Effective Field Theories

Lecture 3

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HQET

A.V. Manohar and M.B. Wise:
Heavy Quark Physics, Cambridge University Press (2000)

Will not discuss the theory or its applications in any detail.

Use it to discuss EFT at one-loop, because all the calculations can be found in any field theory textbook which discusses QED at one loop.

The EFT describes a heavy quark with mass m_Q interacting with gluons and light quarks with momentum $k \ll m_Q$

Expansion in $1/m_Q$

Applicable to B mesons, D mesons, $\Lambda_{b,c}$ baryons, etc.

The full theory is QCD, the heavy quark part is

$$\mathcal{L}_Q = \bar{Q}\left(i\not{D} - m_Q\right) Q$$

In the limit $m_Q \to \infty$, the heavy quark does not move when interacting with the light degrees of freedom. Even though for finite m_Q, the quark does recoil, the EFT is constructed as a formal expansion in powers of $1/m_Q$, expanding about the $m_Q \to \infty$ limit. Recoil effects are taken care of by $1/m_Q$ corrections.

Quark moving with fixed four-velocity v^μ, $v^2 = 1$,

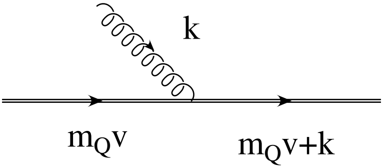
$$p = m_Q v^\mu + k \qquad k \ll m_Q$$

In the rest frame

$$v^\mu = (1, 0, 0, 0)$$

Quark Propagator

Look at the quark propagator:



$$i\frac{\not{p}+m_Q}{p^2-m_Q^2+i\epsilon}$$

$p = mv + k$, where k is called the residual momentum, and is of order $\Lambda_{\rm QCD}$.

k from interactions with other degrees of freedom, e.g. the $\overline{q}$ in a $b\overline{q}$ meson.

HQET Propagator

$$i\frac{m_Q\slashed{v} + \slashed{k} + m_Q}{(m_Q v + k)^2 - m_Q^2 + i\epsilon}$$

Expanding this in the limit $k \ll m_Q$ gives

$$i\frac{1+\slashed{v}}{2k\cdot v + i\epsilon} + \mathcal{O}\left(\frac{k}{m_Q}\right) = i\frac{P_+}{k\cdot v + i\epsilon} + \mathcal{O}\left(\frac{k}{m_Q}\right),$$

with a well defined limit.

$$P_\pm \equiv \frac{1 \pm \slashed{v}}{2}$$

In the rest frame,

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gluon Vertex

The quark-gluon vertex

$$-igT^a\gamma^\mu \to -igT^a v^\mu,$$

using the spinors and keeping the leading terms in $1/m_Q$.

Check this using the P_+ projectors on either side of γ^μ

In the rest frame: the coupling is purely that of an electric charge.

HQET $\mathcal{L}_0$

HQET Lagrangian:

$$\mathcal{L} = \bar{h}_v(x)\,(iD\cdot v)\,h_v(x)\,,$$

$h_v(x)$ is the quark field in the effective theory and satisfies

$$P_+ h_v(x) = h_v(x)\,.$$

h_v annihilates quarks with velocity v, but does not create antiquarks

Mainfest spin-flavor symmetry of $\mathcal{L}$

$$\mathcal{L} \;=\; \bar{b}_v\,(x)\,(iD\cdot v)\,b_v\,(x) + \bar{c}_v\,(x)\,(iD\cdot v)\,c_v\,(x)$$

$$\mathcal{L} = \bar{\Psi}_v\,(x)\,(iD\cdot v)\,\Psi_v\,(x)$$

$$\Psi_v = \begin{pmatrix} b_{v,1} \\ b_{v,2} \\ c_{v,1} \\ c_{v,2} \end{pmatrix}$$

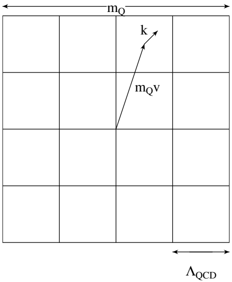
$SU(4)$, or in general, $SU(2N_F)$ symmetry, since $D\cdot v \propto \mathbf{1}$

QCD:

$$\mathcal{L} \;=\; \bar{b}\left(i\not{D} - m_b\right)b + \bar{c}\left(i\not{D} - m_c\right)c$$

Dividing up momentum space

v appears explictly in the HQET Lagrangian.
h_v describes quarks with velocity v, and momenta within $\Lambda_{\rm QCD}$ of $m_Q v$.



quarks with velocity $v' \neq v$ are far away in the EFT.

