Effective Field Theories

Lecture 2

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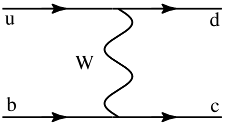
Low energy weak interactions

$$-\frac{ig}{\sqrt{2}}\; V_{ij}\; \bar{q}_i\, \gamma^\mu\, P_L\, q_j,$$



$$A = \left(\frac{ig}{\sqrt{2}}\right)^2 V_{cb} V_{ud}^* \;\left(\bar{c}\,\gamma^\mu\, P_L\, b\right) \left(\bar{d}\,\gamma^\nu\, P_L\, u\right) \left(\frac{-ig_{\mu\nu}}{p^2 - M_W^2}\right),$$

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \ldots\right),$$

and retaining only a finite number of terms.

$$A = \frac{i}{M_W^2}\left(\frac{ig}{\sqrt{2}}\right)^2 V_{cb}V_{ud}^* \ (\bar{c}\,\gamma^\mu P_L\, b)\,(\bar{d}\,\gamma_\mu P_L\, u) + \mathcal{O}\left(\frac{1}{M_W^4}\right).$$

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb}V_{ud}^* \ (\bar{c}\,\gamma^\mu P_L\, b)\,(\bar{d}\,\gamma_\mu P_L\, u) + \mathcal{O}\left(\frac{1}{M_W^4}\right),$$

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8M_W^2}.$$

Effective Lagrangian for μ decay

$$L = -\frac{4G_F}{\sqrt{2}}\ \left(\bar{e}\,\gamma^\mu\, P_L\, \nu_e\right)\left(\bar{\nu}_\mu\,\gamma^\mu\, P_L\,\mu\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right),$$

Gives the standard result for the muon lifetime at lowest order,

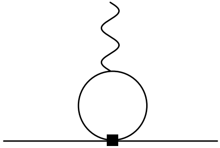
$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}\ f\left(\frac{m_e^2}{m_\mu^2}\right)$$

EFT gives the full dependence on low energy parameters.

$$f(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln \rho$$

The advantages of EFT show up in higher order calculations

Loops



Gives a contribution

$$\int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \frac{1}{k^2 - M_W^2} \frac{1}{k^2 - m^2} \sim \frac{1}{M_W^2} \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \frac{1}{k^2 - m^2} \sim \frac{\Lambda^2}{M_W^2} \sim \mathcal{O}(1)$$

Similarly, a dimension eight operator has vertex k^2/M_W^4, and gives a contribution

$$I' \sim \frac{1}{M_W^4} \int d^4k \, \frac{k^2}{k^2 - m^2} \sim \frac{\Lambda^4}{M_W^4} \sim \mathcal{O}(1)$$

Would need to know the entire effective Lagrangian, since all terms are equally important. The reason for this breakdown is using a cutoff procedure with a dimensionful parameter Λ.

More generally, need to make sure that dimensionful parameters at the high scale do not occur in the numerator after evaluating Feynman diagrams.

In doing weak interactions, one should not have M_G or M_P appear in the numerator.

Need a renormalization scheme which maintains the power counting.

Dimensional Regularization

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\left(k^2\right)^a}{\left(k^2 - m^2\right)^b}$$
$$= \frac{1}{(4\pi)^{d/2}} \frac{(-1)^{a-b}\Gamma(d/2+a)\Gamma(b-a-d/2)}{\Gamma(d/2)\Gamma(b)} \left(M^2\right)^{d/2+a+b}$$

Integral defined by analytic continuation.

Convert all integrals to this form using Feynman parameters for the denominator.

$\overline{\text{MS}}$

Need to use a mass independent subtraction scheme such as $\overline{\text{MS}}$:

μ can only occur in logarithms, so

$$\frac{1}{M_W^2}\int d^4k\,\frac{1}{k^2-m^2}\sim\frac{m^2}{M_W^2}\log\frac{\mu^2}{m^2},$$

$$\frac{1}{M_W^4}\int d^4k\,\frac{k^2}{k^2-m^2}\sim\frac{m^4}{M_W^4}\log\frac{\mu^2}{m^2},$$

Expanding $1/(k^2-M_W^2)$ in a power series ensures that there is no pole for $k\sim M_W$, and so M_W cannot appear in the numerator.

