On spectral geometry approach to Horava-Lifshitz gravity

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What is wrong with Quantum Gravity?
Lifshitz’s idea
Horava-Lifshitz Gravity
  - Projectable
  - Non-projectable
  - Detailed balance condition
  - Healthy (or natural) extension
  - Problems
Spectral dimension
  - Random walk
  - Weyl’s theorem
Spectral geometry approach
  - Spectral Action
  - Geodesic motion?
Conclusions
Problems with the quantization of gravity

\[ \lambda = \delta \text{ in momentum units} \]

\[ D = d - (d/2 - 1) E - n\delta \]

- \( D \) - superficial degree of divergence
- \( d \) - space-time dimension
- \( E \) - number of the external legs
- \( n \) - number of vertices

We can expect renormalizability only when \( \delta \geq 0 \)
As the result, the effective dimensionless constant is given by

\[ S_{EH} = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} R \implies \delta \equiv [G] = 2 - d \]

for \( d = 4 \), \( \delta = -2 < 0 \)

As the result, the effective dimensionless constant is given by

\[ GE^2 := \left( \frac{E}{M_p} \right)^2 \text{ where } M_p = \sqrt{\frac{\hbar c}{G}} = 1.22 \times 10^{19} \text{ GeV} \]

i.e. when \( E \ll M_p \)
(Super)string theory: contains a spin-2 massless mode => has to describe gravity. GR is recovered in long-wave regime. But, the predictive power is quite poor: the string theory landscape has $10^{500}$ vacua.

Loop quantum gravity: one can perform non-perturbative quantization. Among problems, the difficulty of the recovery quasiclassical space.

Some other approaches treat gravity as an emergent phenomenon (e.g., entropic gravity).
Horava’s idea

- Lifshitz model (Lifshitz 1941)

\[ S = \int dt d^n x \left( \dot{\phi}^2 - g (\Delta \phi)^2 + c^2 \phi \Delta \phi \right) \]

\([x] = -1, [t] = -2, [c] = 1\]

The propagator has the form:

\[ G(\omega, \vec{k}) \propto \frac{1}{\omega^2 - c^2 \vec{k}^2 - g \vec{k}^4} \]
UV: \[ \frac{1}{\omega^2 - c^2 k^2 - g^2 k^4} = \frac{1}{\omega^2 - g^2 k^4} + \frac{1}{\omega^2 - g^2 k^4} \frac{c^2 k^2}{\omega^2 - g^2 k^4} + ... \]

IR: \[ \frac{1}{\omega^2 - c^2 k^2 - g^2 k^4} = \frac{1}{\omega^2 - c^2 k^2} + \frac{1}{\omega^2 - c^2 k^2} \frac{g^2 k^4}{\omega^2 - c^2 k^2} + ... \]

I.e. we have two fixed points: UV, which corresponds to \( z=2 \) and has significantly improved behavior and IR, in which by the time rescaling we can set \( c=1 \) and restore relativistic invariance, \( z=1 \)
Why to break Lorentz invariance?

Let us consider the same type of the modification, but when the higher derivatives are added in the Lorentz invariant way.

\[ S = \int d^4x \left( \partial_\mu \phi \partial^\mu \phi + g (\partial_\mu \phi \partial^\mu \phi)^2 \right) \]

The propagator takes the form:

\[ G(\omega, \vec{k}) \propto \frac{1}{k^2 - gk^4} = \frac{1}{k^2 (1 - gk^2)} = \frac{1}{k^2} - \frac{1}{k^2 - 1/g} \]
\[ ds^2 = g_{ij}(dx^i + N^i \, dt)(dx^j + N^j \, dt) - (N c dt)^2 \]

\[ S_{EH} = \frac{1}{16\pi G} \int dtd^3x N \sqrt{g} \left( K_{ij} K^{ij} - K^2 + \frac{3}{2} R \right) \]

where \( K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \) - second fundamental form

ADM
We take ADM slicing as fundamental, i.e. instead of considering just a manifold, we endow it with the foliation structure:

\[ \tilde{x}^i = \tilde{x}^i(\tilde{x}, t), \quad \tilde{t} = \tilde{t}(t) \]

These are foliation - preserving diffeos or FDiff s.