EFT: look at only one box.
Full: All of momentum space.

$$\mathcal{L} = \overline{Q} \left(i\not{D} - m_Q \right) Q$$

$$\begin{aligned}
Q(x) &= e^{-i m_Q v \cdot x} \left[h_v(x) + \mathcal{H}_v(x) \right] \\
h_v(x) &= P_+ h_v(x) \\
\mathcal{H}_v(x) &= P_- \mathcal{H}_v(x)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &= \left[\overline{h}_v(x) + \overline{\mathcal{H}}_v(x) \right] \left(i\not{D} + m_Q \not{v} - m_Q \right) \left[h_v(x) + \mathcal{H}_v(x) \right] \\
&= \overline{h}_v(x) \left(iv \cdot D \right) h_v(x) - \overline{\mathcal{H}}_v(x) \left(2m_Q + iv \cdot D \right) \mathcal{H}_v(x) \\
&+ \overline{\mathcal{H}}_v(x) \left(i\not{D}_\perp \right) h_v(x) + \overline{h}_v(x) \left(i\not{D}_\perp \right) \mathcal{H}_v(x)
\end{aligned}$$

Expand any vector A as

$$A^\mu = (A \cdot v)\, v^\mu + A_\perp^\mu$$

In the rest frame,

$$A^\mu = A^0\,(1, \mathbf{0}) + (0, \mathbf{A}_\perp)$$

$$\begin{aligned}\mathcal{L} &= \overline{h}_v(x)\,(iv \cdot D)\,h_v(x) - \overline{\mathcal{H}}_v(x)\,(2m_Q + iv \cdot D)\,\mathcal{H}_v(x) \\ &\quad + \overline{\mathcal{H}}_v(x)\,\left(i\not{D}_\perp\right)\,h_v(x) + \overline{h}_v(x)\,\left(i\not{D}_\perp\right)\,\mathcal{H}_v(x)\end{aligned}$$

h_v has a kinetic term of order k
$\mathcal{H}_v$ has a kinetic term of order $2m_Q \gg k$

Integrate out the antiparticles $\mathcal{H}_v$.

$$\mathcal{L} = \bar{h}_v(x)\left[iv\cdot D + i\not{D}_\perp \frac{1}{2m_Q + iv\cdot D} i\not{D}_\perp\right] h_v(x)$$

$$= \bar{h}_v(x)\left[iv\cdot D + i\not{D}_\perp \left(\frac{1}{2m_Q} + \ldots\right) i\not{D}_\perp\right] h_v(x)$$

$$\not{D}_\perp \not{D}_\perp = D_\perp^2 + \frac{1}{2} g\sigma^{\mu\nu} G_{\mu\nu}$$

$$\mathcal{L} = \bar{h}_v\left(iv\cdot D\right)h_v + \frac{1}{2m_Q}\bar{h}_v\left(iD_\perp\right)^2 h_v - \frac{g}{4m_Q}\bar{h}_v\sigma_{\alpha\beta}G^{\alpha\beta}h_v$$

$$+ \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

$1/m_Q$ Lagrangian

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D\right) h_v + c_k \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v$$

The $(iD_\perp)^2$ term violates flavor symmetry at order $1/m_Q$

$g\bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v$ term violates spin and flavor symmetry at order $1/m_Q$

The coefficients c_k, c_F are fixed by matching, and are one at tree-level.

c_k is the usual $\mathbf{p^2}/(2m_Q)$ kinetic energy term and c_F is the usual magnetic moment term (with $g = 2$ when $c_F = 1$)

$c_k = 1$ to all order in α_s due to reparametrization invariance (RPI)

One can carry out the expansion to higher order in $1/m_Q$.

