Multi-layered networks and emergence of spatio-temporal order

Evolutionary game theory approach

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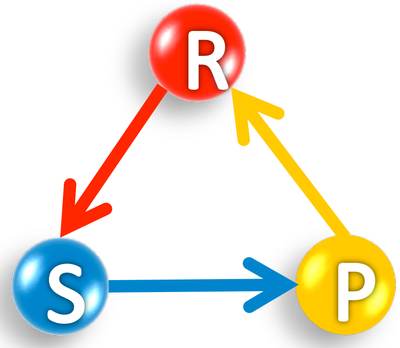
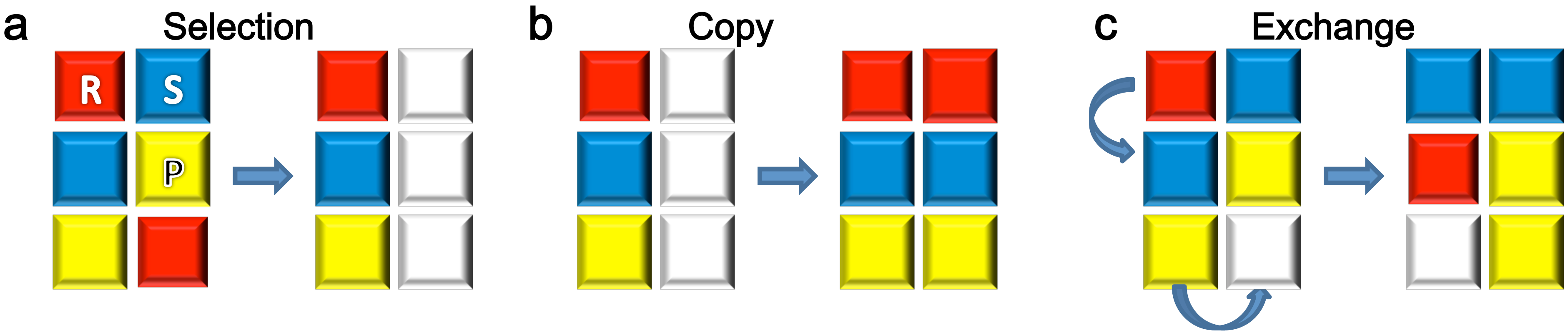
Prof. Ying-Cheng Lai, Arizona and Aberdeen

RPS Game Model

Reichenbach, Mobilia & Frey, Nature (London) 448, 1046, (2007).





Prey **Reproduce** **Mobile**

Role of Epidemic Spreading in Coexistence

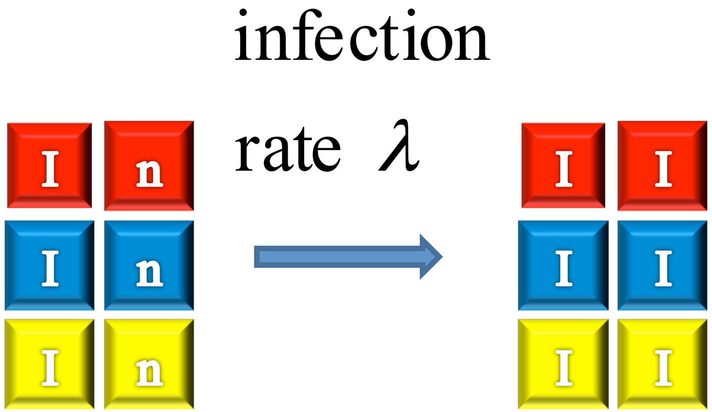
Model

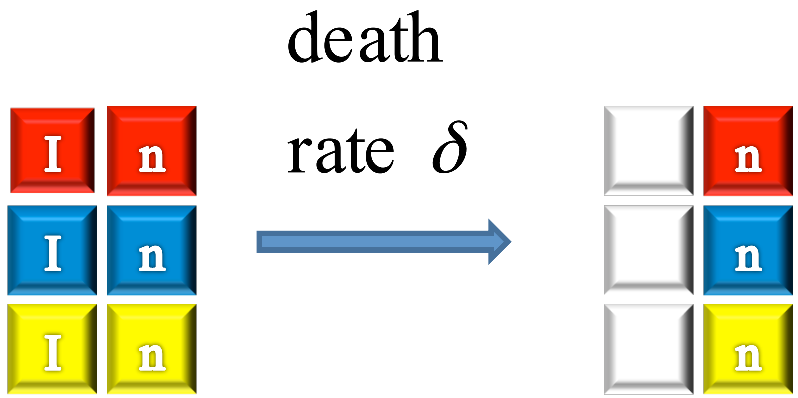
- SIR mode: portion of infected individuals will die
- Spatial neighbors can infect each other
- Offspring is always healthy
- **Our findings**

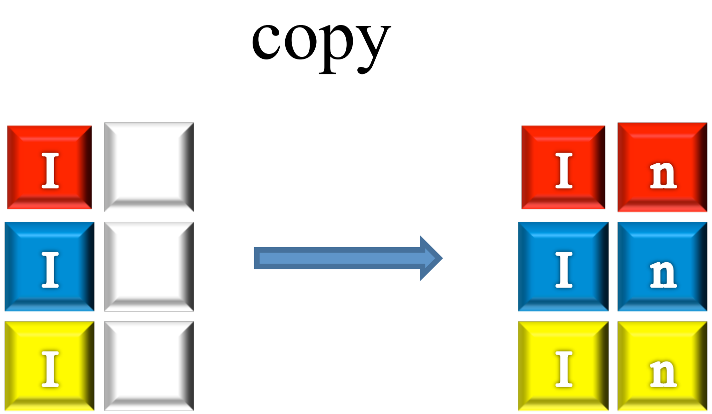
➢ **Virus spread within the same species promotes species coexistence;**

➢ **Spread among different species jeopardizes species coexistence.**

SIR Spreading Within Species







Epidemic spreading does not affect predation and mobility

Offspring is always normal

RPS Model with Epidemic Spreading on 2D Lattice

Cyclic competition, reproduction and mobility:

$$ab \xrightarrow{u} a\varnothing, \quad bc \xrightarrow{u} b\varnothing, \quad ca \xrightarrow{u} c\varnothing, \tag{1}$$

$$a\varnothing \xrightarrow{\sigma} aa, \quad b\varnothing \xrightarrow{\sigma} bb, \quad c\varnothing \xrightarrow{\sigma} cc, \tag{2}$$

$$a\odot \xrightarrow{\epsilon} \odot a, \quad b\odot \xrightarrow{\epsilon} \odot b, \quad c\odot \xrightarrow{\epsilon} \odot c, \tag{3}$$

Epidemic spreading: infection and death

Intra-species $$I_a S_a \xrightarrow{\lambda} I_a I_a, \quad I_b S_b \xrightarrow{\lambda} I_b I_b, \quad I_c S_c \xrightarrow{\lambda} I_c I_c, \tag{4}$$

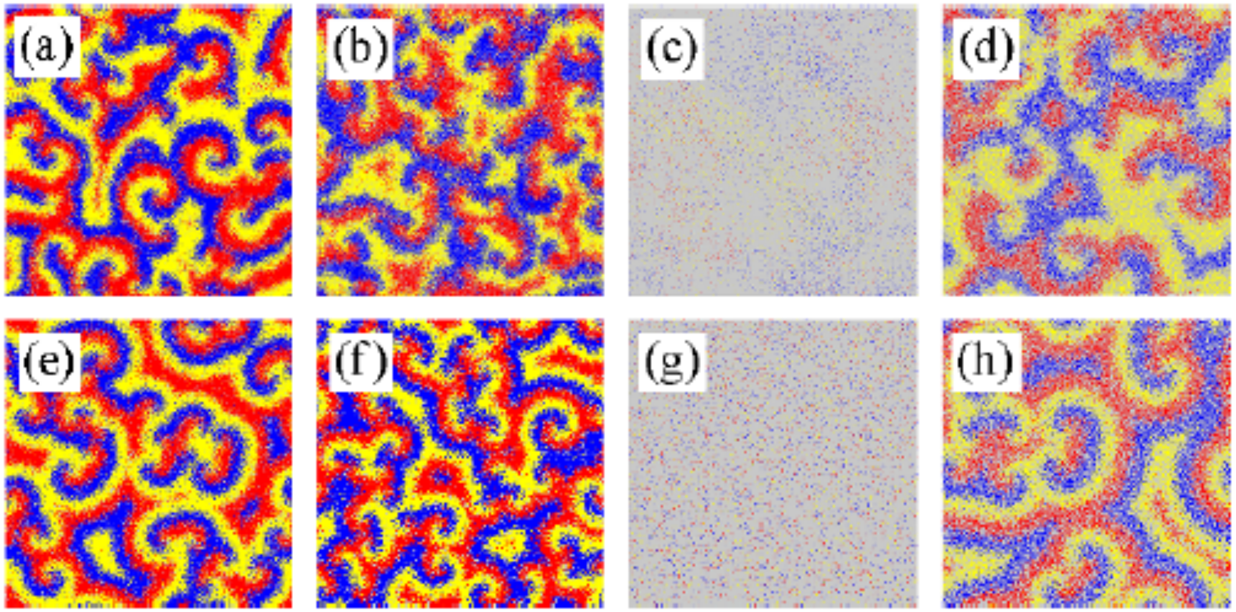
Inter-species $$ⒾⓈ \xrightarrow{\lambda} ⒾⒾ, \tag{5}$$

