EFT Activity Day 3

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May 31, 2013

Today we will match the Standard Model onto a low-energy effective theory of the electroweak interactions in which the heavy W boson is integrated out.

1 Tree-level matching

If you did exercises 10 and 11 yesterday, you are already done with this part. If you did not, or want to check your results, perform the following two exercises.

Exercise 1 Compute the tree-level Feynman diagram for $d \to ue\bar{\nu}_e$ mediated by a W. Then, expand it in powers of $1/M_W$.

The relevant interaction terms in the SM Lagrangian are:

$$\mathcal{L}_{EW} = \frac{g_2}{\sqrt{2}}(V_{ud}\bar{u}_L\gamma^\mu W_\mu^+ d_L + \bar{\nu}_L^e\gamma^\mu W_\mu^+ e_L) + \text{h.c.}\,, \tag{1}$$

where $\psi_L = P_L\psi$ where $P_L = (1-\gamma_5)/2$ projects out the left-handed components of the Dirac spinor ψ.

Exercise 2 Write a dimension-6 operator built from fermion fields (u, d, e, ν_e) generated by integrating out the W boson from the diagram in Exercise 1, to leading order in $1/M_W$. Normalize the coefficient of the operator as:

$$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud}\hat{O}_6\,, \tag{2}$$

and derive G_F in terms of g_2, M_W. You should obtain:

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}\,. \tag{3}$$

2 1-loop matching

In Exercise 2 you should have found the dimension-6 effective operator:

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud} [\bar{u}\gamma_\mu(1-\gamma_5)d]\,[\bar{e}\gamma^\mu(1-\gamma_5)\nu_e] + \text{h.c.} \tag{4}$$

Now we will compare amplitudes for $d \to ue\bar{\nu}_e$ in the full and effective theories at 1-loop in QED, and calculate the appropriate matching coefficient to $\mathcal{O}(\alpha)$ (where $\alpha = e^2/4\pi$), e being the renormalized QED coupling. We ignore QCD and weak corrections for simplicity.

Exercise 3 **(a)** Draw all the 1-loop QED diagrams that correct the tree-level $d \to ue\bar{\nu}_e$ amplitude you drew in Exercise 1. Draw the set of similar 1-loop corrections to the effective theory amplitude computed from the effective Lagrangian in Eq. (4).

(b) If you drew any full theory diagrams with the photon connecting to the W^+ boson in part (a), explain why we do not have to consider them at leading order in $1/M_W$. What would the effective theory diagrams corresponding to these full theory graphs look like?

(c) Consider the set of wavefunction renormalization graphs you drew in part (a) (that is, those with a photon loop only correcting one of the fermion propagators). Convince yourself that they cancel in the difference of full and effective theory diagrams.

(d) Consider the set of full theory diagrams in (a) that correct one of the W vertices (e.g. those with a photon connecting u, d), and the effective theory diagram corresponding to it. Convince yourself that these also cancel in the difference of the full and effective theories.

The only diagrams left that will not cancel out in our matching calculation are those with a photon "traversing" the W propagator, i.e. with a photon connecting u to e or d to e. These are the only diagrams that "know" about both the light and heavy scales in the full theory graphs, i.e. $m_{u,d,e}$ and M_W.

For simplicity, set $m_{u,d} = 0$ for the remainder of this activity, and keep $m_e \neq 0$ only in the electron propagator denominator. This is not kinematically realistic for d decay, but will simplify your calculations while giving the correct result, as the dependence on the light scales will cancel anyway in the matching coefficient. The full and effective theories must have the same dependence on the light scales, and differ only in their dependence on UV scales such as M_W that we integrate out. You can also view m_e as an infrared regulator we use in a theory of massless u, d, e. Dependence on IR regulators must also cancel in matching coefficients.

Exercise 4 **(a)** Write down expressions for the 1-loop amplitudes for $d \to u e \nu_e$ in the full theory that are still remaining at the end of Exercise 3, which will not cancel in the matching to EFT. Don't fully simplify them yet, we'll do that in the next steps (unless you don't need any of my hints!).