Field Redefinitions

Field redefinition

$$h_v \to \left[1 + \frac{1}{m_Q} X\right] h_v$$

changes the effective Lagrangian to

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D\right) h_v + \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v + \frac{1}{m_Q} h_v \left[\left(iv \cdot D\right) X + X^\dagger \left(iv \cdot D\right)\right] h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

For $X = 1/4\left(iv \cdot D\right)$, one can replace

$$D_\perp^2 \to D_\perp^2 + \left(v \cdot D\right)^2 = D^2$$

Usually used in the opposite direction to remove $v \cdot D$ terms (time-derivatives).

Field redefinition change off-shell amplitudes, but not S-matrix elements.

Follows from the LSZ reduction formula. Source term

$$\overline{\eta}\, h_v \to \overline{\eta} \left[1 + \frac{1}{m_Q} X \right] h_v$$

but choice of interpolating field is irrelevant, as long as the field can make a h particle.

The only thing that the effective theory and full theory have to agree on are S-matrix elements.

Effective Lagrangians which look different can reproduce the same S-matrix.

No direct connection between the full theory and EFT field.

In a renormalizable field theory, the only field redefinition used is a rescaling $\psi \to Z^{1/2}\psi$ to keep the kinetic term properly normalized. Non-trivial field redefinitions would induce non-renormalizable terms with dimension > 4.

There is more freedom in an EFT, since we already have higher dimension terms.

Make field redefinitions consistent with the power counting.

Equations of Motion

Field redefinitions are related to using the equations of motion. The shift in $\mathcal{L}$ was proportional to $(v \cdot D)\psi$, which is the equation of motion from the leading order Lagrangian.

By making a transformation on a field,

$$\phi \to \phi + \xi\, f(\phi),$$

one shifts the action by

$$\xi\, f(\phi)\, \delta S/\delta\phi + \mathcal{O}(\xi^2).$$

So generically, one can eliminate terms proportional to the equation of motion by a field redefinition.

If one works to higher orders in power counting, and/or terms with multiple factors of the equations of motion, such as

$$\mathcal{L} = \ldots + \phi\, g(\phi) \left(\partial^2 + m^2\right)^2 \phi$$

best to use field redefinitions to systematically eliminate operators.

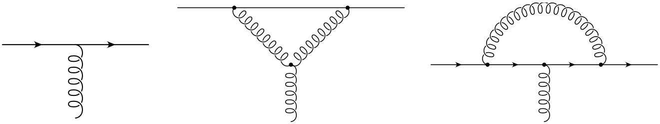
Note that one is using the classical equations of motion in the quantum theory (to all orders in $\hbar$).

Okay for the S-matrix.

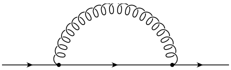
One loop matching

AM: PRD56 (1997) 230

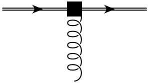
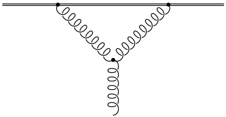
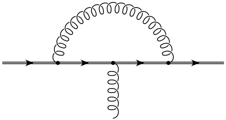
Look at the gluon coupling to one-loop in the full theory:



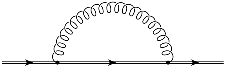
Note that we are matching an *S*-matrix element, the scattering of a quark off a low-momentum gluon, so the wavefunction graph must be included.



The effective theory graphs are:

and wavefunction renormalization



They look the same, but now

$$\gamma^\mu \to v^\mu, \qquad \frac{1}{\not{p} - m_Q} \to \frac{1}{v \cdot k}$$

The full theory amplitude is

$$\Gamma = -igT^a \, \bar{u}\left(p'\right) \left[F_1\left(q^2\right)\gamma^\mu + iF_2\left(q^2\right)\frac{\sigma^{\mu\nu}q_\nu}{2m}\right] u\left(p\right),$$

by current conservation, so we are computing $F_{1,2}$ at one loop.