Spatial patterns

$M < M_c$ $M > M_c$ $M < M_c$

Simulation

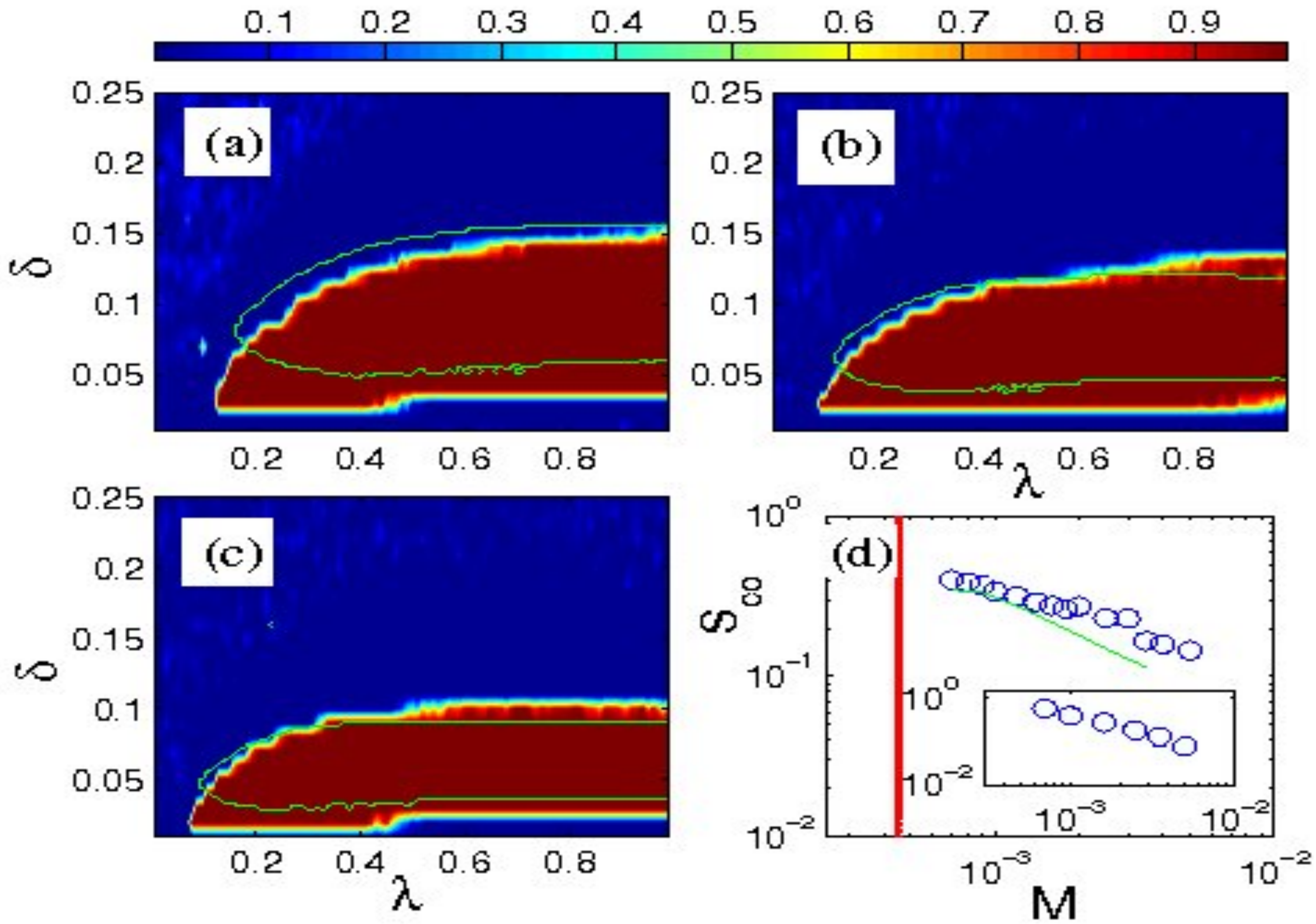
Theoretical model



Intra-species

Inter-species

Parameter-Space Characterization



$(a) M = 9\times 10^{-4} > M_c,$

$(b) M = 1.2\times 10^{-3} > M_c,$

$(c) M = 1.6\times 10^{-3} > M_c$

Intra-species virus spreading

Green boundaries – Prediction from PDE model

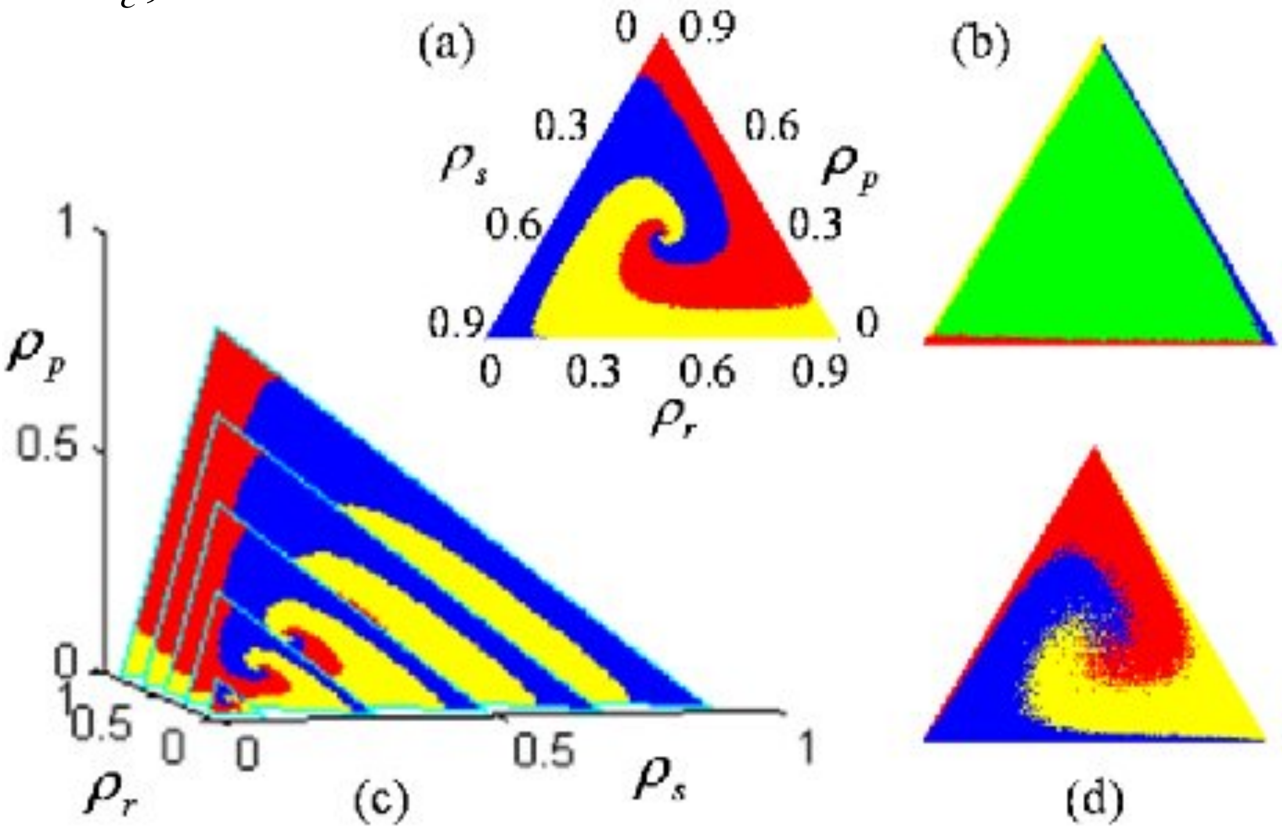
W.-X. Wang, Y.-C. Lai, and C. Grebogi, **Effect of epidemic spreading on species coexistence in rock-paper-scissors games**, Phys. Rev. E **81**,046113(1-4) (2010)

Intra-Species Virus Spreading Induces Coexistence

$M = 10^{-3} > M_c,$

$\lambda = 0.6$

$\delta = 0.13$



(a) No virus spreading; (b) With **intra-species** virus spreading;

(c) Under **inter-species** spreading, basin structure predicted by PDE model for different values of initial density of empty sites;

(d) Basin structure under **inter-species** virus spreading (from direct simulation)

Theoretical Model

Infection densities: $I_a(\mathbf{r},t), I_b(\mathbf{r},t)$ and $I_c(\mathbf{r},t)$

Susceptible densities: $S_a(\mathbf{r},t), S_b(\mathbf{r},t)$ and $S_c(\mathbf{r},t)$

Spatial site: $\mathbf{r} = (r_1, r_2)$ for 2d lattice

Neighboring location: $\mathbf{r} \pm \delta r \cdot \mathbf{e}_i$, where $\mathbf{e}_i$ is the basis of 2d lattice

$$\rho_a(\mathbf{r},t) \equiv I_a(\mathbf{r},t) + S_a(\mathbf{r},t);$$

$$\rho_b(\mathbf{r},t) \equiv I_b(\mathbf{r},t) + S_b(\mathbf{r},t);$$ density of species

$$\rho_c(\mathbf{r},t) \equiv I_c(\mathbf{r},t) + S_c(\mathbf{r},t);$$

$$\rho(\mathbf{r},t) \equiv \rho_a(\mathbf{r},t) + \rho_b(\mathbf{r},t) + \rho_c(\mathbf{r},t);$$ total density of species

rescaled probabilities of

infection and death

$$\alpha = \frac{\lambda}{\varepsilon + u + \sigma}, \qquad \beta = \frac{\delta}{\varepsilon + u + \sigma}$$

Evolutionary Equations

$$\partial_t I_a(\mathbf{r},t) = \frac{1}{z}\sum_{\pm,i=1}^{2}\left\{2\varepsilon[I_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t) - I_a(\mathbf{r},t)] + \alpha I_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t)S_a(\mathbf{r},t) - \beta I_a(\mathbf{r},t) - uI_a(\mathbf{r},t)\rho_c(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t)\right\},$$

$$\partial_t S_a(\mathbf{r},t) = \frac{1}{z}\sum_{\pm,i=1}^{2}\{2\varepsilon[S_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t) - S_a(\mathbf{r},t)] - \alpha I_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t)S_a(\mathbf{r},t) - uS_a(\mathbf{r},t)\rho_c(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t)$$
$$+ \sigma\rho_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t)\big[1-\rho(\mathbf{r},t)\big]\}$$

For $N \to \infty$ and lattice size fixed to 1, $\delta r \to 0$. Thus $\mathbf{r}$ can be treated as a continuous variable together with the following expansion for I_a and S_a :

$$I_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t) = I_a(\mathbf{r},t) \pm \delta r\partial_i I_a(\mathbf{r},t) + \frac{1}{2}\delta r^2\partial_i^2 I_a(\mathbf{r},t) + o(\delta r^2),$$

$$S_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t) = S_a(\mathbf{r},t) \pm \delta r\partial_i S_a(\mathbf{r},t) + \frac{1}{2}\delta r^2\partial_i^2 S_a(\mathbf{r},t) + o(\delta r^2).$$

Up to second order : $$\frac{2\varepsilon}{z}\sum_{\pm,i=1}^{2}[I_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t) - I_a(\mathbf{r},t)] = \frac{\varepsilon}{2}\delta r^2\partial_i^2 I_a(\mathbf{r},t),$$

$$\frac{2\varepsilon}{z}\sum_{\pm,i=1}^{2}[S_a(\mathbf{r}\pm\delta r\cdot\mathbf{e}_i,t) - S_a(\mathbf{r},t)] = \frac{\varepsilon}{2}\delta r^2\partial_i^2 S_a(\mathbf{r},t).$$

Partial Differential Equations

Rescaled exchange rate $\varepsilon = 2MN \Rightarrow \varepsilon\delta r^2 = 2M$.

