New variables and kinematical quantization

→ Gravity in first order tetrad variables

→ Symplectic structure
From vector to connection variables

→ Boundaries
Non comutativity of metric

→ Poisson brackets and constraints

→ Dirac program

→ quantization: Hilbert space

→ Quantum geometry:
quantization of the area operator.

Gravity in the first order formulation

$$S(e_a^I, w_a^{IJ}) = -\frac{1}{2K}\int_{\mathcal{M}} \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega)$$

$\Rightarrow$ Equations of motion $\left\langle \begin{array}{l} d_\omega e^I = 0 \Rightarrow \omega = \omega(e) \\ \epsilon_{IJKL}\, e^J \wedge F^{KL}(\omega) = 0 \end{array}\right.$ $\rightarrow$ $R_{ab} = 0$

vaccuum Einsteins equations

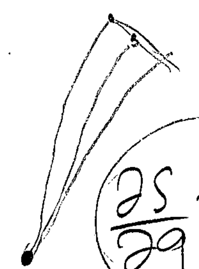
$K = 8\pi G$

$\boxed{\text{We will ignore matter couplings}}$!

Symplectic structure: for a generic $S(\phi)$

$$\delta S = \int_{\mathcal{M}} (EOM) \times \delta\phi + \int_{\partial M} \pi\, \delta\phi$$

Recall CM



$$\frac{\partial S}{\partial q} = \pi$$

Symplectic potential $\Theta(\delta)$

Symplectic structure $\Omega(\delta_1, \delta_2) = \delta_1(\Theta(\delta_2)) - \delta_2(\Theta(\delta_1))$

$$\Omega(\delta_1, \delta_2) = \int_{\partial M} \delta_{[1}\pi \wedge \delta_{2]}\phi \iff \{\pi(x), \phi(y)\} = \delta(x, y)$$

Back to gravity

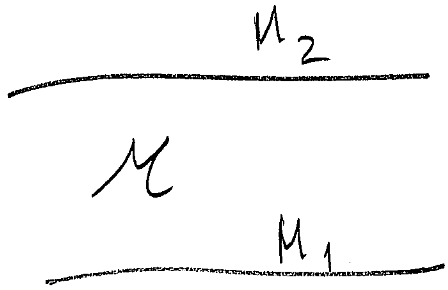
$$S(e,\omega) = -\frac{1}{2\kappa}\int_{\mathcal{M}} d_\omega\, {}^*\!(e\wedge e)^{IJ}\,\delta\omega_{IJ} \; +$$

$$-\;\frac{1}{\kappa}\int_{\mathcal{M}} \epsilon_{IJKL}\,\delta e^I\wedge e^J\wedge F(\omega)^{KL} \; +$$

$$+\frac{1}{2\kappa}\int_{\mathcal{M}} d\left({}^*\!(e\wedge e)^{IJ}\,\delta\omega_{IJ}\right)$$

$$\parallel$$

$$\Theta(\delta) = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} {}^*\!(e\wedge e)^{IJ}\,\delta\omega_{IJ}$$



$$\partial\mathcal{M} = M_1 \cup M_2$$

Gravity symplectic form:

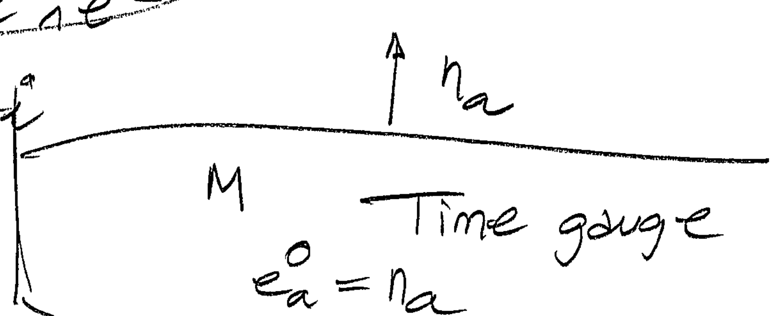
$$\Omega(\delta_1,\delta_2) = \frac{1}{2\kappa}\int_M \delta_{[1}\, {}^*\!(e\wedge e)^{IJ}\wedge\delta_{2]}\,\omega_{IJ}$$