Dimensional regularization is like doing integrals using residues. Relevant scales given by poles of the denominator.

Toy Model (Integral)

Rather than do an explicit EFT example, look at a simple integral which illustrates what happens

Integral arises as a one-loop graph in a field theory, has some couplings in front.

$$I_F = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)}$$

Expanding does not commute with loop integration

Do the integral exactly:

$$\begin{aligned} I_F &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)} \\ &= \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} - \log\frac{M^2}{\mu^2} + \frac{m^2 \log(m^2/M^2)}{M^2 - m^2} + 1\right] \end{aligned}$$

Expand, do the integral term by term, and then sum up the result:

$$\begin{aligned} I_{\rm eft} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)}\left[-\frac{1}{M^2} - \frac{k^2}{M^4} - \ldots\right] \\ &= \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\frac{m^2}{M^2 - m^2} + \frac{m^2}{M^2 - m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2 - m^2}\right] \end{aligned}$$

Very Important Points

Missing the non-analytic terms in M.

The $1/\epsilon$ terms do not agree, they are cancelled by counterterms which differ in the full and EFT.

The two theories have different anomalous dimensions.

The term non-analytic in the IR scale, $\log(m^2)$ agrees in the two theories. This is the part which must be reproduced in the EFT.

The analytic parts are local, and can be included as matching contributions to the Lagrangian.

Sum $\log M^2/m^2$ terms using RG evolution.

No Non-Analytic Terms in M

$$\log \frac{m^2}{M^2} = \log \frac{m^2}{\mu^2} - \log \frac{M^2}{\mu^2}$$

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2 - m^2} \log \frac{m^2}{\mu^2} - \frac{M^2}{M^2 - m^2} \log \frac{M^2}{\mu^2} + 1\right]$$

$$I_{\rm eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon} \frac{m^2}{M^2 - m^2} + \frac{m^2}{M^2 - m^2} \log \frac{m^2}{\mu^2} - \frac{m^2}{M^2 - m^2}\right]$$

$1/\epsilon$ terms are different

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{\rm eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

Each theory has its own counterterms (renormalization).

Different anomalous dimensions

Full theory:

$$\frac{1}{\epsilon}$$

The amplitude has an anomalous dimensions

EFT:

$$-\frac{1}{\epsilon}\,\frac{m^2}{M^2 - m^2} = -\frac{1}{\epsilon}\,\frac{m^2}{M^2} - \frac{1}{\epsilon}\,\frac{m^4}{M^4} + \ldots$$

Each EFT order in $1/M$ has its own anomalous dimension.

Non-analytic Terms in m Agree

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{\rm eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

The EFT reproduces the complete low-energy limit of the full theory, including all the dependence on low energy (IR) scales.

Infinite parts cancelled by counterterms.

The difference between the finite parts of the two results is

$$\begin{aligned}
&\frac{i}{16\pi^2}\left[\log\frac{\mu^2}{M^2}+\frac{m^2\log(\mu^2/M^2)}{M^2-m^2}+\frac{M^2}{M^2-m^2}\right]\\
=\ &\frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right)+\frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right)+\ldots\right]
\end{aligned}$$

The terms in parentheses are matching coefficients to a coefficient of order 1, order $1/M^2$, etc. They are analytic in m.

Note:

$$\log\frac{m}{M}\to-\log\frac{M}{\mu}+\log\frac{m}{\mu}$$

with the first part in the matching, and the second part in the EFT.

Summing Large Logs

The full theory has $\log M^2/m^2$ terms. At higher orders, get

$$\alpha_s^n \log^n M^2/m^2$$

- If $M \gg m$, perturbation theory breaks down as $\alpha_s \log M/m \sim 1$.
- Full theory involves two widely separated scales.
- Calculations become very difficult at higher orders.