$$\partial_t I_a(\mathbf{r},t) = M\nabla^2 I_a(\mathbf{r},t) + \alpha\psi_a(\mathbf{r},t)S_a(\mathbf{r},t) - \beta I_a(\mathbf{r},t) - uI_a(\mathbf{r},t)\rho_c(\mathbf{r},t);$$

$$\partial_t S_a(\mathbf{r},t) = M\nabla^2 S_a(\mathbf{r},t) - \alpha\psi_a(\mathbf{r},t)S_a(\mathbf{r},t) - uS_a(\mathbf{r},t)\rho_c(\mathbf{r},t) + \sigma\rho_a(\mathbf{r},t)[1-\rho(\mathbf{r},t)];$$

$$\partial_t I_b(\mathbf{r},t) = M\nabla^2 I_b(\mathbf{r},t) + \alpha\psi_b(\mathbf{r},t)S_b(\mathbf{r},t) - \beta I_b(\mathbf{r},t) - uI_b(\mathbf{r},t)\rho_a(\mathbf{r},t);$$

$$\partial_t S_b(\mathbf{r},t) = M\nabla^2 S_b(\mathbf{r},t) - \alpha\psi_b(\mathbf{r},t)S_b(\mathbf{r},t) - uS_b(\mathbf{r},t)\rho_a(\mathbf{r},t) + \sigma\rho_b(\mathbf{r},t)[1-\rho(\mathbf{r},t)];$$

$$\partial_t I_c(\mathbf{r},t) = M\nabla^2 I_c(\mathbf{r},t) + \alpha\psi_c(\mathbf{r},t)S_c(\mathbf{r},t) - \beta I_c(\mathbf{r},t) - uI_c(\mathbf{r},t)\rho_b(\mathbf{r},t);$$

$$\partial_t S_c(\mathbf{r},t) = M\nabla^2 S_c(\mathbf{r},t) - \alpha\psi_c(\mathbf{r},t)S_c(\mathbf{r},t) - uS_c(\mathbf{r},t)\rho_b(\mathbf{r},t) + \sigma\rho_c(\mathbf{r},t)[1-\rho(\mathbf{r},t)];$$

where for intra - species transmission

$$\psi_a(\mathbf{r},t) = I_a(\mathbf{r},t),\quad \psi_b(\mathbf{r},t) = I_b(\mathbf{r},t),\quad \psi_c(\mathbf{r},t) = I_c(\mathbf{r},t),$$

for inter - species transmission

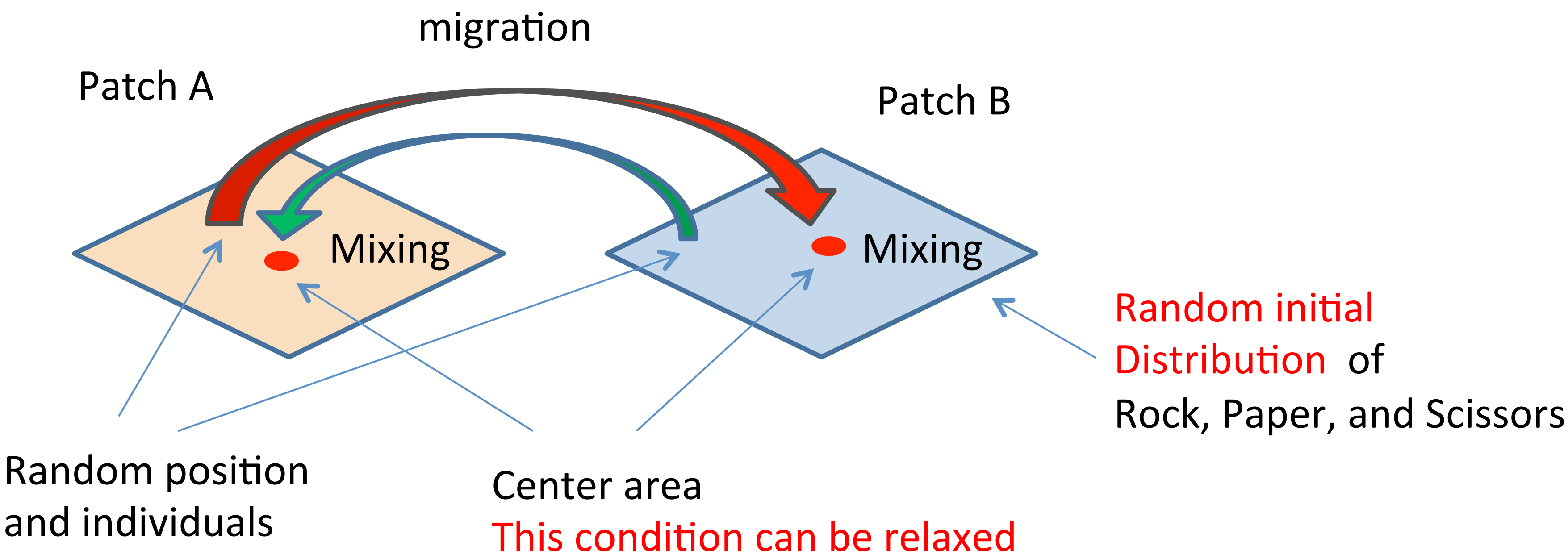
$$\psi_a(\mathbf{r},t) = \psi_b(\mathbf{r},t) = \psi_c(\mathbf{r},t) = I_a(\mathbf{r},t) + I_b(\mathbf{r},t) + I_c(\mathbf{r},t).$$

Inter-Patch Migration: Pattern Formation and Synchronization

- Target waves
- Synchronization and lag synchronization
- Coexistence of target wave and spiral waves
- Multi-target waves

Pattern formation, synchronisation and outbreak of biodiversity in cyclically competing games, W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, Phys. Rev. E **83**, 011917(1-9) (2011)

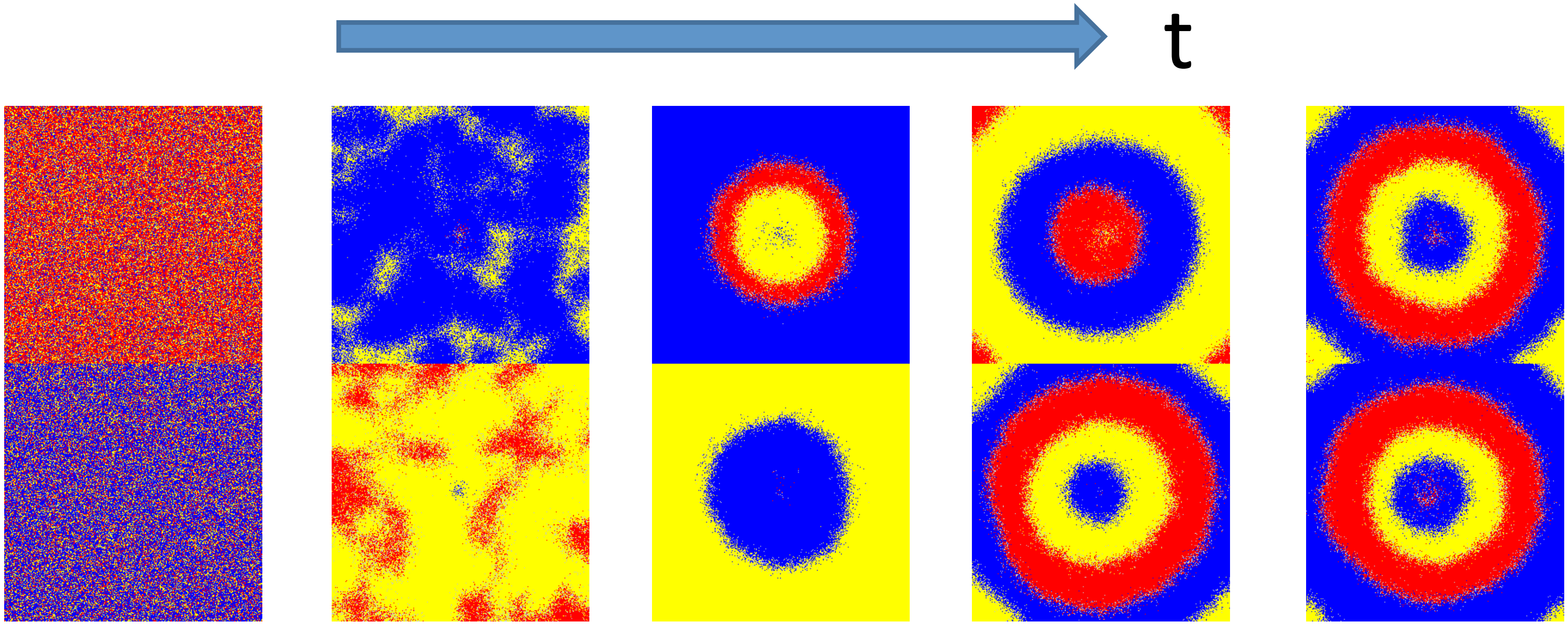
Inter-Patch Migration Model for Two Patches



$$ab \xrightarrow{u} a\varnothing, \quad bc \xrightarrow{u} b\varnothing, \quad ca \xrightarrow{u} c\varnothing, \qquad (1)$$

$$a\varnothing \xrightarrow{\sigma} aa, \quad b\varnothing \xrightarrow{\sigma} bb, \quad c\varnothing \xrightarrow{\sigma} cc, \qquad (2)$$