$$= \frac{1}{2\kappa}\int_M \delta_{[1}\Sigma^{IJ}\wedge\delta_{2]}\,\omega_{IJ}$$

with $\Sigma^{IJ} = \epsilon^{IJ}{}_{KL}\, e^K\wedge e^L$

$$\underleftarrow{\Sigma}^{i0} = \epsilon^{i}{}_{jk}\, e^j\wedge e^k \equiv \Sigma^i$$

$$\underleftarrow{\Sigma}^{ij} = 0$$



Time gauge

$e^0_a = n_a$

In the time gauge

$$\Omega(\delta_1,\delta_2) = \frac{1}{2k}\int \delta_{[1}\Sigma^i \wedge \delta_{2]}K_i$$

related through EOM with K_{ab}

$$\Sigma^i \equiv \epsilon^i{}_{JK}\, e^J \wedge e^K \qquad K^i \equiv \omega^{i0}$$

$$\{\Sigma^i_{ab}, K^J_c\} = \overset{\circ}{\epsilon}_{abc}\, \delta^{iJ}\, \delta(x,y)$$

Ashtekar variables:

$$\theta(\delta) = \frac{1}{2k}\int_M {}^*(e\wedge e)^{IJ}\wedge \delta\omega_{IJ}$$

$$\overset{\text{time gauge}}{=} \frac{1}{2k}\int_M \Sigma^i \wedge \delta K_i$$

$$= \frac{1}{2k\gamma}\int_M \Sigma^i \wedge \delta(\gamma K_i)$$

$\gamma \equiv$ Imirzi parameter

$$= \frac{1}{2k\gamma}\int \Sigma^i \wedge \delta(\gamma K_i + \Gamma_i) - \Sigma^i \wedge \delta\Gamma_i$$

where $\Gamma^i \equiv$ spin connection solution of

$$de^i + \epsilon^i{}_{JK}\, \Gamma^J \wedge e^K = 0$$

SU(2) connection field.

Exercise: Show that $\Sigma_i \delta\Gamma^i = d(e_i \wedge \delta e^i)$

$$de^i + \epsilon^i{}_{JK}\, \Gamma^J \wedge e^K = 0 \qquad (\mathrm{I})$$

$$d(\delta e^i) + \epsilon^i{}_{JK}\, \delta\Gamma^J \wedge e^K + \epsilon^i{}_{JK}\, \Gamma^J \wedge \delta e^K = 0$$

$$d(\delta e^i) \wedge e_i + \epsilon^i{}_{JK}\, \delta\Gamma^J \wedge e^K \wedge e_i + \epsilon^i{}_{JK}\, \Gamma^J \wedge \delta e^K \wedge e_i = 0$$

$$d(\delta e^i) \wedge e_i + \delta\Gamma^i \wedge \Sigma_i + \epsilon^K{}_{Ji}\, \Gamma^J \wedge e^i \wedge \delta e_K = 0$$

$$d(\delta e^i) \wedge e_i + \delta\Gamma^i \wedge \Sigma_i - d(e^K) \wedge \delta e_K = 0$$

$$e_i \wedge d(\delta e^i) - d(e^i) \wedge \delta e_i + \Sigma_i \wedge \delta\Gamma^i = 0$$

$$\Longleftrightarrow \boxed{+ d(e^i \wedge \delta e_i) = \Sigma_i \wedge \delta\Gamma^i}$$

$$\Theta(\delta) = \frac{1}{2k\gamma} \int_M \left[\Sigma^i \wedge \delta A_i - d(e^i \wedge \delta e_i)\right]$$

$$\Theta(\delta) = \frac{1}{2k\gamma} \int_M \Sigma^i \wedge \delta A_i - \int_{\partial M} e^i \wedge \delta e_i$$

$$\Omega(\delta_1, \delta_2) = \frac{1}{2k\gamma}\left(\int_M \delta_{[1}\Sigma^i \wedge \delta_{2]} A_i - \int_{\partial M} \delta_1 e^i \wedge \delta_2 e_i \right)$$

Bulk term

Boundary term.