The graphs do not have this form unless you use background field gauge, which respects gauge invariance on the external gluon.

On-shell, the graphs are IR divergent.

What we want is the matching condition. It is given by the difference of the full and EFT amplitudes, and takes care of the fact that the two theories differ in the UV.

The EFT reproduces the IR behavior of the full theory, and so has the same IR divergences. Thus the matching corrections to the Lagrangian coefficients are IR finite, and well-defined.

One way to proceed: Assume the quarks are off-shell, $p^2 \neq m^2$ and compute full and EFT theory graphs.

$$p^2 - m_Q^2 \to (m_Q v + k)^2 - m_Q^2 \sim 2 m_Q v \cdot k$$

Matching using a IR Regulator

The graphs involve parameter integrals of the form

$$\int \frac{1}{m^2 z - q^2 x(1-x)z^2 - p_1^2 xz(1-z) - p_2^2(1-x)z(1-z)}$$

Starts to look messy, because there are 4 scales in it.

The EFT graph is also messy. It involves q^2, $v \cdot k_1$ and $v \cdot k_2$ but not m.

Matching using Dim Reg for the IR

There is a much better procedure using dimensional regularization

The form factors are functions $F(q^2/m^2, \mu^2/m^2)$, and can have non-analytic terms such as $\log q^2$. These non-analytic terms arise from IR physics and so are the same in the full and EFT theory.

Compute $F(q^2/m^2, \mu^2/m^2, \epsilon)$ at finite ϵ and first expand in q^2/m^2 and then take the limit $\epsilon \to 0$. This drops all non-analytic terms in q, but we don't care since the effective Lagrangian is analytic in momentum.

$$z^\epsilon = z^\epsilon\Big|_{x=0} + \epsilon z^{\epsilon-1}\Big|_{z=0} + \ldots$$

In dim reg, all the terms are zero.
Then

$$F = F(0) + q^2 \frac{\mathrm{d}F(q^2)}{\mathrm{d}q^2}\Big|_{q^2=0} + \ldots$$

The derivatives of F are integrals of the form

$$F^{(n)}(0) \quad = \quad \int \mathrm{d}^4 k \; f(k,m)$$

and depend on only a single scale.

$$\int \frac{1}{m^2 z - q^2 x(1-x) z^2 - p_1^2 xz(1-z) - p_2^2 (1-x) z(1-z)}$$

becomes (with $p_1^2 = p_2^2 = m^2$)

$$\int \frac{1}{\left[\left(m^2 - q^2 x(1-x)\right) z^2\right]^{1+\epsilon}}$$

$$\to \quad \int \frac{1}{\left[\left(m^2\right) z^2\right]^{1+\epsilon}}, \qquad (1+\epsilon) \int \frac{x(1-x) z^2}{\left[\left(m^2\right) z^2\right]^{2+\epsilon}}$$

These are easy to evaluate

Try doing both forms of the integral.

Full Theory Computation

$$F^{(n)}(0) = \frac{A_n}{\epsilon_{UV}} + \frac{B_n}{\epsilon_{IR}} + (A_n + B_n) \log \frac{\mu^2}{m^2} + D_n$$

UV divergences are cancelled by counterterms. IR divergences are real, and are an indication that you are not computing something sensible.

The derivatives of F in the EFT are integrals of the form

$$F^{(n)}(0) = \int \mathrm{d}^4 k \; f(k, v)$$

and are scaleless

EFT Computation

Scaleless integrals vanish in dim regularization.

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{k^2 \left(k^2 - m^2\right)} - \frac{m^2}{k^4 \left(k^2 - m^2\right)} \right]$$

$$= \quad \frac{i}{16\pi^2} \left[\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right] = 0,$$

So the EFT form factor computation is

$$F^{(n)}(0) \quad = \quad \frac{E_n}{\epsilon_{UV}} - \frac{E_n}{\epsilon_{IR}} = 0$$

There can be no log terms and no constants.