(b) To simplify the Dirac structures, use the identity

$$\gamma_\mu \gamma_\nu \gamma_\lambda = g_{\mu\nu}\gamma_\lambda + g_{\nu\lambda}\gamma_\mu - g_{\mu\lambda}\gamma_\nu - i\epsilon_{\mu\nu\lambda\rho}\gamma^\rho\gamma_5\,, \tag{5}$$

where $\epsilon_{0123} = 1$. You will also need to recall $\epsilon_{\alpha\beta\lambda\rho}\epsilon^{\alpha\beta\lambda\sigma} = -6\delta^\sigma_\rho$. Use these to derive:

$$\begin{aligned} (\gamma_\mu\gamma_\nu\gamma_\lambda P_L) \otimes (\gamma^\mu\gamma^\nu\gamma^\lambda P_L) &= 16(\gamma^\mu P_L) \otimes (\gamma_\mu P_L) \\ (\gamma_\mu\gamma_\nu\gamma_\lambda P_L) \otimes (\gamma^\lambda\gamma^\nu\gamma^\mu P_L) &= 4(\gamma^\mu P_L) \otimes (\gamma_\mu P_L) \end{aligned} \tag{6}$$

(c) You will also encounter a loop integral of the form:

$$\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{k^4(k^2 - M_W^2)(k^2 - m_e^2)}\,. \tag{7}$$

Recall a trick from yesterday's activity: you can make the replacement

$$k^\mu k^\nu f(k^2) \longrightarrow \frac{1}{d}k^2 g^{\mu\nu} f(k^2) \tag{8}$$

inside the k integral. Now, Eq. (7) is finite in $d = 4$ dimensions, but it has two scales, and might be somewhat cumbersome to calculate in the time available today. You can do it very quickly by splitting it into two divergent integrals in $d = 4 - 2\epsilon$ dimensions and using results from yesterday. First, derive:

$$\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k^2 - \Delta)} = \frac{i}{16\pi^2}\frac{\Gamma(\epsilon)}{1-\epsilon}\left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon. \tag{9}$$

Divide Eq. (7) into a difference of two integrals of this form, expand them in powers of ϵ, keep the finite result and take $\epsilon \to 0$.

The result of Exercise 4 should give you for the $d \to u e \bar{\nu}_e$ amplitude to $\mathcal{O}(\alpha)$ and leading order in $1/M_W$,

$$\mathcal{M}_{\text{full}} = -\frac{ig_2^2}{8M_W^2}V_{ud}\bar{u}\gamma^\alpha(1-\gamma_5)d\,\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e\left[1 - \frac{\alpha}{4\pi}(4q_u - q_d)\ln\frac{m_e^2}{M_W^2}\right], \tag{10}$$

where $q_u = 2/3$ and $q_d = -1/3$ are the electric charges of u, d in units of e.

Exercise 5 Compute the corresponding 1-loop diagrams in the effective theory. You should encounter a loop integral of the form of Eq. (9). Note its divergence now does not cancel with anything. Expand the result to $\mathcal{O}(\epsilon^0)$. (Make sure you keep $d = 4 - 2\epsilon$ when using Eq. (8)!) The result for the EFT amplitude to $\mathcal{O}(\alpha)$ should be:

$$\mathcal{M}_{\text{eff}} = -\frac{iG_F}{\sqrt{2}} V_{ud} \bar{u}\gamma^\alpha(1-\gamma_5)d\, \bar{e}\gamma_\alpha(1-\gamma_5)\nu_e \left[1+\frac{\alpha}{4\pi}(4q_u - q_d)\left(\frac{1}{\epsilon}+c_1+\ln\frac{\mu^2}{m_e^2}\right)\right], \quad (11)$$

where c_1 is a constant you will derive. I have rescaled μ so that this result is in the $\overline{\text{MS}}$ scheme.