$$a\odot \xrightarrow{\varepsilon} \odot a, \quad b\odot \xrightarrow{\varepsilon} \odot b, \quad c\odot \xrightarrow{\varepsilon} \odot c, \qquad (3)$$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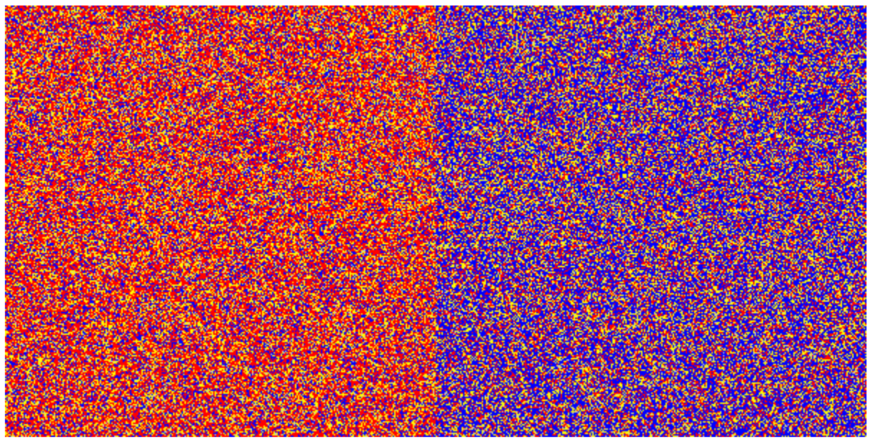
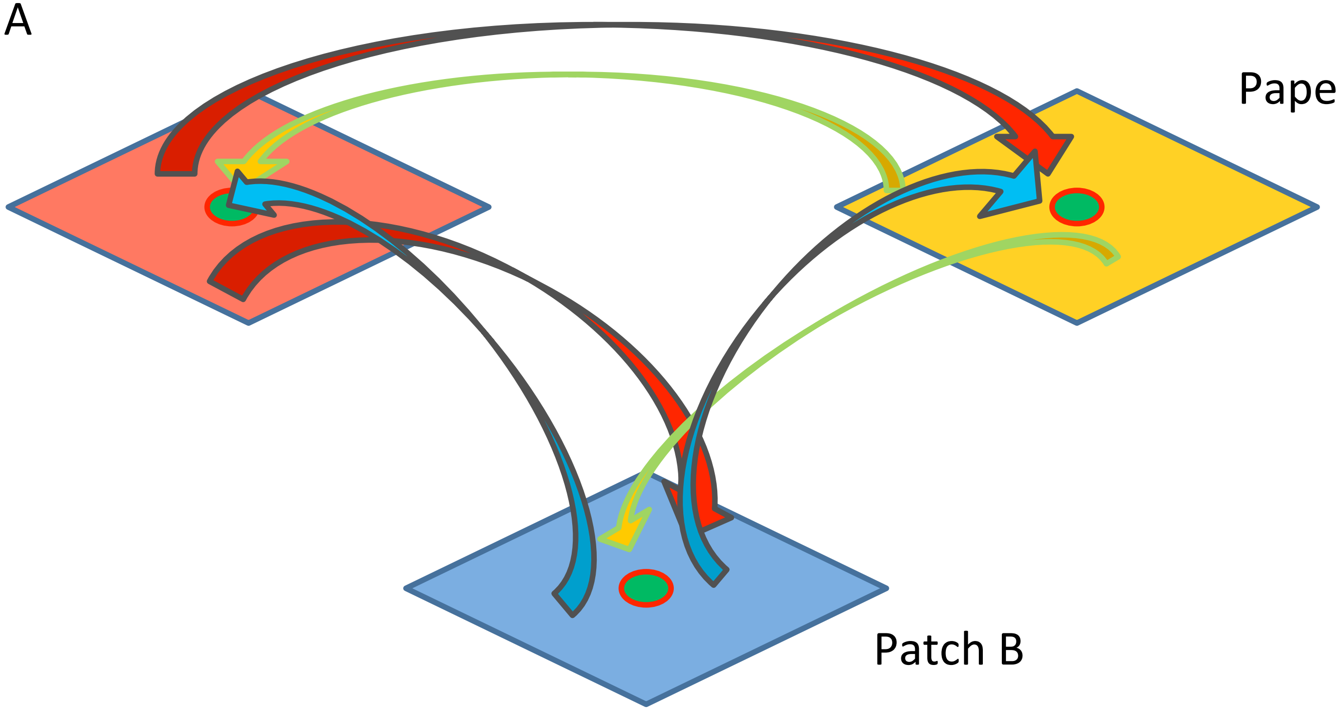
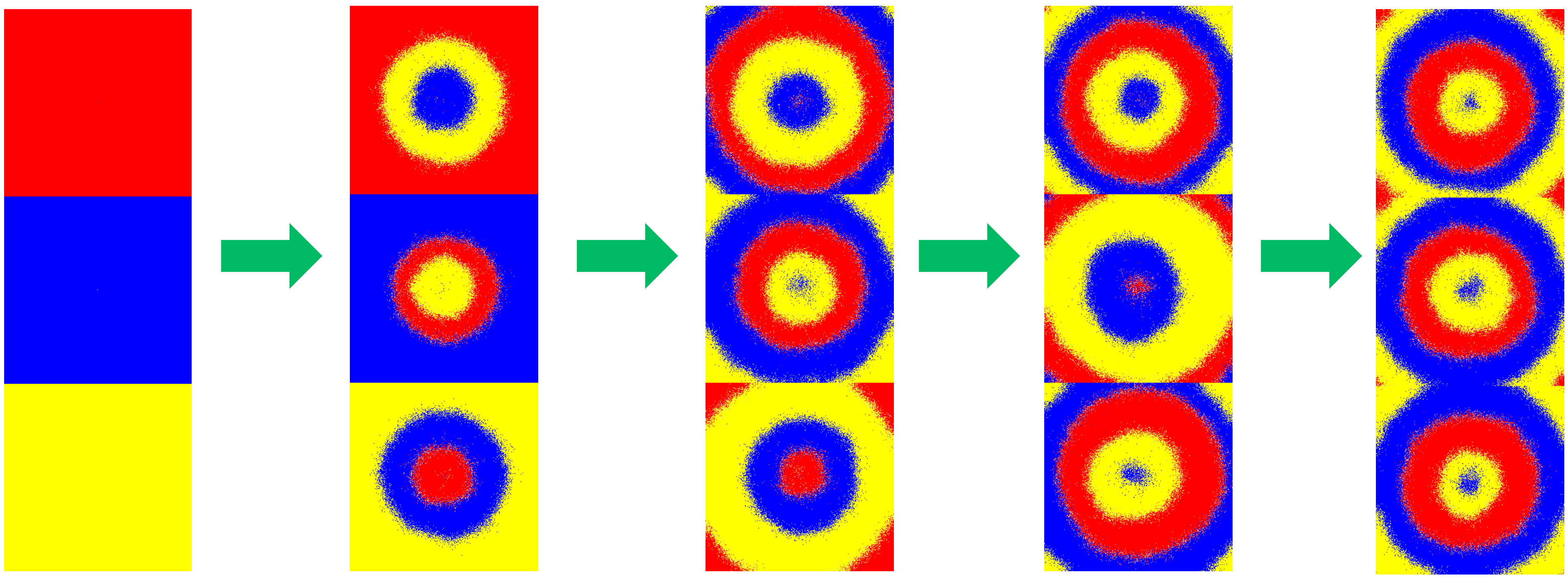


Migration Among Three Patches

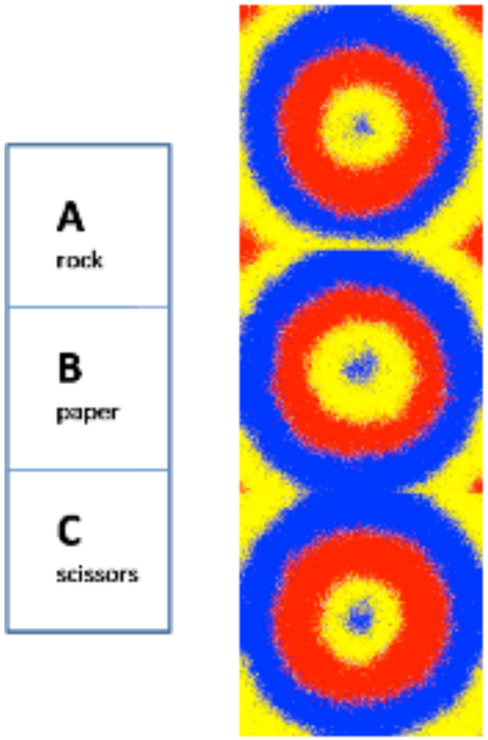


Random origin of migration
Center destination point
Condition of destination can be relaxed



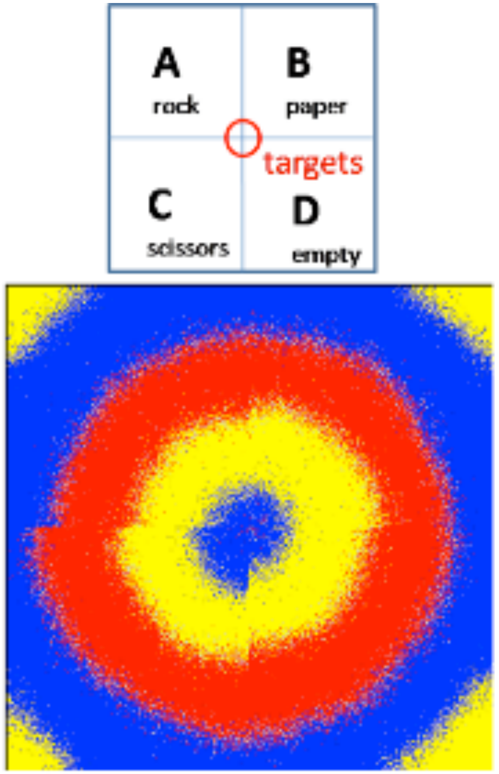
Target Waves and Synchronisation

Three centred



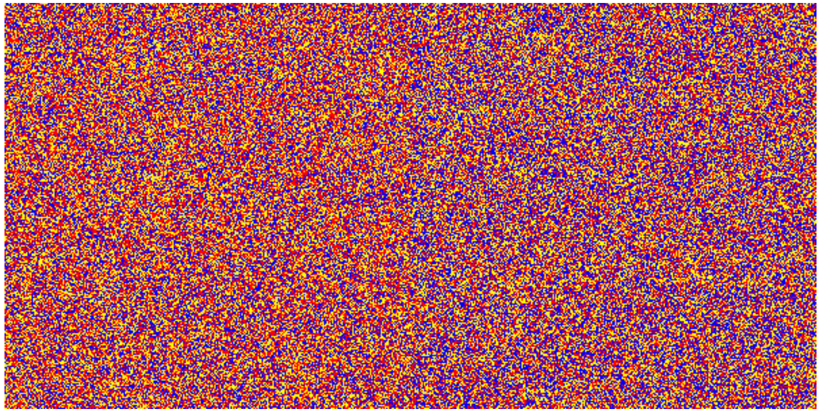
(a)

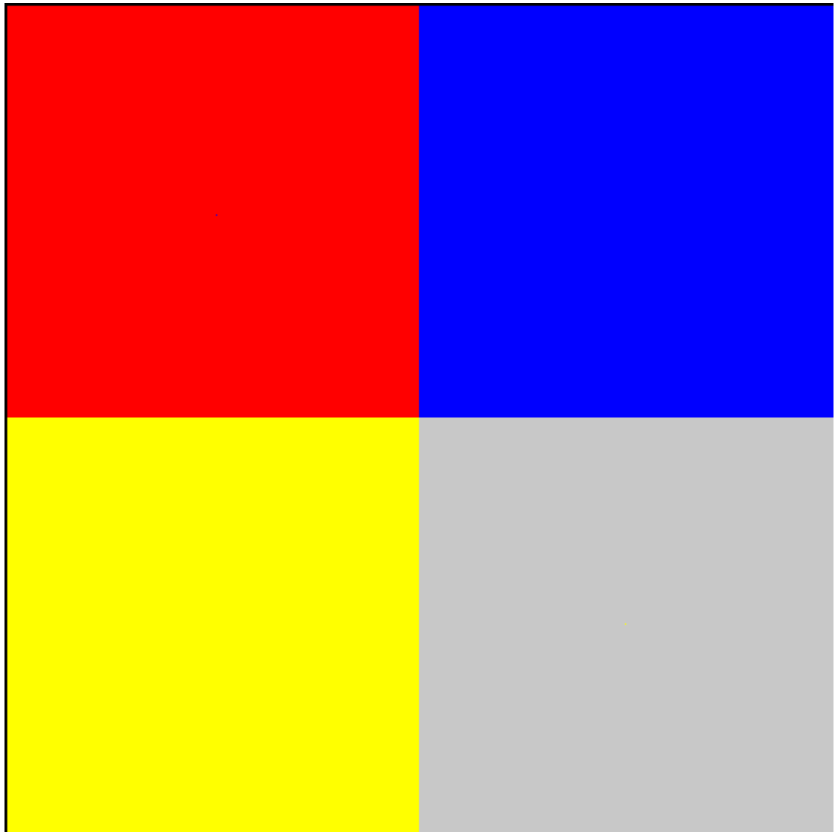
Four de-centred



(b)