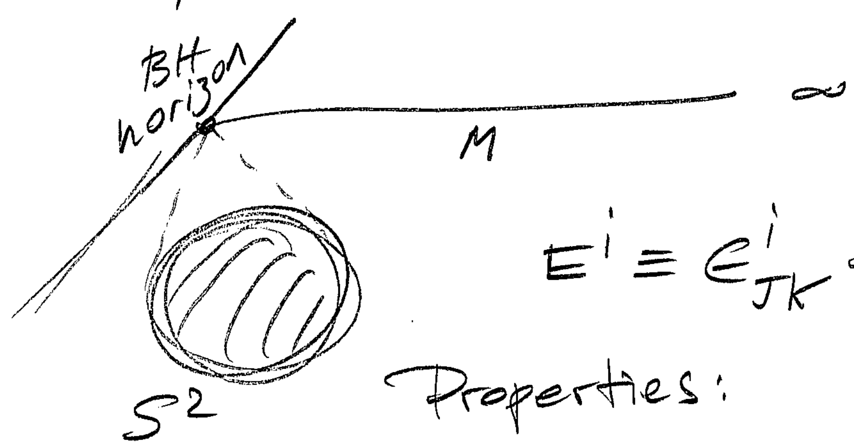
Poisson brackets

$$A^i = \Gamma^i + \gamma K^i$$

$$\{ \Sigma^i_{ab}(x), A^J_c(y) \} = 8\pi G \gamma \epsilon_{abc} \delta^{iJ} \delta^{(3)}(x,y)$$

$$\{ e^i_a, e^J_b \} = 8\pi G \gamma \epsilon_{ab} \delta^{iJ} \delta^{(2)}(x,y)$$

Digression: quantum geometry on the horizon



$$E^i \equiv \epsilon^i_{JK} e^J_a e^K_b \epsilon^{ab}$$

Properties:

(1) $E^i E_i = \det(g^{(2)})$

$\Rightarrow \text{Area}(S^2) = \int \sqrt{E^i E_i}\, dx^2$

(2) $\{E^i(x), E^J(y)\} = \epsilon^{iJ}{}_K \delta^{(2)}(x,y) E^K(x)\, 8\pi\gamma G$

"Area fluxes" are non-commuting and reproduce the SU(2) Lie algebra. Like angular momentum operators we expect them to have a discrete spectrum upon quantization.

$$\hat{A}|\Psi\rangle = \int \sqrt{\hat{E}^i \hat{E}_i}\, |\Psi\rangle = \left(8\pi \ell_p^2 \gamma \sum_p \sqrt{J(J+1)}\right)|\Psi\rangle$$

Constraints (neglect boundary subtleties)

$$\{\Sigma^i_{ab}(x), A^J_c(y)\} = \epsilon_{abc}\,\delta^{iJ}\,\delta^{(3)}(x,y)$$

$$G^i \equiv d_A \Sigma^i_{ab} = 0 \qquad 3$$

$$S \equiv \frac{\epsilon^{abc}\,\epsilon^{def}\,\Sigma^i_{ab}\,\Sigma^J_{de}\,F^K_{ef}\,\epsilon_{iJK}}{\det(\Sigma)} \qquad 1$$

$$V_d = \epsilon^{abc}\,\Sigma^i_{bc}\,F^J_{ad}\,\delta_{iJ} \equiv 0 \qquad \frac{3}{7}$$

Kinematical Phase space has 18 d.o.f per point

$-$ 7 constraints $-$ 7 gauge orbits $= 4$

d.o.f of the Physical phase space of GR.

