Recent development in random planar maps: exercises for lecture I

Jérémie Bouttier and Linxiao Chen

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Abstract

The purpose of this exercise session is to enumerate rooted planar triangulations with a simple boundary, by solving Tutte's equation. The results will be useful for lecture IV. For some questions, a computer algebra system (Mathematica, Maple, Sage...) might be helpful.

1 Preliminaries

A *planar triangulation* is a planar map whose all faces have degree 3. We may actually distinguish different classes of triangulations, depending on their possible "singularities": general triangulations may have loops and multiple edges, and we refer to them as *type I* triangulations, *type II* triangulations may have multiple edges but no loops, finally *type III* triangulations have neither loops nor multiple edges.

Question 1 (Optional)**.** Show that a triangulation is of type II if and only if it is 2-connected, and of type III if and only if it is 3-connected. (Recall that a map or a graph is said *k-connected* if it remains connected whenever one removes at most $(k-1)$ of its vertices. Note that being 1-connected is the same as being connected, and this is the case for any map by definition.)

More generally, a *planar triangulation with a simple boundary* is a rooted planar map such that every non-root face has degree 3, and such that the root face (whose degree k is arbitrary) is simple (i.e. is incident to k distinct vertices, in other words there are no "pinch points"). Type I, II and III triangulations with a simple boundary are defined as above (no loops for type II, neither loops nor multiple edges for type III). The main purpose of this exercise is to enumerate planar triangulations of type II, by solving Tutte's equation.

2 Tutte's equation

For n, m nonnegative integers, we denote by $F_{n,m}$ the number of type II planar triangulations with a simple boundary of length $m+2$ and n internal vertices. It is convenient to view the *link map* (map reduced to a single edge, two vertices and one face) as such a triangulation with $n = m = 0$, hence by convention we set $F_{0,0} = 1$.

Question 2 (Optional)**.** For $n \geq 3$, let T_n be the number of rooted type II planar triangulations (without boundary). Explain why

$$T_n = F_{n-2,0} = F_{n-3,1}.$$

Would these equalities hold in type I or III?

2.1 Derivation

We introduce the generating functions

$$F \equiv F(t,z) = \sum_{n\geq 0}\sum_{m\geq 0} F_{n,m} t^n z^m, \qquad F_0 \equiv F_0(t) = \sum_{n\geq 0} F_{n,0} t^n.$$

Question 3. Derive Tutte's equation

$$F(t,z) = 1 + t\frac{F(t,z) - F_0(t)}{z} + zF(t,z)^2. \tag{1}$$

2.2 Solution I

We first derive a closed form expression for the number of triangulations without a boundary. We set $P(F, F_0, t, z) = F - 1 - t(F - F_0)/z - zF^2$ so that Tutte's equation amounts to

$$P(F(t,z), F_0(t), t, z) = 0. \tag{2}$$

Question 4. Show that there is a unique power series $U \equiv U(t) = 1 + o(t)$ such that

$$\frac{\partial P}{\partial F}(F(t, tU(t)), F_0(t), t, tU(t)) = 0. \tag{3}$$

Show that it also satisfies

$$\frac{\partial P}{\partial z}(F(t, tU(t)), F_0(t), t, tU(t)) = 0. \tag{4}$$

Question 5. By elimination, derive the algebraic equation satisfied by U. Express F_0 in terms of U. (In a more educated language, this is a rational parametrization of the spectral curve.)

We now recall the Lagrange inversion formula, see e.g. [1, Section A.6]: if $\phi \equiv \phi(y)$ is a power series whose constant coefficient is nonzero, then there is a unique power series $u \equiv u(t)$ satisfying $u = t\phi(u)$, and its coefficients read explicitly

$$[t^n]u(t) = \frac{1}{n}[y^{n-1}]\phi(y)^n, \qquad n \geq 1.$$

(Here $[t^n]u(t)$ denotes the coefficient of t^n in $u(t)$, etc.) Furthermore, for an arbitrary function H, we have

$$[t^n]H(u(t)) = \frac{1}{n}[y^{n-1}]\left(H'(y)\phi(y)^n\right), \qquad n \geq 1.$$

Question 6. Apply the Lagrange inversion formula to compute $[t^n]U(t)$ and $[t^n]F_0(t)$. (Hint: take $u(t) = U(t) - 1$.)

2.3 Solution II

With a bit more work it is possible to derive a bivariate closed form expression for $F_{n,m}$.

Question 7. Replace t and $F_0(t)$ by their expressions in terms of U in Tutte's equation, to obtain an algebraic equation relating F, U and z. What do you observe about its discriminant with respect to F? Show that

$$F = \frac{1 - U + 2zU^3 - (1 - U + 2zU^2)\sqrt{1 - 4zU^2}}{4z^2U^3}. \tag{5}$$

Question 8. Deduce a closed form expression for $[z^m]F(t,z)$ in terms of U. (Hint: $\sqrt{1-4x} = 1-2xC(x)$ where $C(x)$ is the generating function of Catalan numbers, $C(x) = \sum_{k\geq 0} \frac{(2k)!}{k!(k+1)!}x^k$.)

Question 9. By the Lagrange inversion formula, obtain a closed form expression for $F_{n,m} = [t^n z^m]F(z,t)$.

3 Asymptotics

Question 10. Show that the radius of convergence of $F_0(t)$ and, more generally, $[z^m]F(t,z)$ for any m, is

$$t_c = \frac{2}{27}.$$

Question 11. Compute $[z^m]F(t_c, z)$. (This is the partition function of the "free distribution" on rooted triangulations of the $(m+2)$-gon, useful when studying local limits.)

Question 12. Show that, for any fixed $m \geq 0$, we have as $n \to \infty$

$$F_{n,m} \sim C_m t_c^{-n} n^{-5/2}$$

and compute C_m as well as its asymptotic behaviour as $m \to \infty$.

References

[1] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[2] G. Schaeffer, *Conjugaison d'arbres et cartes combinatoires aléatoires*, PhD thesis, Université Bordeaux I (1998).

[3] M. Bousquet-Mélou and A. Jehanne, *Polynomial equations with one catalytic variable, algebraic series and map enumeration*, Journal of Combinatorial Theory, Series B 96 (2006) 623–672