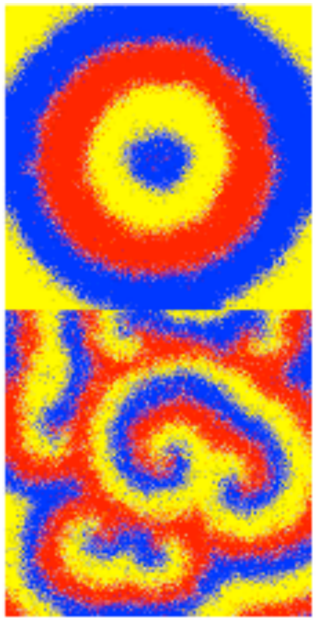
Coexistence of Target and Spiral Waves





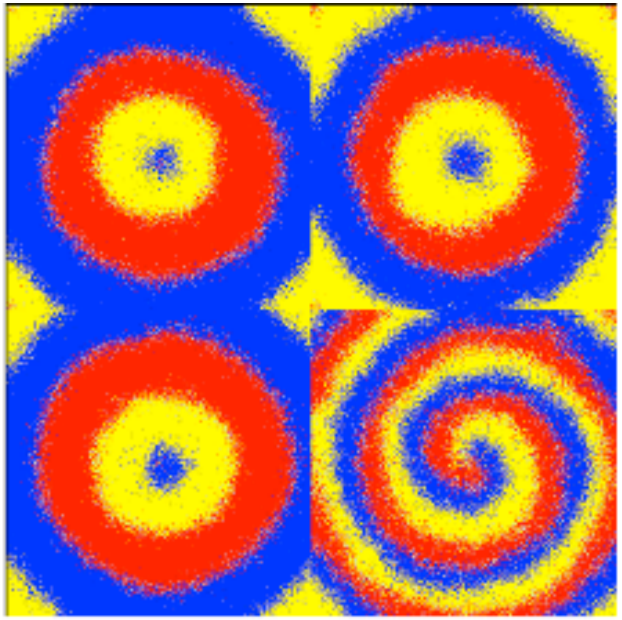
Coexistence of Target and Spiral Waves

Two-patch system



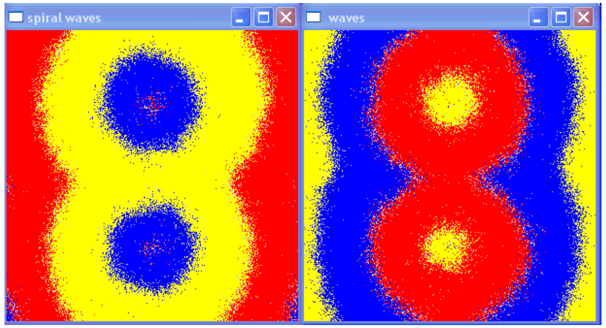
(a)

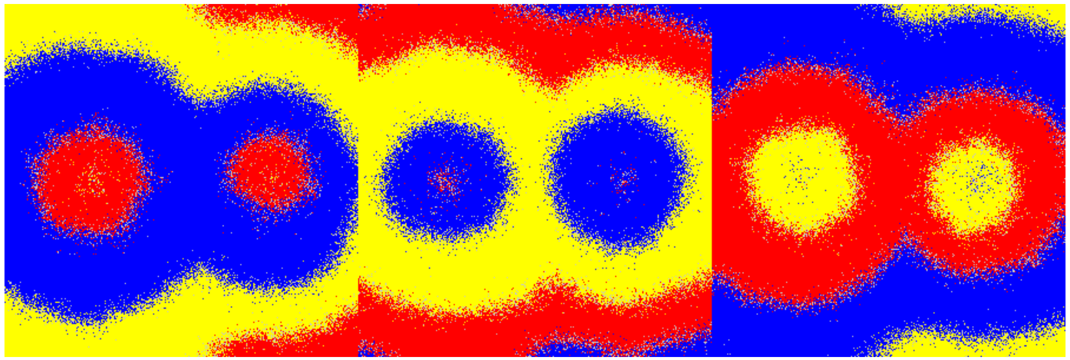
Four-patch system



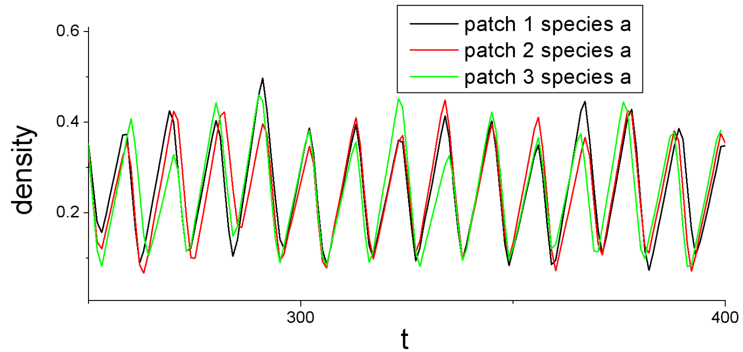
(b)

Multi-Target Waves



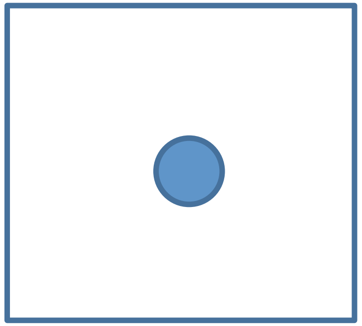


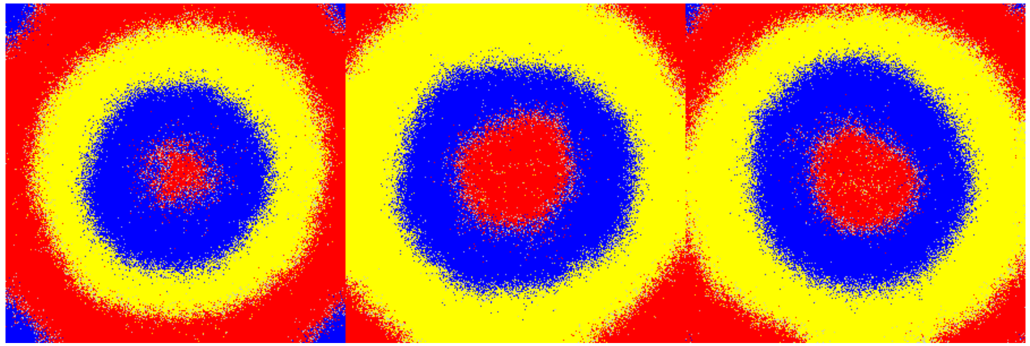
Migration to a target range



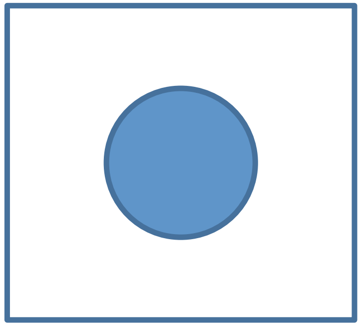
synchronization

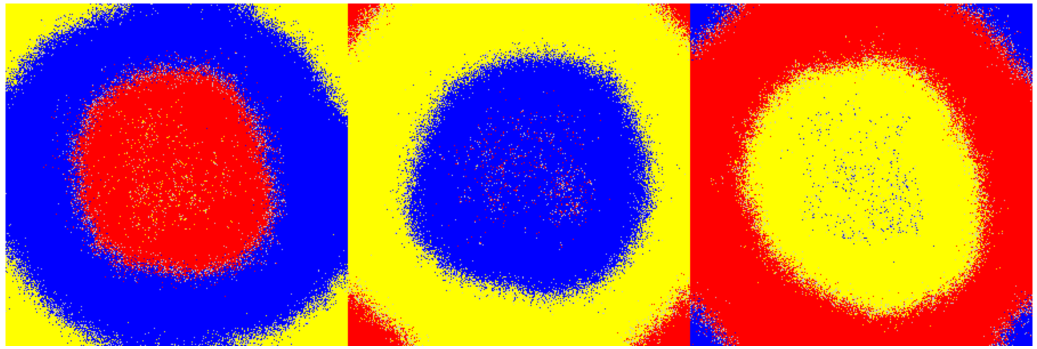
Small target range



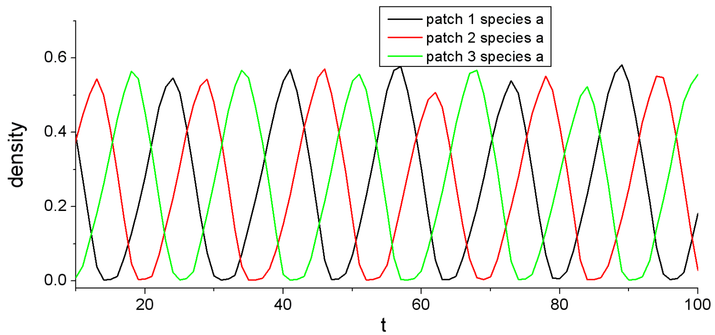


Large target range





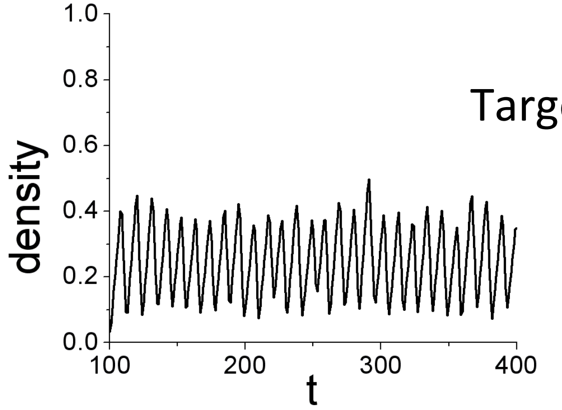
Lag synchronization

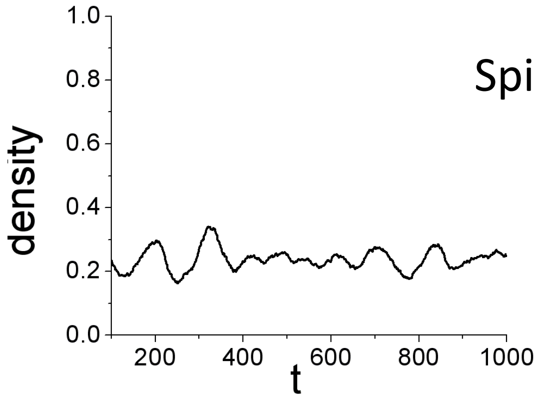


Quantitative Characterization

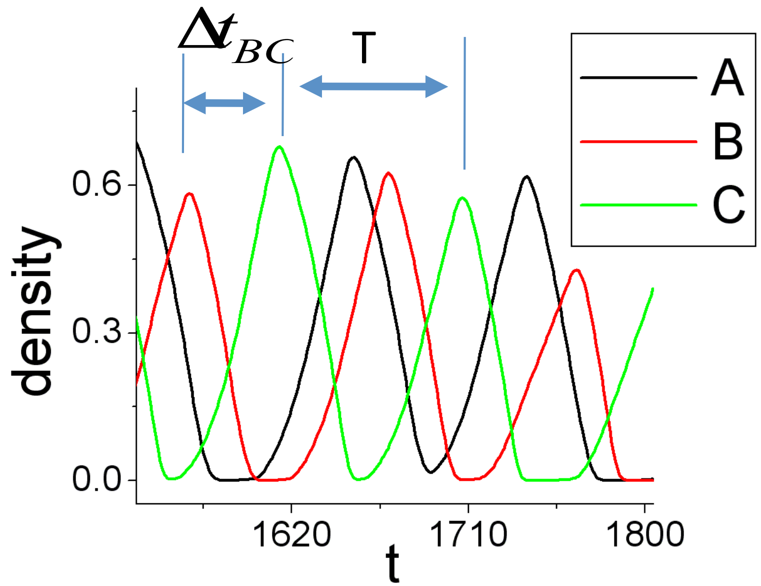
Distinguish target waves from spiral waves

Fourier transform





Order parameter for
phase synchronization of target waves



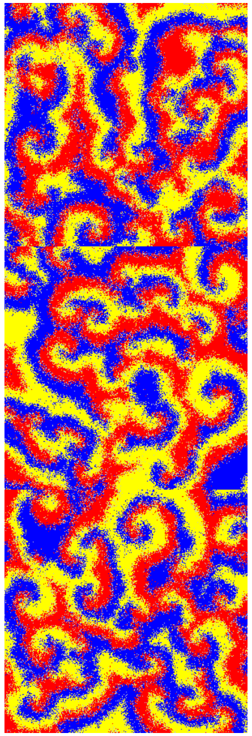
$$\eta_{BC} = 1 - \frac{\langle \min(\Delta t_{BC}, \langle T \rangle - \Delta t_{BC}) \rangle}{\left\langle \frac{T}{2} \right\rangle},$$