Divide one calculation into two calculations, each involving one scale.

- Each calculation much easer since it involves a single scale
- For the matching to be accurate, want $\mu = M$.
- For the EFT to be accurate, want $\mu = m$.

- For the matching use $\mu = M$
- For the EFT calculation, pick $\mu = m$.
- Use the EFT renormalization group to convert the Lagrangian from $\mu = M$ to $\mu = m$.
- RG perturbation theory valid as long as α_s small. Do not need α_s log to be small.

Matching

$$I_M = \frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right)+\frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right)+\ldots\right]$$

We computed the matching from $I_F - I_{\rm eft}$.

But there is an easier way which does not involve computing the two scale integral I_F.

I_M is analytic in m. Therefore, we can compute

$$I_F(m=0) = \int \frac{{\rm d}^d k}{(2\pi)^d}\,\frac{1}{(k^2)(k^2-M^2)}$$

$$\frac{\partial I_F}{\partial m^2}(m=0) = \int \frac{{\rm d}^d k}{(2\pi)^d}\,\frac{1}{(k^2)^2(k^2-M^2)}$$

Keep only the finite terms. More and more IR divergent.

Power Counting Formula

Manifest power counting in p/M.

Loop graphs consistent with the power counting, since one can never get any M's in the numerator.

If the vertices have $1/M^a$, $1/M^b$, etc. then any amplitude (including loops) will have

$$\frac{1}{M^a}\frac{1}{M^b}\ldots = \frac{1}{M^{a+b+\ldots}}$$

Correct dimensions due to factors of the low scale in the numerator, represented generically by p. (Could be a mass)

Power Counting Formula

Only a finite number of terms to any given order in $1/M$.

Order $1/M$: L_5 at tree level

Order $1/M^2$: L_6 at tree level,
or loop graphs with two insertions of L_5.

General power counting result:

- you can count the powers of M.
- you can count powers of p

[Weinberg power counting formula for χPT]

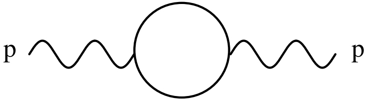
$$A \sim p^r, \qquad r = \sum_k n_k (k-4)$$

where n_k is the number of vertices of order p^k.

Decoupling of Heavy Particles

Heavy particles decouple from low energy physics.
Obvious?

Not explicit in a mass independent scheme such as $\overline{MS}$.



$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \int_0^1 dx\; x(1-x)\;\log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right]$$

and we want to look at $p^2 \ll m^2$.

The graph is UV divergent.

Momentum Subtraction Scheme

Note that renormalization involves doing the integrals, and then performing a subtraction using some scheme to render the amplitudes finite.

Subtract the value of the graph at the Euclidean momentum point $p^2 = -M^2$ (the $1/\epsilon$ drops out)

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\; x(1-x)\;\log\frac{m^2-p^2x(1-x)}{m^2+M^2x(1-x)}\right].$$

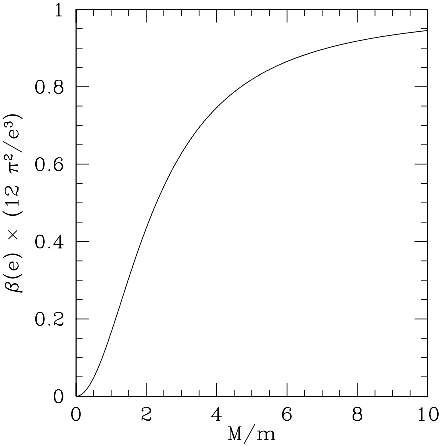
$$\begin{aligned}\beta\left(e\right) &= -\frac{e}{2}M\frac{\mathrm{d}}{\mathrm{d}M}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\; x(1-x)\;\log\frac{m^2-p^2x(1-x)}{m^2+M^2x(1-x)}\right]\\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\; x(1-x)\;\frac{M^2x(1-x)}{m^2+M^2x(1-x)}.\end{aligned}$$

