i} \frac{dU_e}{dt} &= -\epsilon (E + U_e \times B) - \beta_e \nabla p_e + \frac{\eta}{\epsilon} J \\
J &= \nabla \times B = \frac{1}{\epsilon} (U_i - U_e)
\end{align*}
\]

we replace

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \quad \text{and} \quad B = \nabla \times A
\]

to obtain

\[
\frac{\partial}{\partial t} \left( A - \epsilon^2 \nabla^2 A - \frac{\epsilon^2}{\epsilon} U \right) = (U - \epsilon J) \times (B - \epsilon^2 \nabla^2 B - \frac{\epsilon^2}{\epsilon} w) - \nabla (\phi - \epsilon \beta_e p_e - \frac{\epsilon^2}{\epsilon} \frac{U_e^2}{2}) + \eta \nabla^2 A
\]

Electron inertia is quantified by the dimensionless parameter \( \epsilon_e = \sqrt{\frac{m_e}{m_i}} \epsilon = \frac{c}{\omega_{pe} L_0} \)

Just as the Hall effect introduces the new spatial scale \( k_H = \frac{1}{\epsilon} \) (the ion skin depth), electron inertia introduces the electron skin depth \( k_e = \frac{1}{\epsilon_e} \) which satisfies

\[
k_e = \sqrt{\frac{m_i}{m_e}} k_H \gg k_H
\]
We now express the EIHMHD equations in 2.5D geometry. I.e. for simplicity we assume \( \partial_z = 0 \) and therefore

\[
\mathbf{B}_\perp = \nabla \times [\mathbf{a}(x, y, t)] + \mathbf{b}(x, y, t)
\]

\[
\mathbf{U} = \nabla \times [\mathbf{\phi}(x, y, t)] + \mathbf{u}(x, y, t)
\]

The equations for these four scalar fields are

\[
\begin{align*}
\partial_t a' &= [\phi - \varepsilon b, a'] + \eta \nabla^2_\perp a \\
\partial_t \omega &= [\phi, \omega] - [a, j] + \nu \nabla^2_\perp \omega \\
\partial_t b' &= [\phi - \varepsilon b, b'] + [u - \varepsilon j, a'] + \eta \nabla^2_\perp b \\
\partial_t u &= [\phi, u] - [a, b] + \nu \nabla^2_\perp u
\end{align*}
\]

where

\[
a' = (1 - \varepsilon^2 \nabla^2_\perp) a - \frac{\varepsilon^2}{\varepsilon} u \quad \text{and} \quad b' = (1 - \varepsilon^2 \nabla^2_\perp) b - \frac{\varepsilon^2}{\varepsilon} w
\]
If we linearize our equations around an equilibrium characterized by a uniform magnetic field, we obtain the following dispersion relation:

\[
\left( \frac{\omega}{k \cdot B_0} \right)^2 \pm \frac{k \varepsilon}{1 + \varepsilon_e^2 k^2} \left( \frac{\omega}{k \cdot B_0} \right) - \frac{1}{1 + \varepsilon_e^2 k^2} = 0
\]

Asymptotically, at very large k, we have two branches

- \( \omega \xrightarrow{k \to \infty} \omega_{ce} \cos \theta \)
- \( \omega \xrightarrow{k \to \infty} \omega_{ci} \cos \theta \)

while for very small k, both branches simply become Alfven modes, i.e.

- \( \omega \xrightarrow{k \to 0} k \cos \theta \)

Different approximations, just as one-fluid MHD, Hall-MHD and electron-inertia HMHD can clearly be identified in this diagram.
For each species $s$ in the incompressible and ideal limit

$$
\nabla \cdot \left( m_s n_s \left( \partial_t U_s - U_s \times W_s \right) \right) = q_s n_s \left( E + \frac{1}{c} U_s \times B \right) - \nabla \left( p_s + m_s n_s \frac{U_s^2}{2} \right)
$$

Using that

$$
J = \frac{c}{4\pi} \nabla \times B = \sum_s q_s n_s U_s \quad \text{and} \quad E = -\frac{1}{c} \partial_t A - \nabla \phi
$$

we can readily show that energy is an ideal invariant, where

$$
E = \int d^3r \left( \sum_s m_s n_s \frac{U_s^2}{2} + \frac{B^2}{8\Pi} \right)
$$

We also have a helicity per species which is conserved, where

$$
H_s = \int d^3r \left( \nabla \cdot \left( A + \frac{cm_s}{q_s} U_s \right) \right) \cdot \left( B + \frac{cm_s}{q_s} W_s \right)
$$
EIHMHD simulations

- We perform 512x512 simulations of the EIHMHD equations in 2.5D geometry to study magnetic reconnection.
- We force an external field with a double hyperbolic tangent profile to drive reconnection at two X points.
- At three successive times we show the current density in the background, the proton flow in the left half of each frame, and the electron flow on the right half.
- Although at large scales both flows look quite similar, in the vicinity of the X points, electrons tend to move much faster, close to the Alfven velocity.
The total reconnected flux at the X-point is the magnetic flux through the perpendicular surface that extends from the O-point to the X-point.

We compare the total reconnected flux between a run that includes electron inertia and another one that does not.

The reconnection rate is the time derivative of these two curves.

The apparent saturation is just a spurious effect stemming from the dynamical destruction of the X-point.
Conclusions

- In this presentation, we integrated the Hall-MHD equations numerically, to study magnetic reconnection. Even though the Hall effect does not produce reconnection, its role is to enhance the Ohmic reconnection rate.

- The existence of parallel electric fields can provide particle acceleration.

- We extended the Hall-MHD equations to include electron inertia, leading to what we call the EIHMHD equations.

- Integrating the EIHMHD equations in a 2.5D setup, we show that electron inertia leads to efficient magnetic reconnection, even in the absence of magnetic resistivity.

- The ideal invariants of a multi-species plasma are the total energy and also one helicity per species.