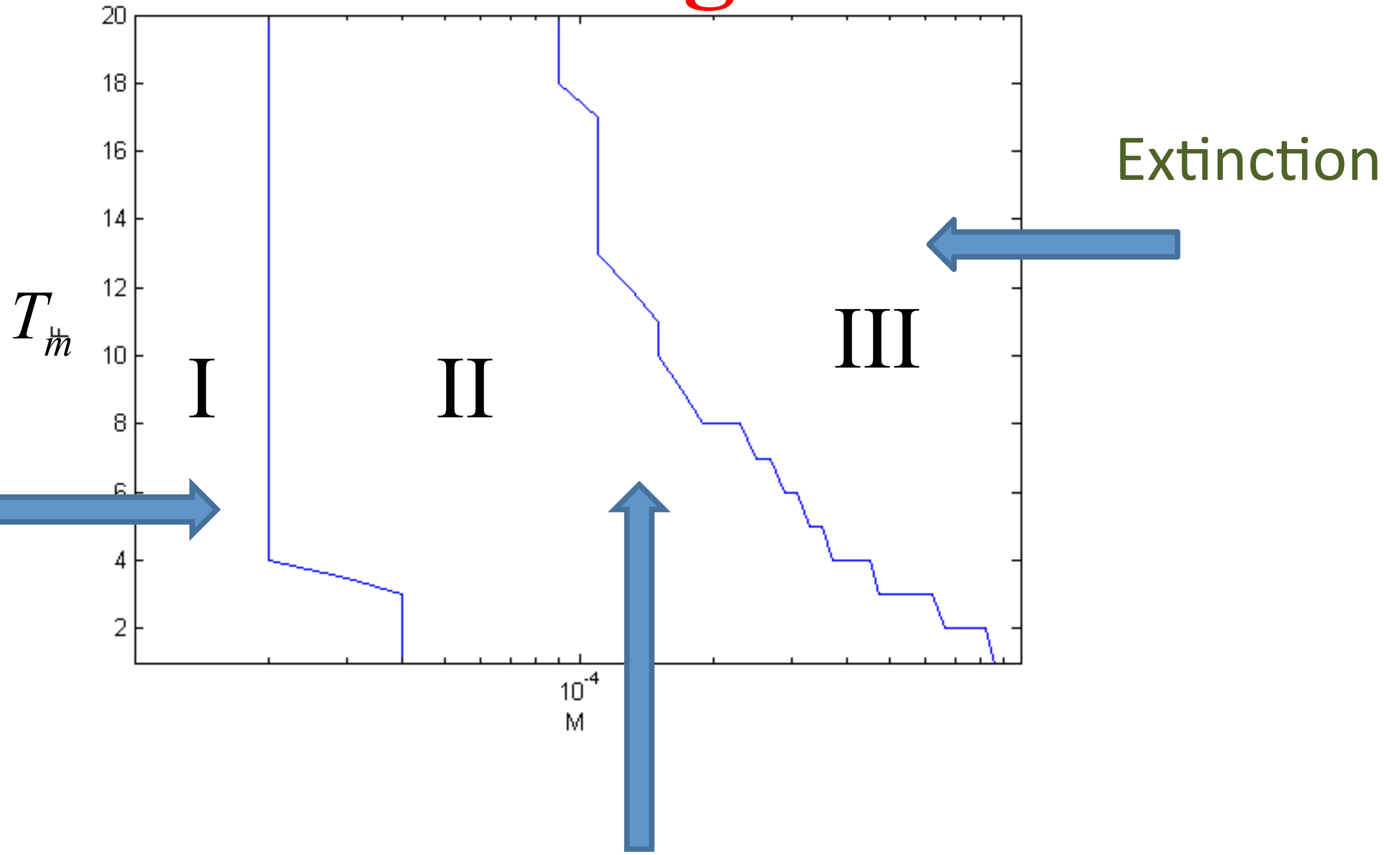
$$\eta = \frac{\eta_{AB} + \eta_{BC} + \eta_{AC}}{3} \Leftarrow \text{order parameter}$$

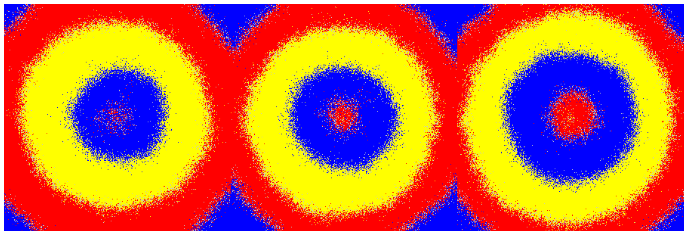
$$0 < \eta < 1$$

$$\eta = 1: \text{complete synchronization}$$

Phase diagram

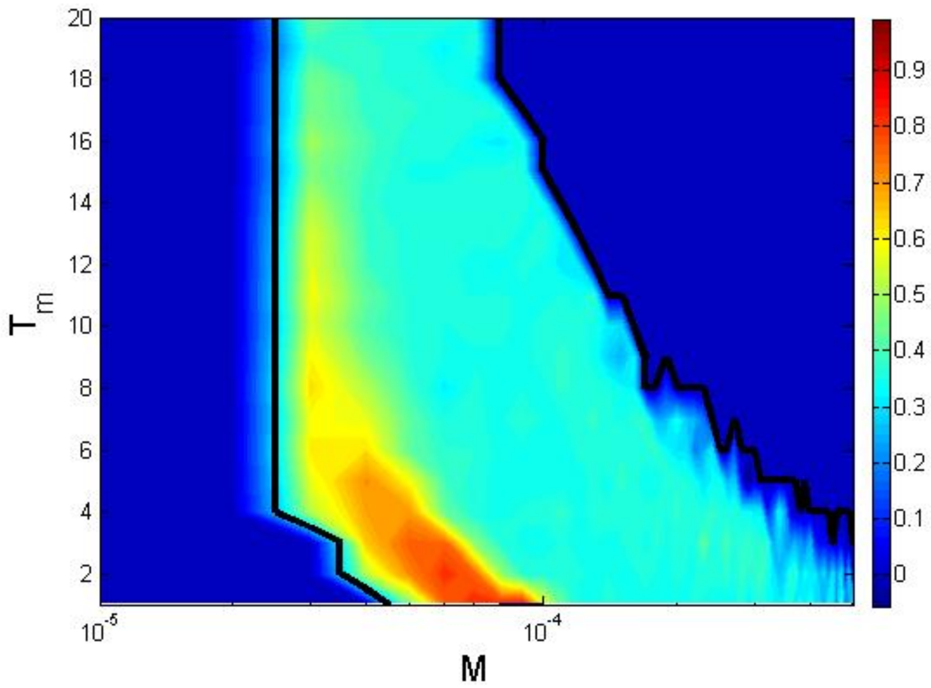




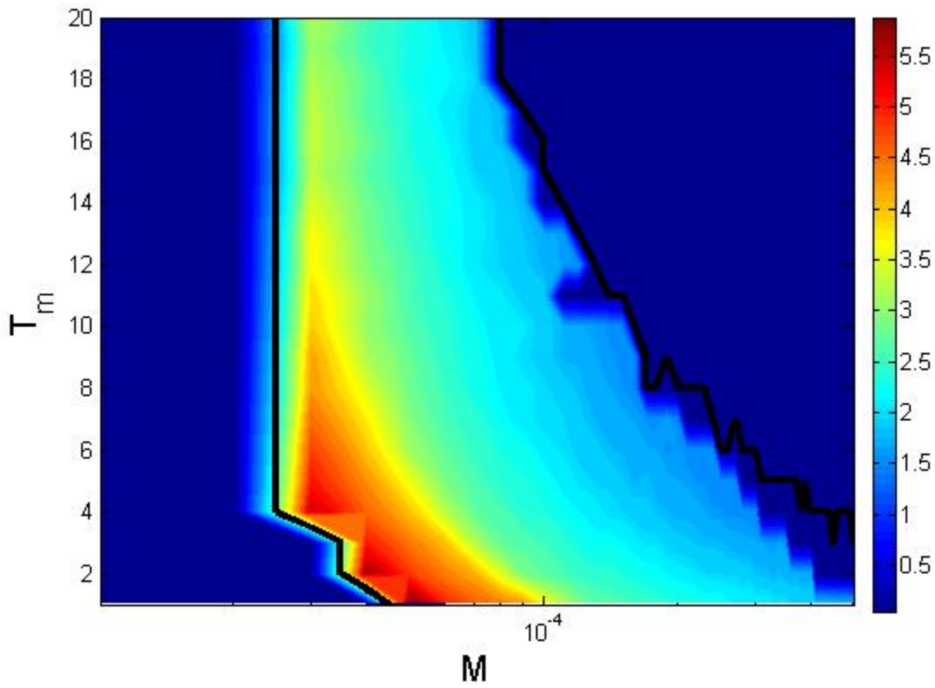


Synchronisability and Number of Rings

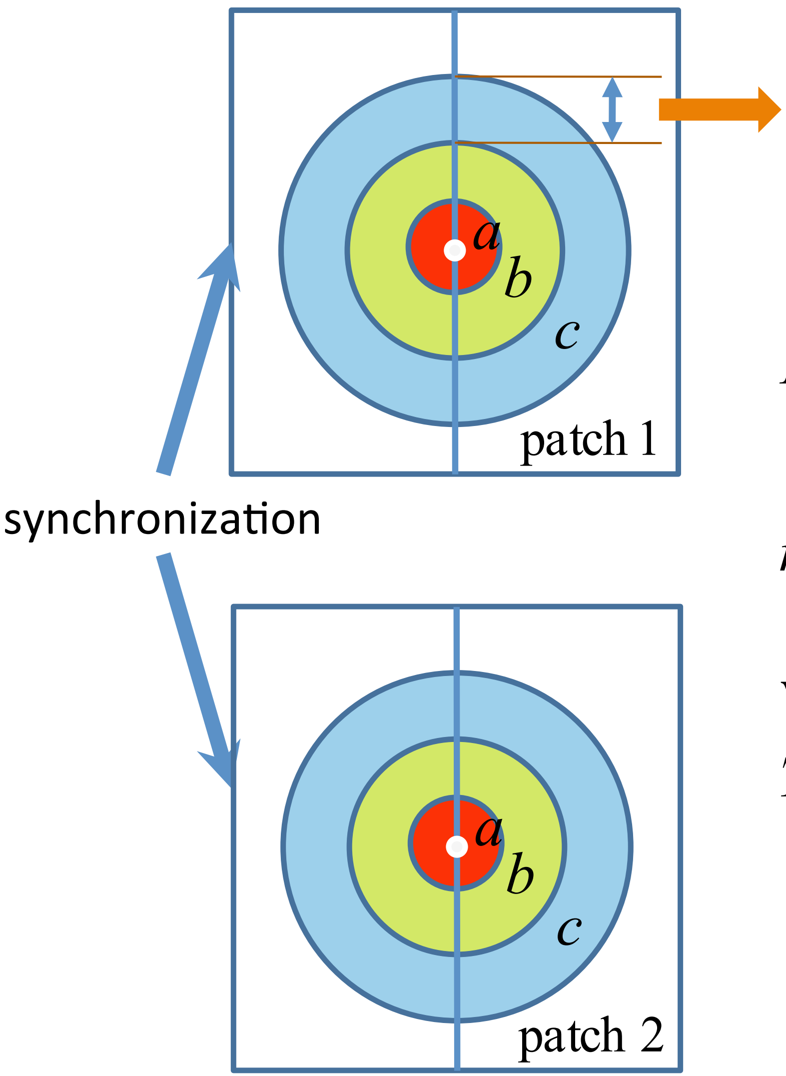
Synchronisability



Number of rings



Predicting the Number of Rings



L_r : length of rings

$$L_r = \frac{T_m}{\langle \rho_c \rangle_{patch2} - \langle \rho_b \rangle_{patch2}} \cdot V,$$

$$n_r = \frac{L}{\sqrt{2L_r}},$$

where V is the velocity of front propagation, and T_m is the period of inter-patch migration.