Matching

Matching:

$$\text{full} = \text{eft} + c$$

which gives

$$\frac{A_n}{\epsilon_{UV}} + \frac{B_n}{\epsilon_{IR}} + (A_n + B_n)\log\frac{\mu^2}{m^2} + D_n + c.t. = \frac{E_n}{\epsilon_{UV}} - \frac{E_n}{\epsilon_{IR}} + c_n + c.t.$$

The ϵ_{UV} terms are cancelled by the counterterms, which are different in the two theories ($A_n \neq E_n$)

The IR divergence are the same, so $B_n = -E_n$.

$$c_n = (A_n + B_n)\log\frac{\mu^2}{m^2} + D_n$$

Summary

The anomalous dimension in the full theory is $-2A_n$

The anomalous dimension in the EFT is $-2E_n = 2B_n$

The IR divergence in the full theory is equal to the UV divergence in the EFT

The matching coefficient is given by the finite part of the full theory graph dropping both UV and IR $1/\epsilon$ terms.

At higher loops, not always clear what is ϵ_{UV} and ϵ_{IR}. Just compute with ϵ and drop $1/\epsilon$ terms.

The coefficient of the log in the matching computation is the difference of the anomalous dimensions in the full and EFT theories.

Consistency condition: Matching at μ, so change in matching with μ must be the difference of anomalous dimensions.

Summing Logs

Suppose there is an infrared scale in the problem, λ^2. In the presence of this scale, the full theory IR divergence is cut off at λ^2, so the IR log term becomes

$$B \log \frac{\lambda^2}{m^2} = B \log \frac{\mu^2}{m^2} + B \log \frac{\lambda^2}{\mu^2}$$

This log of a ratio of scales is split into two logs, each involving a single scale. The first is in the matching at m, and the second is in the anomalous dimension in the EFT.

By matching at $\mu = m$, and then running from m to λ, one finds no large logs in the matching coefficient c.

The IR logs in the full theory are summed by the RGE in the effective theory.

Schematic Calculation

So how does one compute B decay?

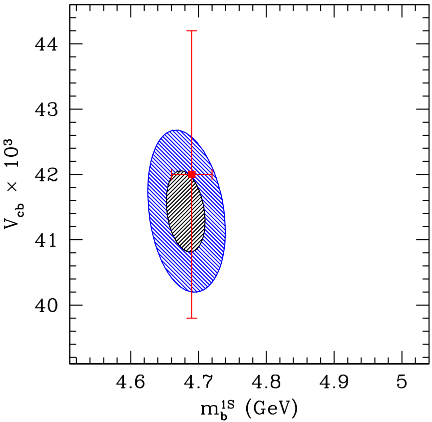
1. Match onto the Fermi theory at M_W —Expansion in $\alpha_s(M_W)$
2. Run the 4-quark operators to m_b —Sum $(\alpha_s \log M_W/m_b)^n$.
3. Match onto HQET at m_b — Expansion in $\alpha_s(m_b)$
4. Run in HQET to low scale $\mu \sim \Lambda_{\rm QCD}$ — Sum $(\alpha_s \log m_b/\Lambda_{\rm QCD})^n$
5. Evaluate non-perturbative matrix elements of operators in $\mathcal{L}$ — Get power corrections $\Lambda_{\rm QCD}/m_b$.

- One computation has been broken up into several much simpler calculations, each of which involves a single scale.
- Can sum the logarithms using RG improved perturbation theory, rather fixed order perturbation theory (which often breaks down)
- Both short distance and long distance corrections can be included in a systematic way to arbitrary accuracy (assuming you work hard enough).
- There is no model dependence, a systematic QCD calculation.

Include corrections to order $1/m^3$. Radiative corrections included are order α_s^2 for the order 1 term and α_s for the $1/m$ term. Order unity for the remaining terms.

Application to B Decays

Bauer et al. PRD 70 (2004) 094017



$$m_b^{1S} = 4.68 \pm 0.03 \text{ GeV}, \quad V_{cb} = (41.4 \pm 0.7) \times 10^{-3}$$