<u>Constraint algebra:</u>

$$\{G(\alpha), G(\beta)\} = G([\alpha,\beta])$$

$$\{V(\sigma), G(\beta)\} = G(\mathcal{L}_\sigma \beta)$$

$$\{V(\sigma), V(w)\} = V(\mathcal{L}_\sigma w)$$

$$\{V(\sigma), S(N)\} = S(\mathcal{L}_\sigma N)$$

$$\left\{\begin{aligned} G(\alpha) &= \int \alpha_i G^i \\ V(\sigma) &= \int \sigma^a V_a \\ S(N) &= \int N S \end{aligned}\right.$$

complicated

$$\{S(N), S(M)\} = V\left(q^{ab}(N\partial_b M - M\partial_b N)\right)$$

Dirac Program in LQG

1) Construct a Hilbertspace representation of basic fields

$$\Psi(A) \in \mathcal{H} \qquad \langle \Psi | \varphi \rangle = \int DA\, \overline{\Psi(A)}\, \varphi(A)$$

$$\Sigma, A \longrightarrow \hat{\Sigma}, \hat{A} \qquad [\ \ ] = i\hbar \{\ \ \}$$

2) Quantize the constraints and reproduce their algebra (no anomalies)

"QUANTUM EINSTEINS EQUATION"

(1) and (2) ~~Theory const.~~

(3) and (4) To use the theory

3) Solve constraints $\hat{C} |\Psi\rangle = 0$

4) Construct enough observables (physical gauge invariant questions)

$$\{O, C\} = 0$$

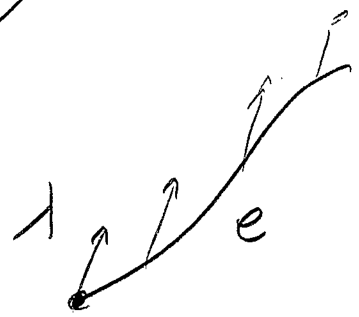
to do physics with.

Construction of the Hilbert space

The holonomy

given $\lambda \in su(2)$ and a curve $e \subset M$ one says that the field λ is parallel transported along e if

$$\dot{X}^{\mu}_{(s)} \underbrace{D_\mu \lambda}_{\partial_\mu \lambda + [A_\mu, \lambda]} = 0$$

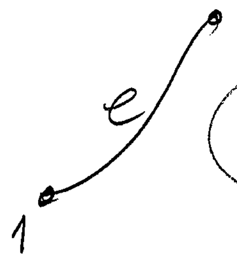


if the initial value of $\lambda = \lambda_0$ then the general solution is

$$\lambda = h(A,s)\, \lambda_0\, h^{-1}(A,s) \qquad h(A,s) \in SU(2)$$

$$\frac{d}{ds} h(A,s) + \dot{X}^{\mu}(s) A_\mu(s)\, h(A,s) = 0$$

with $h(A,0) = \mathbb{1}$.



$$h_e(A) \equiv h(A, s_2)$$

$$h_e(A) = P \exp\left(-\int_e A\right)$$

$$= \mathbb{1} - \int_0^1 \dot{X}^\mu A_\mu(s_1)\, ds_1 + \int_0^1 \int_0^{s_1} \dot{X}^\nu A_\nu(s_2)\, \dot{X}^\mu A_\mu(s_1)$$

$$+ \ldots$$

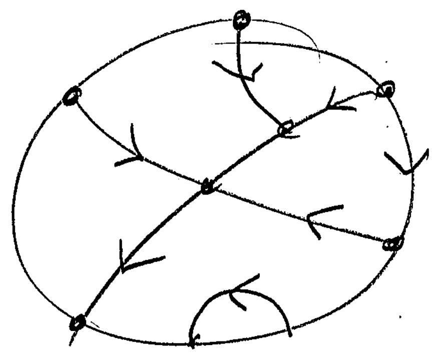
Properties of Holonomy (Exercise)

Choose holonomies as the basic configuration variables

<u>Cylindrical Functions</u>:

(1) $\gamma \equiv$ oriented graph

$N_e \equiv$ number of edges



(2) $F: SU(2)^{N_e} \longrightarrow \mathbb{C}$

square integrable

$$\Psi_{\gamma,f}(A) \in Cyl \subset \mathcal{H}$$

$$\Psi_{\gamma,f}(A) = F(h_1, h_2, \ldots, h_{N_e})$$

There is a natural positive linear functional "A STATE" $\mu_{AL}: Cyl \longrightarrow \mathbb{C}$

$$\mu_{AL}(\Psi_{\gamma,F}) \equiv \int_{SU(2)^{N_e}} \prod_{i=1}^{N_e} dh_i \; F(h_1 \ldots h_{N_e})$$

$$\mu_{AL}(1) = 1$$

$$\mu_{AL}(\bar{\Psi}\Psi) \geq 0 \quad \forall \; \Psi \in Cyl.$$

The kinematical inner product

$$\langle \varphi | \psi \rangle \equiv \mu_{AL}(\bar{\varphi}\psi) \qquad \varphi, \psi \in Cyl$$

$$\langle \varphi_{\gamma, F}, \psi_{\gamma' F'} \rangle =$$

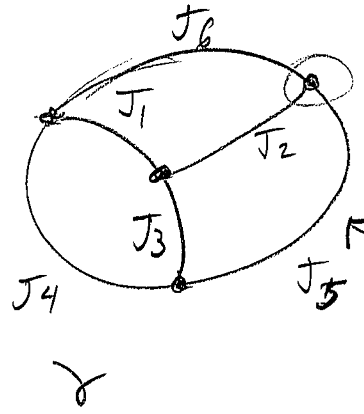
$$= \int \prod_{i=1}^{N_{\gamma \cup \gamma'}} dh_i \quad \bar{F}(h_i) F'(h_i)$$

An orthonormal basis (spin network states)

$$f(g) = \sum_{J, m, m'} F_J^{mm'} D^J_{mm'}(g)$$

Wigner matrices

Spin network states

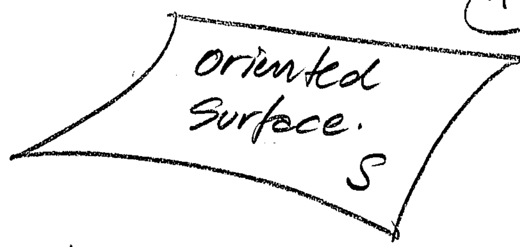


Invariant tensors at nodes (gauge invariance)

Irrep Wigner matrices at edges.

Above states are eigen-states of the quantum geometry of space !,

The conjugate variables



Non Abelian Flux

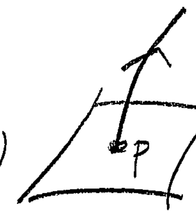
$$E(S,\alpha) = \int_S d\sigma_1 d\sigma_2 \, \frac{\partial X^a}{\partial \sigma_1} \frac{\partial X^b}{\partial \sigma_2} \Sigma^i_{ab} \, \alpha_i$$

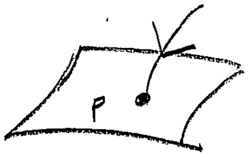
$$\{ \Sigma^i_{ab}, A_c^j \} = \epsilon_{abc} \, \delta^{ij} \, \delta^{(3)}(x,y)$$

$$S : \mathbb{R}^2 \longrightarrow M$$

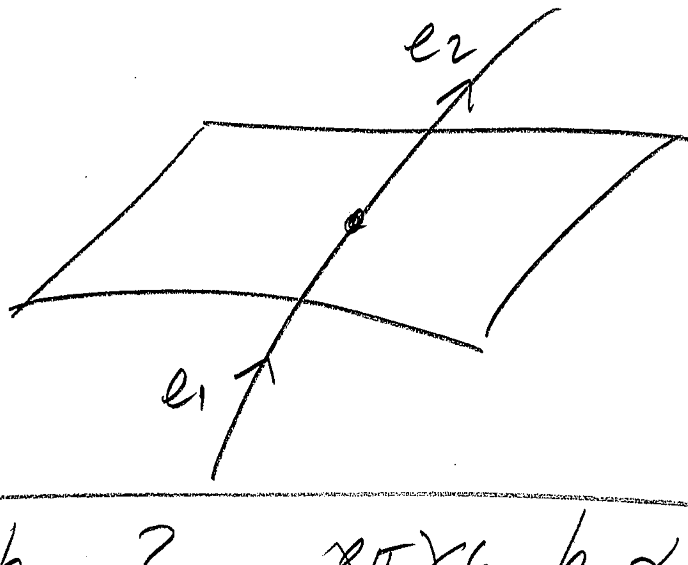
$$\sigma_1, \sigma_2 \longrightarrow X^a(\sigma_1, \sigma_2)$$

$$\{ E(S,\alpha), h_e \} = 4\pi G\gamma \, O(e,S) \begin{cases} h_e \tau^i \alpha_i(p) \\ -\tau^i \alpha_i(p) h_e \end{cases}$$





$$O(e,S) = \begin{cases} 0 & e \text{ tangent to } S \\ 1 & e \text{ above } S \\ -1 & e \text{ below } S \end{cases}$$