$m \ll M$ (light fermion):

$$\beta(e) \approx \frac{e^3}{2\pi^2} \int_0^1 dx\, x(1-x) = \frac{e^3}{12\pi^2}.$$

$M \ll m$ (heavy fermion):

$$\beta(e) \approx \frac{e^3}{2\pi^2} \int_0^1 dx\, x(1-x) \frac{M^2 x(1-x)}{m^2} = \frac{e^3}{60\pi^2} \frac{M^2}{m^2}.$$



cross-over

In the $\overline{\text{MS}}$ scheme:

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\log\frac{m^2-p^2x(1-x)}{\mu^2}\right].$$

$$\begin{aligned}\beta(e) &= -\frac{e}{2}\mu\frac{d}{d\mu}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\ x(1-x)\log\frac{m^2-p^2x(1-x)}{\mu^2}\right]\\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x) = \frac{e^3}{12\pi^2},\end{aligned}$$

Is the first term in the β-function scheme independent?

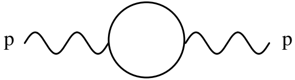
$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\; x(1-x)\log\frac{m^2}{\mu^2}\right],$$

Large logs cancel the wrong β-function contributions.

Explicitly integrate out heavy particles and go to an EFT.

Full theory: Includes fermion with mass m.

EFT: drop the heavy fermion (it no longer contributes to β)



Present in theory above m, but not in theory below m. Assume that $p \ll m$, so

$$\begin{aligned} & \int_0^1 dx\; x(1-x)\; \log \frac{m^2 - p^2 x(1-x)}{\mu^2} \\ = & \int_0^1 dx\; x(1-x)\; \left[\log \frac{m^2}{\mu^2} + \frac{p^2 x(1-x)}{m^2} + \ldots \right] \\ = & \frac{1}{6} \log \frac{m^2}{\mu^2} + \frac{p^2}{30 m^2} + \ldots \end{aligned}$$

So in theory above m:

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \ldots\right] + c.t.$$

Counterterm cancels $1/\epsilon$ term (and also contributes to the β function).

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[-\frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \ldots\right]$$

The log term gives

$$Z = 1 - \frac{e^2}{12\pi^2} \log \frac{m^2}{\mu^2}$$

so that in the effective theory,

$$\frac{1}{e_L^2(\mu)} = \frac{1}{e_H^2(\mu)} \left[1 - \frac{e_H^2(\mu)}{12\pi^2} \log \frac{m^2}{\mu^2} \right]$$

One usually integrates out heavy fermions at $\mu = m$, so that (at one loop), the coupling constant has no matching correction.

The p^2 term gives the dimension six operator

$$-\frac{1}{4}\frac{e^2}{2\pi^2}\frac{1}{30m^2}F_{\mu\nu}\partial^2 F^{\mu\nu}$$

and so on.

Even if the structure of the graphs is the same in the full and effective theories, one still needs to compute the difference to compute possible matching corrections, because the integrals need not have the same value. (next example)

This difference is independent of IR physics, since both theories have the same IR behavior, so the matching corrections are IR finite.

Note that nothing discontinuous is happening to any physical quantity at m.

We have changed our description of the theory from the full theory including m to an effective theory without m. By construction, the EFT gives the same amplitude as the full theory, so the amplitudes are continuous through m.

All m dependence in the effective theory is manifest through the explicit $1/m$ factors and through logarithmic dependence in the matching coefficients (in e_L).

Have to treat the p^2/m^2 term as a perturbation

Otherwise

$$\frac{1}{p^2 - e^2 p^4/(60\pi^2 m^2)}$$

has a pole at

$$p^2 = \frac{60\pi^2 m^2}{e^2} = \frac{15\pi m^2}{\alpha}$$

This new pole will violate the power counting. Also can get ghosts from quantizing a higher derivative theory.