$V \Leftarrow$ CGLE

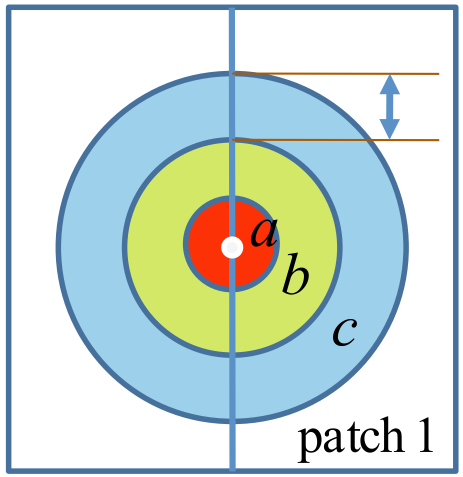
$$\partial_t z(r,t) = M\Delta z(r,t) + (c_1 - i\omega)z(r,t) - c_2(1 - ic_3)|z(r,t)|^2 z(r,t),$$

linearize the CGLE around the state $z = 0$,

$$\partial_t z(r,t) = M\Delta z(r,t) + (c_1 - i\omega)z(r,t) + o(z^2),$$

$$\Rightarrow V = 2\sqrt{c_1 M},$$

where $c_1 = \frac{1}{2}\frac{\mu\sigma}{3\mu+\sigma}$.



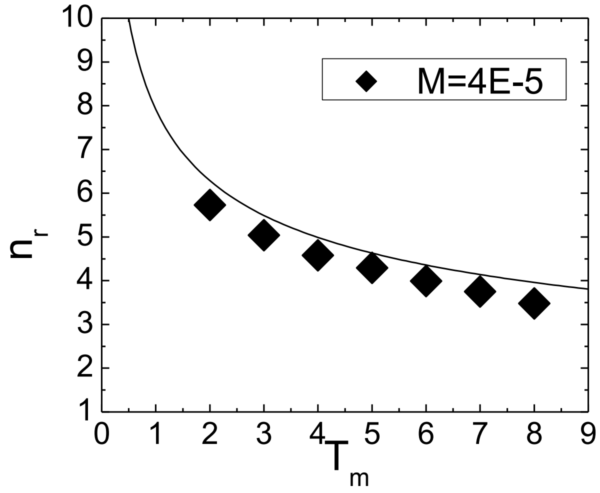
Due to synchronization, $\langle \rho_c \rangle_{patch1} = \langle \rho_c \rangle_{patch2}$,

and $\langle \rho_b \rangle_{patch1} = \langle \rho_b \rangle_{patch2}$.

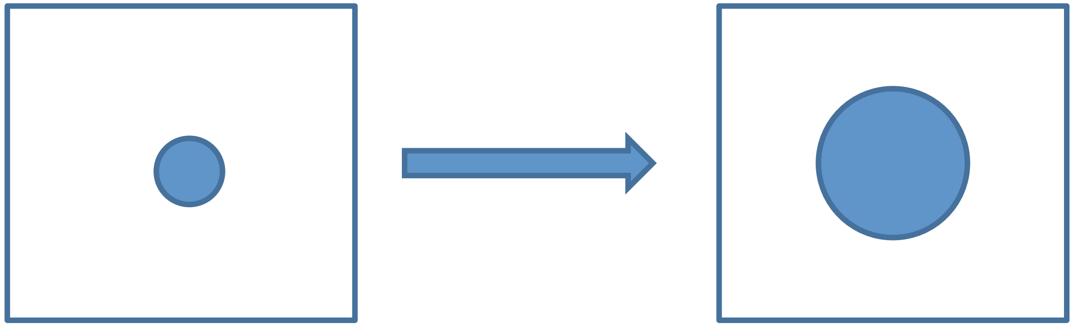
$$\langle \rho_c \rangle_{patch1} = \int_{L_r}^{2L_r} \frac{\pi(x+L_r)^2 - \pi x^2}{L_r \cdot L^2} dx = \frac{4\pi L_r^2}{L^2}.$$

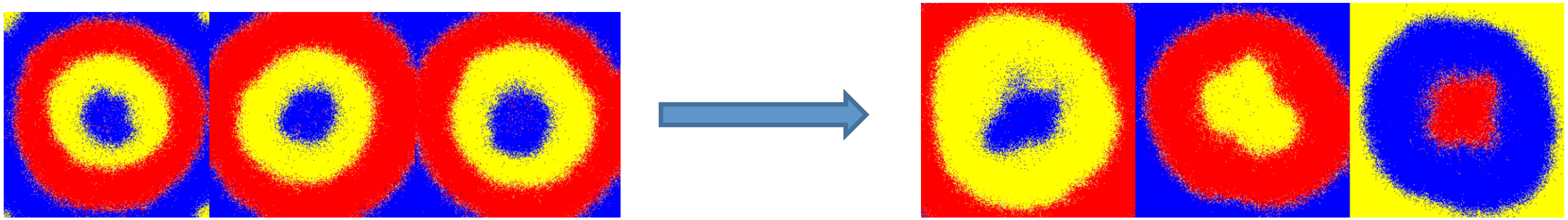
$$\langle \rho_b \rangle_{patch1} = \int_{0}^{L_r} \frac{\pi(x+L_r)^2 - \pi x^2}{L_r \cdot L^2} dx = \frac{2\pi L_r^2}{L^2}.$$

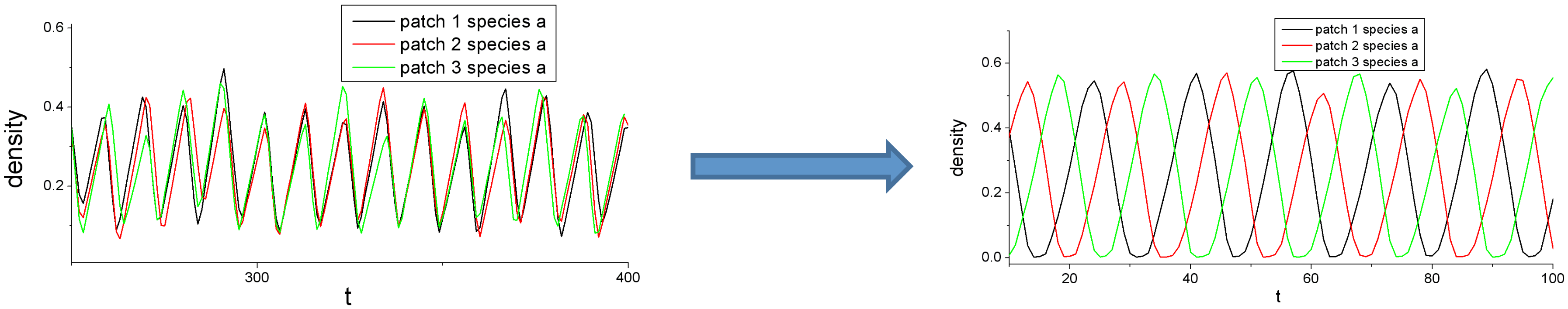
$$\Rightarrow L_r = \left(\frac{T_m L^2}{2\pi} \sqrt{\frac{2\mu\sigma}{3\mu+\sigma} M} \right)^{\frac{1}{3}} \quad \& \quad n_r = \frac{L}{\sqrt{2}L_r}.$$



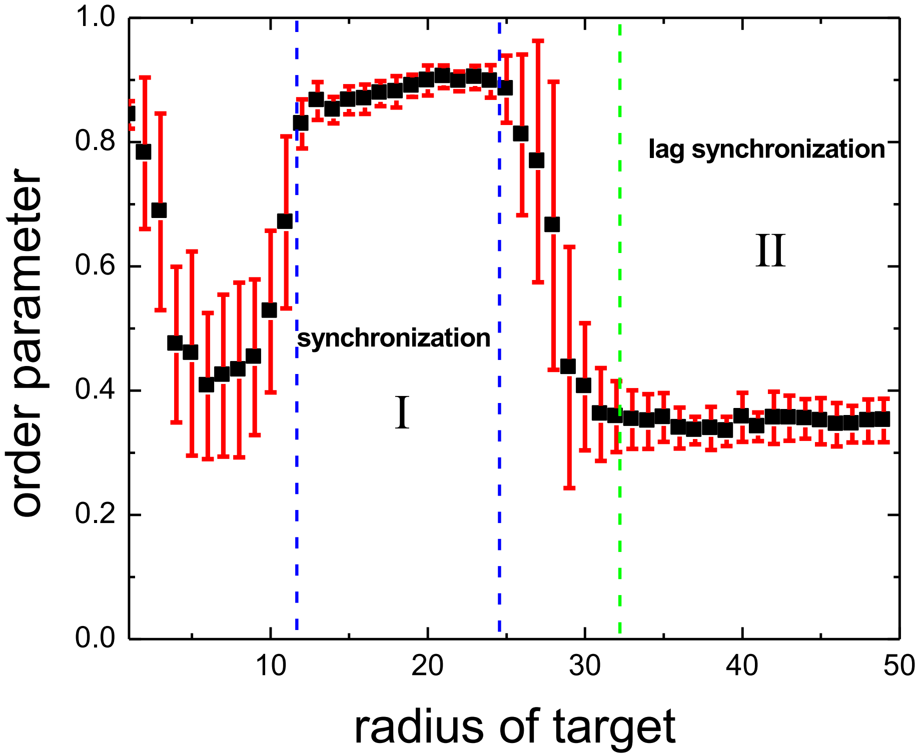
Transition from Complete to Lag Synchronisation