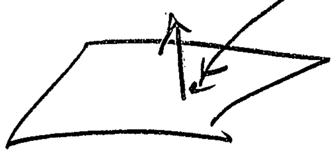


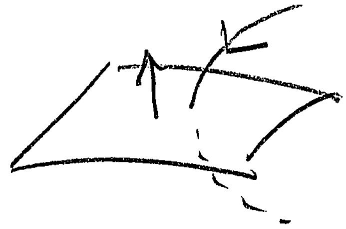
$$\boxed{\{ E(S,\alpha), h_{e_2 e_1} \} = 8\pi\gamma G \; h_{e_2} \alpha \, h_{e_1}}$$

Quantization of the fluxes

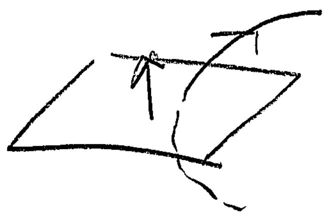
$$\hat{E}(S,\alpha) = \int_S d\sigma^1 d\sigma^2 \frac{\partial X^a}{\partial \sigma_1}\frac{\partial X^b}{\partial \sigma_2}\left(-i\, 8\pi l_p^2 \gamma\, \alpha^i \frac{\delta}{\delta A_c^i}\right)\epsilon_{abc}$$

$$\hat{E}(S,\alpha)\, h_e = -i\, 8\pi\gamma l_p^2 \begin{cases} \frac{1}{2} 0\, h_e \alpha \\ -\frac{1}{2} 0\, \alpha h_e \end{cases}$$





$$\hat{E}(S,\alpha)\, h_{e_2 e_1} = -i\, 8\pi\gamma l_p^2 \begin{cases} h_{e_2}\alpha h_{e_1} \\ - h_{e_1}\alpha h_{e_2} \end{cases}$$



Lemma:

$$\overbrace{\sum_{A=1}^{3} \hat{E}(S,\alpha_A)\hat{E}(S,\alpha_A)}^{\hat{E}\cdot\hat{E}(S)}\, h_{e_2 e_1} =$$

$$= -(8\pi\gamma l_p^2)^2\, h_{e_2}\Big(\underbrace{\sum_{A=1}^{3} \alpha_A^i \alpha_A^j}_{\delta^{ij}}\, \tau_i \tau_j\Big) h_{e_1}$$

$$= -\frac{3}{4}(8\pi\gamma l_p^2)^2\, h_{e_2 e_1}$$

$$\boxed{\hat{E}\cdot\hat{E}(S) \triangleright \overset{j}{\pi}(h_{e_2 e_1}) = (8\pi\gamma l_p^2)^2\, j(j+1)\, \overset{j}{\pi}(h_{e_2 e_1})}$$

Geometric interpretation of E

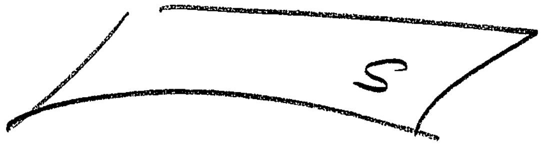
$$g\, g^{\alpha\beta} = \Sigma^i_{ab}\, \Sigma^J_{cd}\, \epsilon^{ab\alpha}\, \epsilon^{cd\beta}\, \delta_{iJ}$$

From $\Sigma^i_{ab} = \epsilon^i_{JK}\, e^J_a \wedge e^K_b$ and

from $g_{ab} = e^i_a (e_b)_i$

Any geometric quantity can be constructed from the Σ's and regularized in terms of infinitesimal surfaces with the fluxes $E(S, \alpha)$

