More exercises: QCD at large N_c

(30/5/13 - C.Schat)

1. Discuss non-planarity for baryon diagrams: Work out the N_c scaling of a two-gluon exchange between two quarks in a baryon, in the box and crossed box case.

2. The 4-meson vertex scales like $1/N_c$ and the baryon mass as N_c. You can obtain some insight into the motivation of considering the baryon as a soliton in large N_c from the following simple(st) example:

   (a) Consider $\lambda\phi^4$ in the broken phase ($\mu^2 > 0$)

   $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{1}$$

   Find the lowest energy state.

   (b) Show that

   $$\phi(x) = v\tanh(\frac{\mu}{\sqrt{2}}x) \tag{2}$$

   with $v = \sqrt{\mu^2/\lambda}$, is a static solution that interpolates between the two vacuum states. This solution is a finite energy solution, known as "the kink".

   (c) Compute the energy of this solution. You should get

   $$E_0 = \frac{4}{3}\frac{\mu^3}{\lambda\sqrt{2}} \tag{3}$$

   Notice that it scales as the inverse of the ϕ^4 coupling constant.

   (d) Consider the time dependent translation $\phi(x,t) = \phi(x - a(t))$ and compute the energy again ($a(t)$ is called a collective coordinate). You will obtain

   $$E = E_0 + \frac{1}{2}E_0\left(\frac{da}{dt}\right)^2 \tag{4}$$

   which you can compare with the motion of a free particle and identify E_0 as the mass of the particle.

   (e) A model that realizes this picture for baryons is the Skyrme model, in particular you can also compute the mass splitting of the N and Δ states in this model and see that it matches the general large N expression (see next problem for the quark model calculation).
   Ref: G.Adkins, C.Nappi, E.Witten, Nucl.Phys. B228 (1983) 552.

3. To order $1/N_c$ and in flavor $SU(2)$ the mass operator is

$$M = N_c c_0 + \frac{1}{N_c} c_1 J^2 \tag{5}$$

Compute c_1 in the quark model. Use $H = H_0 + H_{hyp}$ where

$$\begin{aligned} H_0 &= \sum_{i=3}^{3} \frac{p_i^2}{2m_i} + \frac{K}{2} \sum_{i<j} (r_i - r_j)^2 & (6) \\ H_{hyp} &= A \sum_{i<j} s_i \cdot s_j \delta^{(3)}(r_{ij}) & (7) \end{aligned}$$

Eliminating the center of mass motion using the coordinates

$$\begin{aligned} \rho &= (r_1 - r_2)/\sqrt{2} & (8) \\ \lambda &= (r_1 + r_2 - 2r_3)/\sqrt{6} & (9) \end{aligned}$$

you obtain a harmonic oscillator with ground state

$$\Psi_{00} = \frac{\alpha^3}{\pi^{3/2}} \exp^{-\frac{\alpha^2}{2}(\rho^2 + \lambda^2)} \tag{10}$$

where $\alpha = (3Km)^{1/4}$. Compute $m_\Delta - m_N$. You should obtain

$$m_\Delta - m_N = \frac{3\alpha^3}{4\pi\sqrt{2\pi}} A \tag{11}$$

Solve for c_1.

4. (a) Derive the consistency relations for pion-nucleon scattering

$$[X^{ia}, X^{jb}] = 0 \tag{12}$$

(b) Solve the consistency relations. Using the Wigner-Eckart theorem you can write

$$\langle J'm'|X^{ia}|Jm\rangle = X(J,J')\sqrt{\frac{2J+1}{2J'+1}} \left(\begin{array}{cc|c} J & 1 & J' \\ m & i & m' \end{array} \right) \left(\begin{array}{cc|c} J & 1 & J' \\ \alpha & a & \alpha' \end{array} \right) \tag{13}$$

Introduce a complete set of intermediate states and multiply by

$$\begin{aligned} (-1)^{j+b} \quad & \left(\begin{array}{cc|c} J & 1 & H \\ m & i & h \end{array} \right) \left(\begin{array}{cc|c} J & 1 & K \\ \alpha & a & \eta \end{array} \right) \\ & \left(\begin{array}{cc|c} J' & 1 & H' \\ m' & -j & h' \end{array} \right) \left(\begin{array}{cc|c} J' & 1 & K' \\ \alpha' & -b & \eta' \end{array} \right) \end{aligned} \tag{14}$$

After summing over projections you can use the following useful expression

$$\sum_{m_1,j,i,m',m} \left(\begin{array}{cc|c} J' & 1 & H' \\ m' & j & h' \end{array} \right) \left(\begin{array}{cc|c} J_1 & 1 & J' \\ m_1 & i & m' \end{array} \right) \left(\begin{array}{cc|c} J_1 & 1 & J \\ m_1 & j & m \end{array} \right) \left(\begin{array}{cc|c} J & 1 & H \\ m & i & h \end{array} \right)$$

$$= (-1)^{J+J'} \left\{ \begin{array}{ccc} J & 1 & H \\ J' & 1 & J_1 \end{array} \right\} \sqrt{(2J'+1)(2J+1)} \delta_{HH'} \delta_{hh'} \quad (15)$$

and the orthogonality of the Clebsch's.

(c) Finally, using the identity

$$(2H+1) \sum_{J_1} (2J_1+1) \left\{ \begin{array}{ccc} J & 1 & H \\ J' & 1 & J_1 \end{array} \right\} \left\{ \begin{array}{ccc} J & 1 & K \\ J' & 1 & J_1 \end{array} \right\} = \delta_{HK} \quad (16)$$

you can see that $X(J, J') = 1$ is a solution. This has implications, like the one of the next exercise.

Refs: A.R.Edmonds' book "Angular momentum in quantum mechanics", R.Dashen A.V.Manohar, Phys.Lett. B315 (1993) 425-430 (original calculation).

5. Show that the solution of the consistency equations obtained in the previous exercise predicts $g_{\pi NN}/g_{\pi N\Delta} = \frac{3}{2}$. This is a result also obtained in the quark model or the Skyrme model in the large N_c limit.