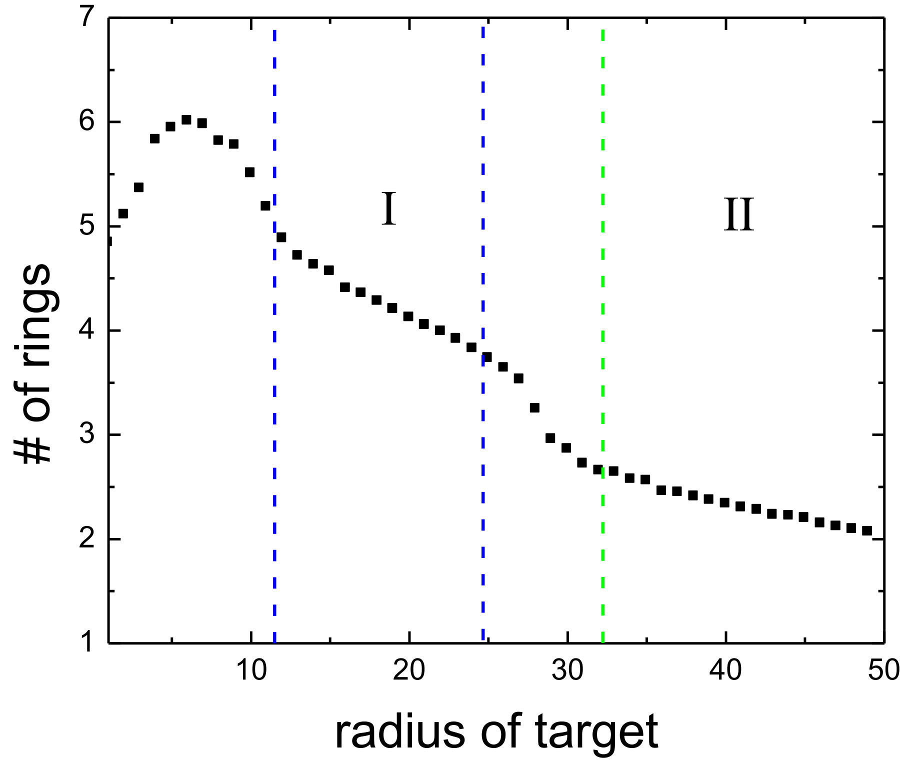


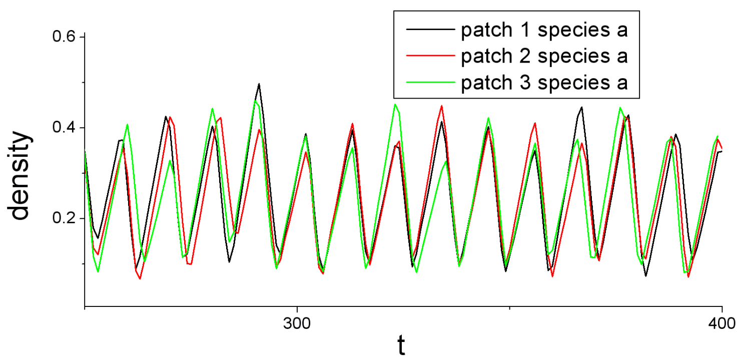


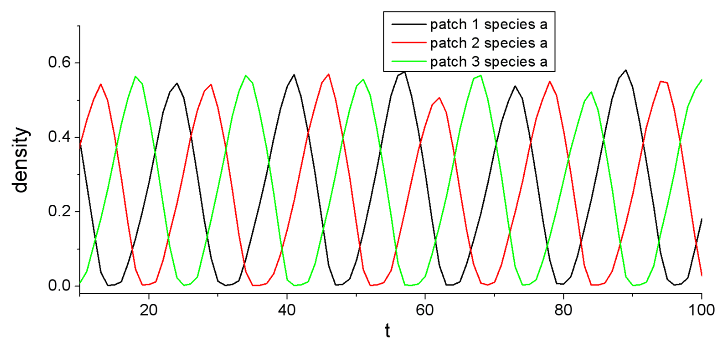


Transition to Lag Synchronisation



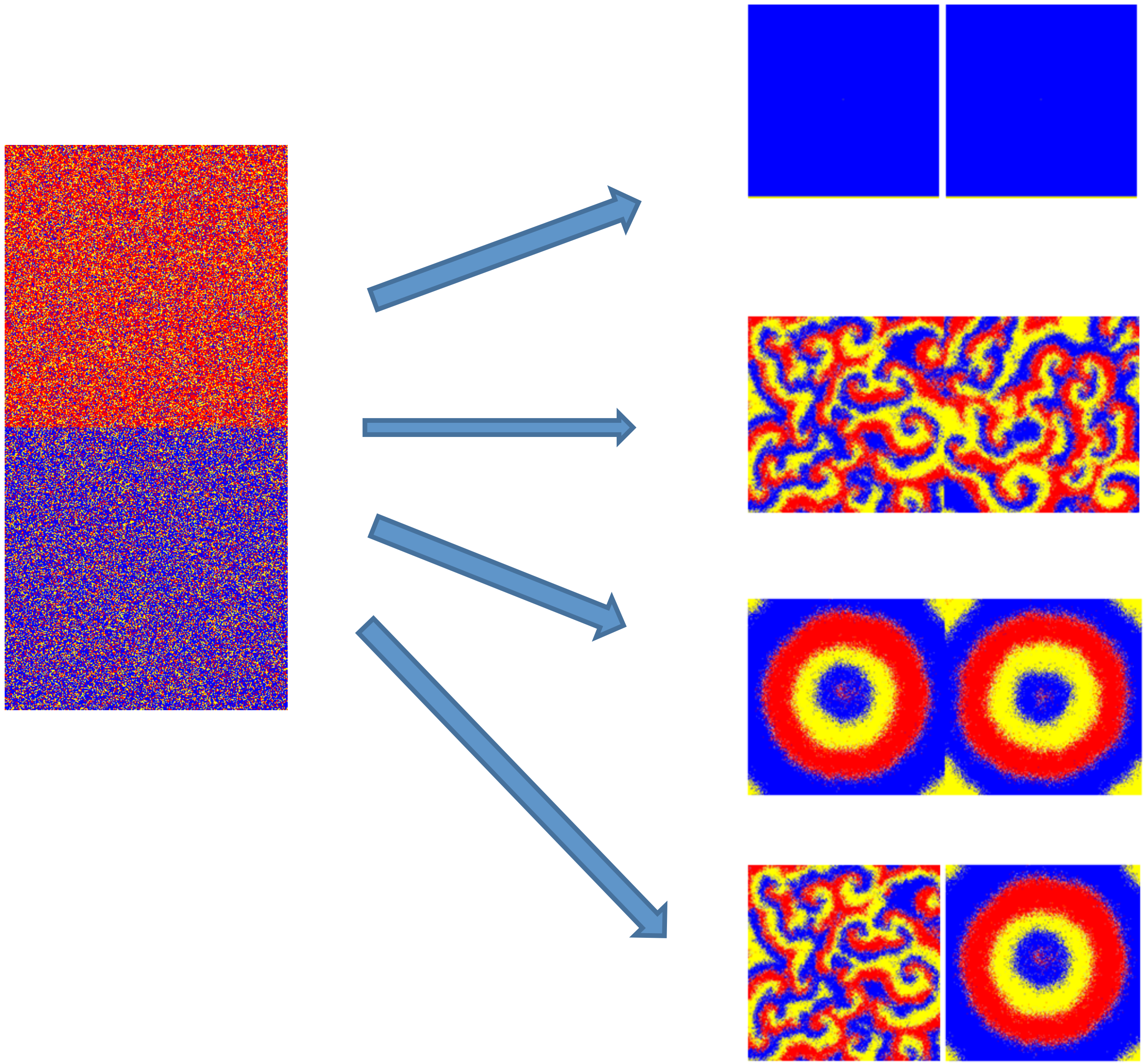




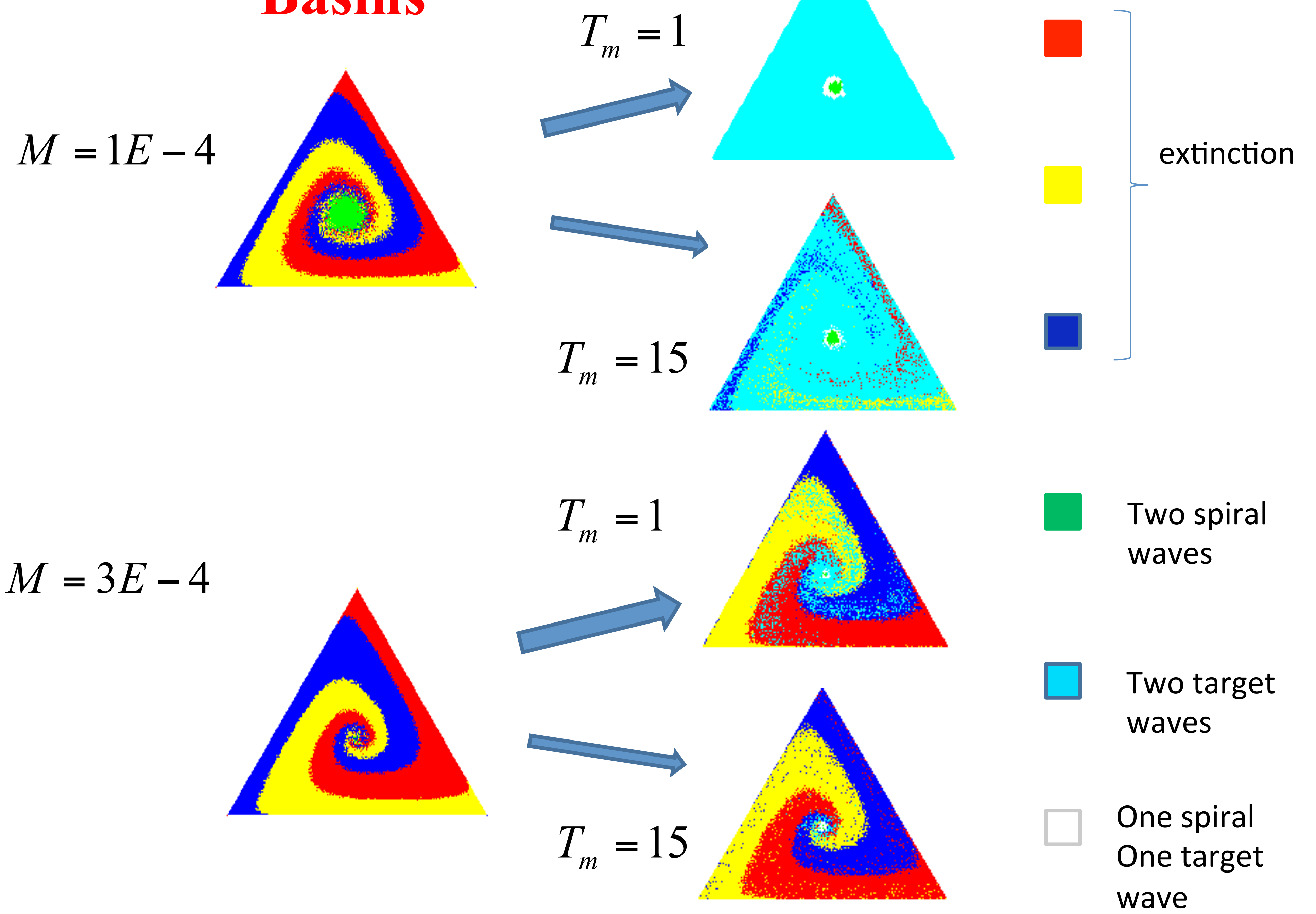


Basins of Extinction and Coexistence

$$\rho_1(a) = \rho_2(b)$$
$$\rho_1(b) = \rho_2(c)$$
$$\rho_1(c) = \rho_2(a)$$



Basins


$M = 1E-4$
$T_m = 1$
$T_m = 15$
$M = 3E-4$
$T_m = 1$
$T_m = 15$
extinction
Two spiral waves
Two target waves
One spiral One target wave

Conclusions

- Virus spreading within species can promote coexistence
- Virus spreading across different species tend to hamper coexistence
- Inter-patch migration can induce coexistence in the form of novel target waves
- All these are obtained from **microscopic** model of evolutionary dynamics