Examples:



AREA

$$A_S = \int_S \sqrt{h}\, d\sigma^1 d\sigma^2$$

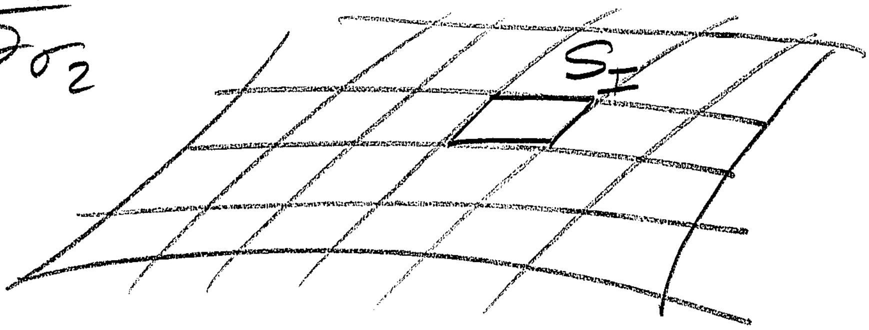
$$= \int_S \sqrt{\underbrace{(\Sigma^i_{ab}\, n_c\, \epsilon^{abc})}_{E^i_a n^a} (\Sigma^J_{a'b'} n_{c'}\, \epsilon^{abc})\, \delta_{iJ}}$$

Volume $V = \int_B \sqrt{g} = \int \sqrt{\epsilon_{iJK}\, \epsilon^{abc}\, E^i_a E^J_b E^K_c}$

The quantization of the area operator

$$A_S = \int \sqrt{\left(\Sigma^i_{ab}\, n_c\, \epsilon^{abc}\right)\cdot\left(\Sigma^i_{a'b'}\, n_{c'}\, \epsilon^{a'b'c'}\right)}$$

$$n_a = \epsilon_{abc} \frac{\partial x^b}{\partial \sigma_1}\frac{\partial x^c}{\partial \sigma_2}$$



$$A_S^N = \sum_I^N \sqrt{\sum_A E(S_I,A)\,E(S_I,A)}$$

$$E(S_I,A) = \int_{S_I} \Sigma^i_{ab}\, n_c\, \epsilon^{abc}\, \alpha_i^A$$

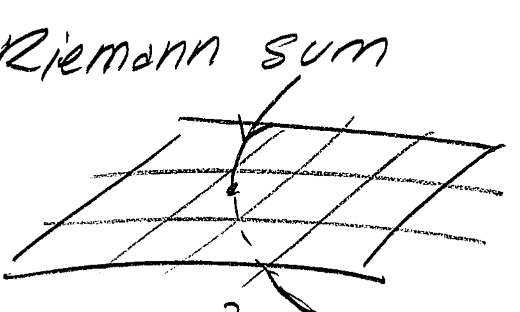
$$\approx \Sigma^i_{ab}\, n_c\, \epsilon^{abc}\, \alpha_i^A\, \Delta\sigma_1 \Delta\sigma_2 + \cdots$$

$$\sqrt{\sum_A E(S_I,A)\,E(S_I,A)} \sim \sqrt{\left(\Sigma^i_{ab}\, n_c\, \epsilon^{abc}\right)\left(\Sigma^i_{a'b'}\, n_{c'}\, \epsilon^{a'b'c'}\right)}\, \Delta\sigma_1\sigma_2$$