Also, we introduce anisotropic scaling between \( x \) and \( t \):

\[ \tilde{x} \rightarrow \alpha \tilde{x}, \quad t \rightarrow \alpha^{-z} t \quad \text{or} \quad [\tilde{x}] = -1, \quad [t] = -z \]

This is equivalent to prescribing the following dimensions:

\[ [c] = z - 1, \quad [N] = [g_{ij}] = 0, \quad [N_i] = z - 1 \implies [G] = 3 - z \]
• Projectable FDiff gravity (Horava 2009)

\[ N = N(t), \quad N \rightarrow N \frac{\partial t}{\partial \tilde{t}} \]

\[ S = \frac{M_p^2}{2} \int d^3 x dt \sqrt{g} N \left( K_{ij} K^{ij} - \lambda K^2 - V_P \right) \]

\[ V_P = 2\Lambda - \xi R + M_*^{-2} \left( A_1 R^2 + A_2 R_{ij} R^{ij} \right) + \]
\[ + M_*^{-4} \left( B_1 R \Delta R + B_2 R_{ij} R^{jk} R^i_k + B_3 \nabla_i R_{jk} \nabla^i R^{jk} + B_4 R R^{jk} R_{jk} + B_5 R^3 \right) \]

• Non-projectable FDiff gravity (Blas et al. 2010)

\[ N = N(t, \tilde{x}), \quad a_i := N^{-1} \nabla_i N \]

\[ S = \frac{M_p^2}{2} \int d^3 x dt \sqrt{g} N \left( K_{ij} K^{ij} - \lambda K^2 - V_{NP} \right) \]

\[ V_{NP} = V_P - \alpha a_i a^i + M_*^{-2} \left( C_1 a_i \Delta a^i + C_2 (a_i a^i)^2 + C_3 a_i a_j R^{ij} \ldots \right) + \]
\[ + M_*^{-4} \left( D_1 a_i \Delta^2 a^i + D_2 (a_i a^i)^3 + D_3 a_k a^k a_i a_j R^{ij} \ldots \right) \]
Horava’s recipe to deal with the large number of terms: Detailed Balance condition:

$$S_v = \frac{K^2}{8} \int dtd^3 x \sqrt{g} NE^{ij} G_{ijkl} E^{kl}, \text{ where}$$

$$\sqrt{g} E^{ij} = \frac{\delta W[g]}{\delta g_{ij}} \text{ and } W \text{ is some lower - dimensional action :}$$

$$W = \frac{1}{\omega^2} \int CS_3(\Gamma) + \mu \int d^3 x \sqrt{g} (R - 2\Lambda_w)$$

$$G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl} \text{ modified DeWitt metric}$$

Even applied to the healthy extended model the detailed balance condition still requires serious fine tuning (Verniery&Sotiriou 2013)
Some properties

• Broken 4d diffeos => Lorentz violation
• Extra scalar mode in addition to two graviton polarizations
• In general the scalar mode does not decouple in IR, this can endanger the renormalizability
• The model with the detailed balance condition does not pass the Solar system tests
• The healthy extension (with $a_i$) has A LOT of free parameters and some of them still require fine tuning
• ...
UV spectral dimension (Horava)

- CDT lattice calculations indicate that $d=2$ in UV (Ambjorn et al. 2005)
- Spectral dimension (Horava 2009)

$$d_s = -2 \frac{d \log P(\sigma)}{d \log \sigma}$$

$P(\sigma)$ - average return probability, $\sigma$ - diffusion time

$$P(\sigma) := \rho(\tau, \bar{x}; \tau, \bar{x}; \sigma)$$

$$\frac{\partial}{\partial \sigma} \rho(\tau, \bar{x}; \tau', \bar{x}'; \sigma) = \left( \frac{\partial^2}{\partial \tau^2} + \Delta \right) \rho(\tau, \bar{x}; \tau', \bar{x}'; \sigma)$$
• Spectral dimension in Horava case

\[ \frac{\partial}{\partial \sigma} \rho(\tau, \vec{x}; \tau', \vec{x}'; \sigma) = \left( \frac{\partial^2}{\partial \tau^2} + (-1)^{z+1} \Delta^z \right) \rho(\tau, \vec{x}; \tau', \vec{x}'; \sigma) \]

\[ \rho(\tau, \vec{x}; \tau, \vec{x}; \sigma) = \frac{C}{\sigma^{(1+D/z)/2}} \]

\[ d_s = 1 + \frac{D}{z} \quad \text{or, in the case of } D = 3, z = 3, d_s = 2 \]

• In IR, the diffusion equation will be dominated by \( z=1 \), leading to \( d_s=4 \)
Spectral geometry

(Connes 1990 and up to now)