Exercise 6 The $1/\epsilon$ pole in Eq. (11) will be canceled by a counterterm for the operator $\hat{\mathcal{O}}_6$ in the EFT Lagrangian, so discard it. Note that the dependence on the light scale m_e is identical in the full and effective theories, as promised. The difference between the two amplitudes is the 1-loop *matching coefficient* for $\hat{\mathcal{O}}_6$, that is, $\mathcal{L}_{\text{eff}}$ in Eq. (4) should look like

$$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud} C \hat{\mathcal{O}}_6 \,. \quad (12)$$

Compute, to $\mathcal{O}(\alpha)$,

$$C(\mu) = 1 - \frac{\alpha(\mu)}{4\pi}(4q_u - q_d)\left(c_1 + \ln\frac{\mu^2}{M_W^2}\right). \quad (13)$$

Note that C depends on the UV scale M_W but not the IR scale m_e. It encodes the effects of physics above the scale M_W that is missing in the EFT. It also depends on the matching scale μ, dependence on which will cancel in physical observables.

3 RG Evolution

We can use renormalization group evolution to sum the large logs of m_e/M_W in the full theory amplitude Eq. (10). The EFT has divided these logs into logs of μ/M_W and μ/m_e (which will appear in matrix elements of $\hat{\mathcal{O}}_6$ computed in the EFT, which we are not doing in this activity). RG equations tell us how to solve for the dependence on μ in each object, summing logs of μ/M_W and μ/m_b in the process.

Exercise 7 The RGE for C is

$$\mu\frac{d}{d\mu}C(\mu) = \gamma_C(\mu)C(\mu) \,. \quad (14)$$

From Eq. (13) compute the *anomalous dimension* γ_C to $\mathcal{O}(\alpha)$, obtaining

$$\gamma_C(\mu) = -\frac{3\alpha(\mu)}{2\pi} \,. \quad (15)$$

Exercise 8 Solve the RGE Eq. (14) for $C(\mu_2)$ at one scale in terms of $C(\mu_1)$ at another scale. The generic solution is given by

$$\ln \frac{C(\mu_2)}{C(\mu_1)} = \int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} \gamma(\mu) \,. \tag{16}$$

Explicitly, you should obtain

$$C(\mu_2) = C(\mu_1) \Big(\frac{\alpha(\mu_2)}{\alpha(\mu_1)} \Big)^{-9/4} \tag{17}$$

To obtain this form easily, change variables on the RHS of Eq. (16) from μ to $\alpha(\mu)$ using the definition of the beta function,

$$\mu \frac{d}{d\mu} \alpha = \beta(\alpha) \,, \tag{18}$$

and use the 1-loop result,

$$\beta(\alpha) = \frac{\beta_0 \alpha^2}{2\pi} \,. \tag{19}$$

Exercise 9 Eq. (17) allows you to express $C(\mu_2)$ at a scale where C may contain large logs in terms of $C(\mu_1)$ at another scale where it has no large logs. What should you choose for μ_1? Write $C(\mu)$ at any μ in terms of this μ_1. All the (leading) large logs of ratios of these two scales are resummed in the ratio of couplings $\alpha(\mu_2)/\alpha(\mu_1)$.

Exercise 10 Matrix elements like $\langle u e \bar{\nu}_e | \hat{\mathcal{O}}_6 | d \rangle$ will also depend on μ in the EFT. They will contain logs of μ/m_e. We know that physical quantities do not depend on μ, in particular the product of C and $\langle \hat{\mathcal{O}}_6 \rangle$,

$$\mu \frac{d}{d\mu} [C(\mu) \langle \hat{\mathcal{O}}_6 \rangle_\mu] = 0 \,. \tag{20}$$

Remember that this is supposed to reproduce the full theory matrix element, which does not know about μ. Use this relation to derive the anomalous dimension of $S(\mu) \equiv \langle \hat{\mathcal{O}}_6 \rangle_\mu$. Solve the RGE for S. (Do not recalculate anything! Use the results of Exercise 8 and 9, appropriately modified.)

Exercise 11 Now use the RG solutions for C and S to resum logs of m_e/M_W in the full theory amplitude. Write:

$$M_{\text{full}} = C(\mu) S(\mu) \,, \tag{21}$$

and convince yourself that it is equivalent to choose $\mu = M_W$ and evolve S from the scale m_e to M_W, or to choose $\mu = m_e$ and evolve C from M_W to m_e. Either method resums the logs of m_e/M_W. This is called the equivalence or consistency of "top-down" or "bottom-up" running.