$$+ \cdots$$

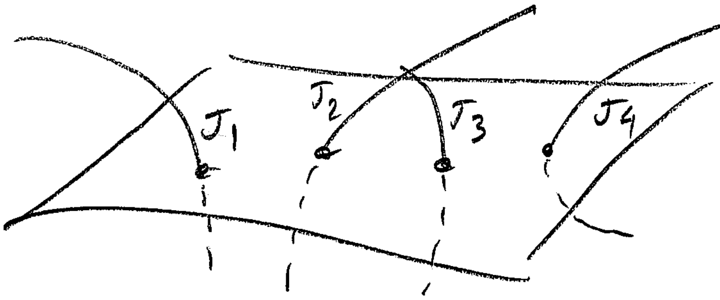
$$A_S = \lim_{N\to\infty} A_S^N$$

Riemann sum



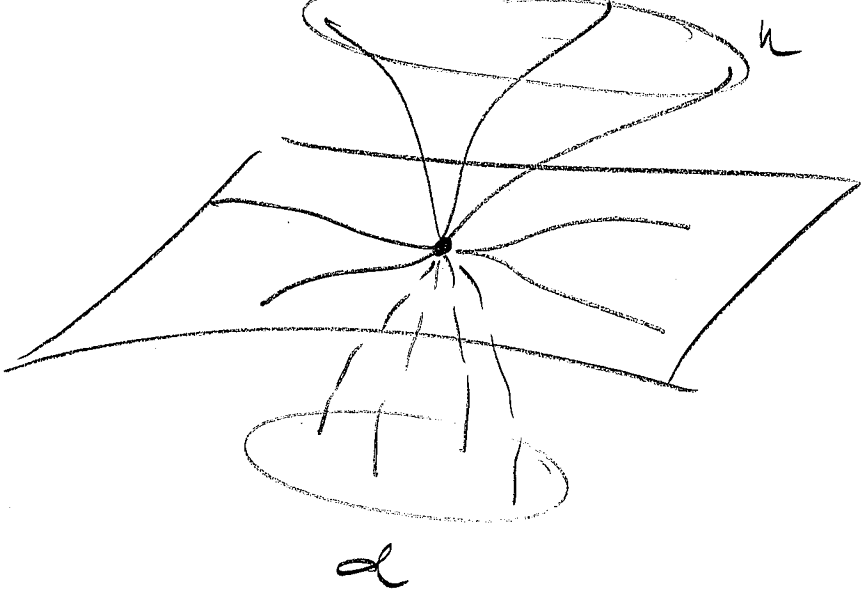
$$\hat{A}_S|\Psi\rangle = \lim_{N\to\infty} \hat{A}_S^N |\Psi\rangle = 8\pi\gamma l_p^2 \sqrt{J(J+1)}\, |\Psi\rangle$$

$$\hat{A}_S |\Psi\rangle = 8\pi l_p^2 \gamma \sum_p \sqrt{J(J+1)}\; |\Psi\rangle$$

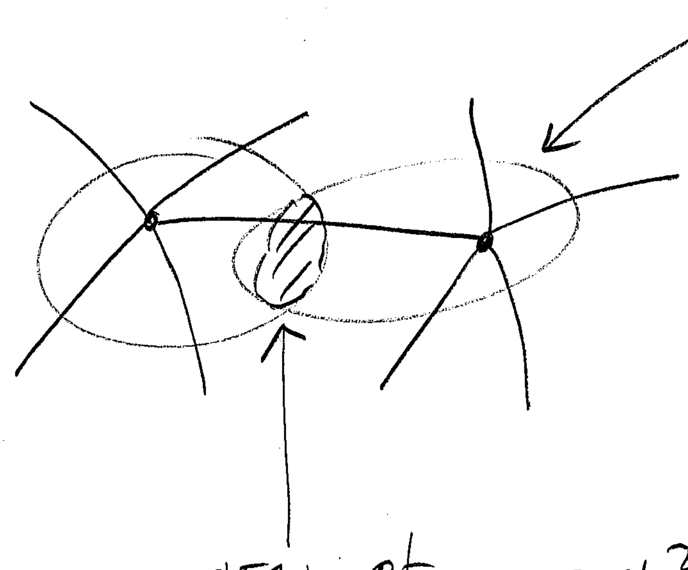


The most general node contribution

$$\hat{A}_S |\Psi\rangle = 4\pi \gamma l_p^2 \sqrt{2 J_u(J_u+1) + 2 J_d(J_d+1) - J_t(J_t+1)}\; |\Psi\rangle$$



Geometric interpretation of spin-network state.



quanta of volume

$V = \int \sqrt{EEE}$

area of interphase $8\pi \gamma l_p^2 \sqrt{J(J+1)}$