\(T=(A,H,D)\) – spectral triple

A Riemannian manifold, \(M\), is completely Recovered from \(T\). In this case

i. \(A=\mathcal{C}^\infty(M)\)

ii. \(H=L^2(M,S)\)

iii. \(D=\gamma^\mu(\partial_\mu+\omega_\mu)\)
\[ d(x, y) = \sup \{ |f(x) - f(y)| : f \in C(M), \| [D, f] \| \leq 1 \} \]

\[ N_{|D|}(\lambda) \rightarrow \frac{2^m \Omega^n}{n(2\pi)^n} \text{Vol}(M) \lambda^n \]

\[ Tr^+(f \mid D)^{-n} = \frac{2^m \Omega^n}{n(2\pi)^n} \int_M f \nu_g , \text{ where } Tr^+A = \lim_{N \rightarrow \infty} \frac{\sigma_N(A)}{\log N} \]

\[ Tr\chi\left( \frac{D^2}{m_0^2} \right) = \frac{N}{48\pi^2} \left[ 12m_0^4 f_0 \int d^4 x \sqrt{g} + m_0^2 f_2 \int d^4 x \sqrt{g} R + \right. \]
\[ \left. + f_2 \int d^4 x \sqrt{g} \left( -\frac{3}{20} C_{\mu \nu \rho \sigma} C^{\mu \nu \rho \sigma} + \frac{1}{10} R ; \mu + \frac{11}{20} R^* R^* \right) + O\left( \frac{1}{m_0^2} \right) \right] \]
The choice of the Dirac operator in the form $D = \gamma^\mu(\partial_\mu + \omega_\mu)$ is not natural anymore.

The foliation structure dictates the following (schematic) form for $D$ (for $z=3$):

$$D = \partial_t + \sigma^\mu \partial_\mu \Delta + M \star \Delta + M \star 2 \sigma^{\mu} \partial_\mu$$

This $D$ should be used to obtain “physical” geometry instead of auxiliary 3+1 dimensional space. (AP 2010, Gregory & AP 2012)
Model calculation

- $M = S^1 \times T^3$, $D^2 = \partial_t^2 + \Delta^3 + M^2_\ast \Delta^2 + M^4_\ast \Delta$
- $sp(D^2) = \{ n^2 + (n_1^2 + n_2^2 + n_3^2)^3 + M^2_\ast (n_1^2 + n_2^2 + n_3^2)^2$
  $+ M^4_\ast (n_1^2 + n_2^2 + n_3^2), n_i \in \mathbb{Z} \}$
- $N_{|D|}(\lambda) = \{ \# \text{ eigenvalues } < \lambda \}$
- when $\lambda \ll M^6_\ast$ the last term dominates:
  $$N_{|D|}(\lambda) \approx \int_0^\lambda dn \int_0 \left( \lambda^2 - n^2 \right)^{1/2} 4\pi \rho^2 d\rho \propto \lambda^4 \implies d = 4$$
- when $\lambda \gg M^6_\ast$ the first term dominates:
  $$N_{|D|}(\lambda) \approx \int_0^\lambda dn \int_0 \left( \lambda^2 - n^2 \right)^{1/6} 4\pi \rho^2 d\rho \propto \lambda^2 \implies d = 2$$
One can do better and go beyond the flat case.

- Define a generalized $\zeta$-function
  \[ \zeta_\Delta(s) := \text{Tr}(\Delta^{-s}) \]
- Now $\Delta$ can be any generalized elliptic operator.
- $\zeta$-function can be extended to a meromorphic function on the whole complex plane with the only poles given by
  \[ \frac{n-p+zp}{2z}, \frac{n-p+z p-1}{2z}, \ldots, \frac{n-p+z p-k}{2z}, \ldots \]
- The first pole is related to the analytic dimension
  \[ \frac{n-p+z p}{2z} = \frac{n_a}{2} \]
- $n=D+1$, $p=1$ we have
  \[ n_a = 1 + \frac{D}{z} \]
Spectral Action (the speculative part)

Part I \[ Tr \chi \left( \frac{D^2}{m_0^2} \right) = \text{Horava - Lifshitz gravity?} \]

- Dirac operator is very complicated:
  \[ D^2 = \Delta_{\tau} + f(\Delta_x), \]
  where \[ \Delta_{\tau} = - \frac{1}{N \sqrt{g}} \partial_{\tau} \left( \frac{\sqrt{g}}{N} \partial_{\tau} \right) \]
  and \[ \Delta_x = \frac{1}{N \sqrt{g}} \partial_i \left( N \sqrt{g} g^{ij} \partial_j \right) \]

- To calculate the trace of this operator one has to find the heat kernel
  \[ \begin{cases} 
  (\partial_s + D^2)K(x, x'; s) = 0 \\
  K(x, x'; +0) = \delta(x, x') 
  \end{cases} \]
Even the flat case is not trivial (Mamiya & AP 2013)

\[
K(x - x'; \tau) = \frac{1}{z(4\pi)^2\tau^{\frac{1}{2}(1+3/z)}} e^{-\frac{(t-t')^2}{4\tau}} \sum_{\{j_k\}=0}^{\infty} \left( \prod_{k=0}^{z-1} \frac{(-\tau \gamma_k)^{j_k}}{j_k!} \right) \left( \tau \gamma_z \right)^{-\sum_k k j_k / z} \times \times_1 \Psi_1 \left[ ((3/2 + \sum_k k j_k) / z, 1 / z); (3/2, 1); -\frac{|x - x'|^2}{4(\tau \gamma_z)^{1/z}} \right].
\]

This allows to perform a completely analytical study of the spectral dimension flow:

\[
d_S = 1 + \frac{3}{z} + 2\gamma \gamma_z^{-\frac{k}{z}} \tau^{1-\frac{k}{z}} \left( 1 - \frac{k}{z} \right) \times_1 \Psi_0 \left[ (\frac{3+2k}{2z}, \frac{k}{z}); -\gamma \gamma_z^{-\frac{k}{z}} \tau^{1-\frac{k}{z}} \right] \times_1 \Psi_0 \left[ (\frac{3}{2z}, \frac{k}{z}); -\gamma \gamma_z^{-\frac{k}{z}} \tau^{1-\frac{k}{z}} \right].
\]

Using the approach of Nesterov&Solodukhin 2010 we hope to show how to recover the HL action as the spectral action.
Part II Matter

- The matter coupling to geometry is restricted only by FPDiff.
- This permits inclusion of the higher spatial derivatives in $S_{\text{matter}}$.
- There is no guiding principle on how to proceed except the control over the amount of Lorentz violation (Pospelov&Shang 2010, Kimpton&Padilla 2013)
- The spectral action approach has the second part (Chamseddinne&Connes 1996)

\[ S_{\text{matter}} \propto \langle \psi | D | \psi \rangle \]

- The operator $D$ is the same that was used for the gravity part!
$S_{\text{geom}} = \text{Tr } D$

$S_{\text{matt}} = \langle \psi | D | \psi \rangle$

- Dirac operator
- Geometry
- Space-time
- Gauge
- Matter
• What happens to the geodesic motion?

\[ \nabla_\mu T^{\mu \nu} = 0 \implies \text{geodesic motion} \]

(Dixon 1970, Hawking & Ellis 1973)

• Now we DO NOT have \( \nabla_\mu T^{\mu \nu} = 0 \)

Instead we do have \( h_\lambda^\nu \nabla_\mu T^{\mu \nu} = 0 \), where \( T^{\mu \nu} \propto \frac{\delta S_{\text{matter}}}{\delta g_{\mu \nu}} \)

• Alternative way to get geodesics:
  • Write a field theory
  • Find field equations
  • Restrict to the 1-particle sector
  • Do quasi-classical analysis
  • Hamilton-Jacobi \( \Rightarrow \) geodesic motion
Immediate result is that “geodesics” change

\[ S = -\frac{1}{2} \int d^4 x \sqrt{-g} \left( g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{m^2 c^2}{\hbar^2} \phi^2 \right) \]

\[ \Box \phi - \frac{m^2 c^2}{\hbar^2} \phi = 0 \]

\[ \phi = A e^{\frac{i}{\hbar} S} \]

\[ \left\{ \begin{array}{c} 2 \nabla_\mu A \nabla^\mu S + A \Box S = 0 \Rightarrow \nabla_\mu (A^2 \nabla^\mu S) = 0 \\ \nabla_\mu S \nabla^\mu S = m^2 c^2 = \hbar^2 \frac{\Box A}{A} \end{array} \right. \]

\[ H = g^{\mu \nu} p_\mu p_\nu + m^2 c^2 \]

\[ \left\{ \begin{array}{c} g^{\mu \nu} p_\mu p_\nu + m^2 c^2 = 0 \\ \dot{x}^\mu = 2N(\tau) g^{\mu \nu} p_\nu \\ \dot{p}_\mu = -N(\tau) \frac{\partial g^{\nu \lambda}}{\partial x^\mu} p_\nu p_\lambda \end{array} \right. \]

\[ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu \lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0, \, \tau \text{ is a proper time} \]
Conclusions/Discussions

- Horava-Lifshitz could provide a UV completion of GR
- For this the original proposal should be modified ("healthy" extension?)
- It would be good to have a more geometrical approach to construct the theory
- What is the choice of the coupling to matter?
- What is the correct physical motion of a test particle? Geodesics?
• What is the underlying geometry? E.g. can one get the physical motion of point particle as geodesical motion in this geometry?
• Gauge sector, matter content (it is more natural now to have fields in reps of SO(3))
• Methods of spectral geometry plus spectra action principle might prove useful.
Science is the best way to satisfy your own curiosity at the government’s expense.

L.A. Artsimovich